Manifolds over Cayley-Dickson algebras and their immersions

Sergey V. Ludkowski

ABSTRACT. *Weakly holomorphic manifolds over Cayley-Dickson algebras are defined and their embeddings and immersions are studied.*

1. Introduction

Real and complex manifolds are widely used in different branches of mathematics [4, 16, 17, 18, 20, 21, 22, 34]. On the other hand, Cayley-Dickson algebras $\mathcal{A}_r$, particularly, the quaternion skew field $\mathbf{H} = \mathcal{A}_2$ and the octonion algebra $\mathbf{O} = \mathcal{A}_3$, have found many-sided applications not only in mathematics, but also in theoretical physics (see [2], [7] - [14], [16, 22, 36, 35] and references therein). Theory of functions of quaternion and octonion variables is presented in these works and cited below. Various classes of such functions and different variants of their super-differentiability were investigated and described depending on needs of mathematics and theoretical physics over quaternions, octonions and some other alternative algebras.

This paper continues previous works of the author, where different theory from the cited above publications was developed. Functions of Cayley-Dickson variables were studied earlier [23, 24, 29, 33]. Their super-differentiability was defined in terms of representing them words and phrases as a differentiation which is real-linear, additive and satisfying Leibniz' rule on an algebra of phrases over $\mathcal{A}_r$ (see in details Chapter 1 §§2.1 and 2.2 in the book [28] or in the articles [23, 29]). That is a weak version of a super-differentiability used in super-analysis. Though such weak super-differentiability over $\mathcal{A}_r$ of a function f on an open domain implies that f is locally analytic in an $\mathcal{A}_r$-variable with $\mathcal{A}_r$-coefficients in power series with definite order of the multiplication in each additive. A super-differentiable function on a domain U in $\mathcal{A}_r^n$ or $l_2(\mathcal{A}_r)$ of $\mathcal{A}_r$-variables is also called $\mathcal{A}_r$-differentiable (or weakly $\mathcal{A}_r$-holomorphic). For $r \geq 4$ the Cayley-Dickson algebras are non-associative and non-alternative. This approach appeared to be effective for investigations of problems of analysis, partial differential equations, operator theory, noncommutative geometry [25], [26] - [32].

This article is devoted to investigations of $\mathcal{A}_r$-differentiable manifolds (weakly holomorphic manifolds). Their embeddings and immersions are studied. Results and notations of previous papers [23, 24, 29, 33] are used below.

Main results of this paper are obtained for the first time.

2. Manifolds over Cayley-Dickson algebras

DEFINITION 2.1. *An* **R** *linear space X which is also left and right $\mathcal{A}_r$ module will be called an $\mathcal{A}_r$ vector space. We present X as the direct sum*

(DS) $X = X_0 i_0 \oplus ... \oplus X_m i_m \oplus ...,$ *where $X_0,...,X_m,...$ are pairwise isomorphic real linear spaces, where $i_0, ..., i_{2^r-1}$ are generators of the Cayley-Dickson algebra $\mathcal{A}_r$ such that $i_0 = 1$, $i_k^2 = -1$ and $i_k i_j = -i_j i_k$ for each $k \geq 1$ and $j \geq 1$ so that $k \neq j$, $2 \leq r$.*

Let X and Y be two **R** *linear normed spaces which are also left and right $\mathcal{A}_r$ modules, where $1 \leq r$, such that*

(1) $0 \leq \|ax\|_X \leq |a|\|x\|_X$ *and* $\|xa\|_X \leq |a|\|x\|_X$ *and*

(2) $\|ax_j\|_X = |a|\|x_j\|_X$ *and*

(3) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$

for all $x, y \in X$ and $a \in \mathcal{A}_r$ and $x_j \in X_j$. Such spaces X and Y will be called $\mathcal{A}_r$ normed spaces.

Suppose that X and Y are two normed spaces over the Cayley-Dickson algebra $\mathcal{A}_v$. A continuous **R** *linear mapping $\theta : X \to Y$ is called an* **R** *linear homomorphism. If in addition $\theta(bx) = b\theta(x)$ and $\theta(xb) = \theta(x)b$ for each $b \in \mathcal{A}_v$ and $x \in X$, then θ is called a homomorphism of $\mathcal{A}_v$ (two sided) modules X and Y.*

If a homomorphism is injective, then it is called an embedding (**R** *linear or for $\mathcal{A}_v$ modules correspondingly).*

If a homomorphism h is bijective and from X onto Y so that its inverse mapping h^{-1} is also continuous, then it is called an isomorphism (**R** *linear or of $\mathcal{A}_v$ modules respectively).*

DEFINITION 2.2. *We say that a real vector space Z is supplied with a scalar product if a bi-* **R** *-linear bi-additive mapping $<,>: Z^2 \to$* **R** *is given satisfying the conditions:*

(1) $< x, x > \; \geq 0$, $< x, x >= 0$ *if and only if $x = 0$;*

(2) $< x, y >=< y, x >$;

(3) $< ax + by, z >= a < x, z > + b < y, z >$ *for each real numbers $a, b \in$* **R** *and vectors $x, y, z \in Z$.*

Then an $\mathcal{A}_r$ vector space X is supplied with an $\mathcal{A}_r$ valued scalar product, if a bi- **R** *-linear bi-$\mathcal{A}_r$-additive mapping $< *, * >: X^2 \to \mathcal{A}_r$ is given such that*

(4) $< f, g >= \sum_{j,k} < f_j, g_k > i_j^* i_k,$

where $f = f_0 i_0 + ... + f_m i_m + ...,$ $f, g \in X$, $f_j, g_j \in X_j$, each X_j is a real linear space with a real valued scalar product, $(X_j, < *, * >)$ is real linear isomorphic

with $(X_k, < *, * >)$ and $< f_j, g_k > \in \mathbf{R}$ for each j, k. The scalar product induces the norm:

(5) $\|f\| := \sqrt{< f, f >}$.

An $\mathcal{A}_r$ normed space or an $\mathcal{A}_r$ vector space with $\mathcal{A}_r$ scalar product complete relative to its norm will be called an $\mathcal{A}_r$ Banach space or an $\mathcal{A}_r$ Hilbert space respectively.

A Hilbert space X over $\mathcal{A}_r$ is denoted by $l_2(\lambda, \mathcal{A}_r)$, where λ is a set of the cardinality $card(\lambda) \geq \aleph_0$ which is the topological weight of X_0, i.e. $X_0 = l_2(\lambda, \mathbf{R})$.

A mapping $f : U \to l_2(\lambda, \mathcal{A}_r)$ can be written in the form

$$f(z) = \sum_{j \in \lambda} f^j(z) e_j,$$

where $\{e_j : j \in \lambda\}$ is an orthonormal basis in the Hilbert space $l_2(\lambda, \mathcal{A}_r)$, U is a domain in $l_2(\psi, \mathcal{A}_r)$, $f^j(z) \in \mathcal{A}_r$ for each $z \in U$ and every $j \in \lambda$. If f is Frechét differentiable over $\mathbf{R}$ and each function $f^j(z)$ is differentiable by each Cayley-Dickson variable $_k z$ on U, then f is called $\mathcal{A}_r$-differentiable on U, where

$$z = \sum_{k \in \psi} {}_k z q_k,$$

while $\{q_k : k \in \psi\}$ denotes the standard orthonormal basis in $l_2(\psi, \mathcal{A}_r)$, $_k z \in \mathcal{A}_r$.

DEFINITION 2.3. Let M be a set such that

(1) $M = \bigcup_j U_j$, M is a Hausdorff topological space,

(2) each U_j is open in M,

(3) $\phi_j : U_j \to \phi_j(U_j) \subset X$ are homeomorphisms, $\phi_j(U_j)$ is open in X for each j,

(4) if $U_i \cap U_j \neq \emptyset$, the transition mapping $\phi_i \circ \phi_j^{-1}$ of charts is bijective and is $\mathcal{A}_r$-differentiable on its domain, while

(5) $\phi_i : M \to X$ with $\phi_i \circ \phi_j^{-1}$ being $\mathcal{A}_r$-differentiable on $\phi_j(U_j)$ for each $i \neq j$;

where X is either $\mathcal{A}_r^m$ with $m \in \mathbf{N}$ or a Hilbert space $l_2(\lambda, \mathcal{A}_r)$ over the Cayley-Dickson algebra $\mathcal{A}_r$. Then M is called an $\mathcal{A}_r$-differentiable manifold (or a weakly holomorphic manifold).

PROPOSITION 2.4. Let M be an $\mathcal{A}_r$- differentiable manifold. Then there exists a tangent bundle TM which has the structure of an $\mathcal{A}_r$- differentiable manifold such that each fibre $T_x M$ is the vector space over the Cayley-Dickson algebra $\mathcal{A}_r$.

Proof. The Cayley-Dickson algebra $\mathcal{A}_r$ has the real shadow, which is the Euclidean space $\mathbf{R}^{2^r}$, since $\mathcal{A}_r$ is the algebra over $\mathbf{R}$. Therefore, a manifold M

has also a real manifold structure. Each $\mathcal{A}_r$- differentiable mapping is infinite differentiable in accordance with Theorems 2.15 and 3.10 in [33, 23]. Then the tangent bundle TM exists, which is C^∞-manifold such that each fibre $T_x M$ is a tangent space, where $x \in M$, T is the tangent functor. If $At(M) = \{(U_j, \phi_j) : j\}$, then $At(TM) = \{(TU_j, T\phi_j) : j\}$, $TU_j = U_j \times X$, where X is the $\mathcal{A}_r$ vector space on which M is modeled, $T(\phi_j \circ \phi_k^{-1}) = (\phi_j \circ \phi_k^{-1}, D(\phi_j \circ \phi_k^{-1}))$ for each $U_j \cap U_k \neq \emptyset$. Each transition mapping $\phi_j \circ \phi_k^{-1}$ is $\mathcal{A}_r$- differentiable on its domain, then its (strong) differential coincides with the super-differential $D(\phi_j \circ \phi_k^{-1}) = D_z(\phi_j \circ \phi_k^{-1})$, since $\tilde{\partial}(\phi_j \circ \phi_k^{-1}) = 0$. Therefore, the super-differential $D(\phi_j \circ \phi_k^{-1})$ is $\mathbf{R}$-linear and $\mathcal{A}_r$-additive, hence it is an automorphism of the $\mathcal{A}_r$ vector space X. But $D_z(\phi_j \circ \phi_k^{-1})$ is $\mathcal{A}_r$- differentiable as well, consequently, TM is the $\mathcal{A}_r$- differentiable manifold.

$\square$

DEFINITION 2.5. A C^1-mapping $f : M \to N$ is called an immersion, if the real rank of df is $rang(df|_x : T_x M \to T_{f(x)} N) = m_M$ for each $x \in M$, where $m_M := dim_{\mathbf{R}} M$. An immersion $f : M \to N$ is called an embedding, if f is a homeomorphism on its image.

THEOREM 2.6. Let M be a compact $\mathcal{A}_r$- differentiable manifold, $dim_{\mathcal{A}_r} M = m < \infty$, where $2 \leq r \in \mathbf{N}$.

(I). Then there exists an $\mathcal{A}_r$- differentiable embedding $\tau : M \hookrightarrow \mathcal{A}_r^{2m+1}$ and an $\mathcal{A}_r$- differentiable immersion $\theta : M \to \mathcal{A}_r^{2m}$ correspondingly.

(II). If M is a paracompact $\mathcal{A}_r$- differentiable manifold with countable atlas on $l_2(\lambda, \mathcal{A}_r)$, where $card(\lambda) \geq \aleph_0$, then there exists a $\mathcal{A}_r$- differentiable embedding $\tau : M \hookrightarrow l_2(\lambda, \mathcal{A}_r)$.

Proof. (I). For the proof of this theorem identities of Cayley-Dickson algebras are used below. This permits to supply the unit sphere of suitable dimension multiple of 2^r with the structure of an $\mathcal{A}_r$ differentiable manifold (see below), where $2 \leq r \in \mathbf{N}$. Then charts of a suitable refined atlas with $\mathcal{A}_r$- differentiable transition mappings are used.

Let at first M be compact. Since M is compact, then it is finite dimensional over the Cayley-Dickson algebra $\mathcal{A}_r$, $dim_{\mathcal{A}_r} M = m \in \mathbf{N}$, such that $dim_{\mathbf{R}} M = 2^r m$ is its real dimension. Take an atlas $At'(M)$ refining the initial atlas $At(M)$ of M such that (U'_j, ϕ_j) are charts of M, where each U'_j is $\mathcal{A}_r$- differentiable diffeomorphic to an interior of the unit ball $Int(B(\mathcal{A}_r^m, 0, 1))$, where $B(\mathcal{A}_r^m, y, \rho) := \{z \in \mathcal{A}_r^m : |z - y| \leq \rho\}$. In view of compactness of the manifold M a covering $\{U'_j : j\}$ has a finite subcovering, hence $At'(M)$ can be chosen finite. Denote for convenience the latter atlas as $At(M)$. Let (U_j, ϕ_j) be the chart of the atlas $At(M)$, where U_j is open in M, hence $M \setminus U_j$ is closed in M.

Consider the space $\mathcal{A}_r^m \times \mathbf{R}$ as the $\mathbf{R}$-linear space $\mathbf{R}^{2^r m+1}$, i.e. its real shadow. The unit sphere $S^{2^r m} := S(\mathbf{R}^{2^r m+1}, 0, 1) := \{z \in \mathbf{R}^{2^r m+1} : |z| = 1\}$

in $\mathcal{A}_r^m \times \mathbf{R}$ can be supplied with two charts (V_1, ϕ_1) and (V_2, ϕ_2) such that $V_1 := S^{2^r m} \setminus \{0, ..., 0, 1\}$ and $V_2 := S^{2^r m} \setminus \{0, ..., 0, -1\}$, where ϕ_1 and ϕ_2 are stereographic projections from poles $\{0, ..., 0, 1\}$ and $\{0, ..., 0, -1\}$ of V_1 and V_2 respectively onto $\mathcal{A}_r^m$. Then the transition mapping between two charts $\phi_2 \circ \phi_1^{-1} : \mathbf{E} \setminus \{0\} \to \mathbf{E} \setminus \{0\}$ is given by the formula $\phi_2 \circ \phi_1^{-1}(y) = y/|y|^2$ where $y = (y_1, ..., y_{2^r m}) \in \mathbf{E} \setminus \{0\}$, $\mathbf{E} = \mathbf{R^{2^r m}}$ (see §1.1.3 [20]). On the other hand the Euclidean space $\mathbf{E}$ is the real shadow of $\mathcal{A}_r^m$. We denote the unit sphere in $\mathcal{A}^m \times \mathbf{R}$ by $S^{2^r m}$ also.

To rewrite a function from the real variables z_j in the z-representation or vice versa the following identities are used:

$$z_j = \frac{1}{2}\left[-zi_j + i_j(2^r - 2)^{-1}\left(-z + \sum_{k=1}^{2^r-1} i_k(zi_k^*) \right) \right] \qquad (1)$$

for each $j = 1, 2, ..., 2^r - 1$,

$$z_0 = \frac{1}{2}\left[z + (2^r - 2)^{-1}\left(-z + \sum_{k=1}^{2^r-1} i_k(zi_k^*) \right) \right] \qquad (2)$$

where $2 \leq r \in \mathbf{N}$, z is a Cayley-Dickson number decomposed as

$$z = z_0 i_0 + ... + z_{2^r-1} i_{2^r-1} \in \mathcal{A}_r \qquad (3)$$

with $z_j \in \mathbf{R}$ for each j, $i_k^* = \tilde{i}_k = -i_k$ for each $k > 0$, $i_0 = 1$, since $i_j(i_j i_k) = -i_k$ and $(i_k i_j)i_j = -i_k$ for each $j > 0$, also $i_j i_k = -i_k i_j$ for each $j \neq k$ with $j > 0$ and $k > 0$, while $i_k(i_0 i_k^*) = 1$ for each k. Formulas (1)-(3) define the real-linear projection operators $\pi_j : \mathcal{A}_r \to \mathbf{R}$ so that

$$\pi_j(z) = z_j \qquad (4)$$

for each Cayley-Dickson number $z \in \mathcal{A}_r$ and every $j = 0, 1, ..., 2^r - 1$.

The conjugation is given by the formula:

$$z^* = -(2^r - 2)^{-1} \sum_{p=0}^{2^r-1} (i_p z) i_p \qquad (5)$$

in $\mathcal{A}_r^m$ due to formulas (1)-(3), which provides z^* in the z-representation, where $i_0, ..., i_{2^r-1}$ are the standard generators of the Cayley-Dickson algebra $\mathcal{A}_r$. Therefore the transition mapping $\phi_2 \circ \phi_1^{-1} : \mathcal{A}_r^m \setminus \{0\} \to \mathcal{A}_r^m \setminus \{0\}$ has the form in the z-representation:

$$\phi_2 \circ \phi_1^{-1}(z) = -\frac{(2^r - 2)z}{\sum_{k=1}^m [_k z \sum_{p=0}^{2^r-1} (i_{p\ k}z)i_p]}, \qquad (6)$$

where $z = (_1z, ..., _mz)$ with $_jz \in \mathcal{A}_r$ for each $j = 1, ..., m$, $z \in \mathcal{A}_r^m \setminus \{0\}$. The transition mapping is presented as the fraction of two polynomials on the domain on which the denominator is non-zero. The fraction of two $\mathcal{A}_r$-differentiable functions is $\mathcal{A}_r$-differentiable on a domain where the denominator is non-zero [24, 29]. Therefore, $\phi_2 \circ \phi_1^{-1}(z)$ is the $\mathcal{A}_r$-differentiable diffeomorphism in $\mathcal{A}_r^m \setminus \{0\}$, i.e. the (weak) super-differential $D_z(\phi_2 \circ \phi_1^{-1})$ exists. Thus in the $\mathcal{A}_r$ realization $\phi_l(V_l) = \mathcal{A}_r^m \setminus \{0\}$ for $l = 1$ and $l = 2$ the unit sphere $S^{2^r m}$ is supplied with the structure of the $\mathcal{A}_r$-differentiable manifold.

If $g : M \to \mathcal{A}_r^N$ is a continuous mapping, then $g(M)$ is compact, since M is compact (see Theorem 3.1.10 [6]). Therefore, $g(M)$ is bounded and closed in $\mathcal{A}_r^N$ (see Theorems 3.1.8 and 3.1.23 [6]). Thus there exists a shift $h(z) = z + q$ on $\mathcal{A}_r^N$ such that $h(g(M))$ does not contain zero and hence $\inf\{|z| : z \in h(g(M))\} > 0$.

We consider $[Int(B(\mathcal{A}_r^m, 0, 1)) + q] \subset \mathcal{A}_r^m \setminus \{0\}$ with $q \in \mathcal{A}_r^m$ such that $|q| > 1$ and $At'(M)$ as above. The finite union of such balls $[Int(B(\mathcal{A}_r^m, 0, 1)) + q]$ is bounded in $\mathcal{A}_r^m \setminus \{0\}$. The shift mapping $z \mapsto z + q$ is $\mathcal{A}_r$-differentiable on $\mathcal{A}_r^m$. On the other hand, the manifold M is compact and each its atlas has a finite subatlas, where an atlas of M satisfies Conditions $3(1-5)$ above.

Simplifying the notation we can choose an atlas $\{(E_j, \xi_j) : j = 1, ..., n\}$ of M with mappings ξ_j satisfying the following properties: each $\xi_j : E_j \to \xi_j(E_j)$ is the $\mathcal{A}_r$-differentiable diffeomorphism onto the subset $\xi_j(E_j)$ in the ball $B(\mathcal{A}_r^m, q, b)$ with $|q| > 4b > 0$, whilst $\xi_j : M \to \mathcal{A}_r^m$ is $\mathcal{A}_r$-differentiable, $cl_M(E_j) \subset H_j$, $E_j \subset H_j$, H_j is open in M for each j, the restriction $\xi_j|_{H_j}$ is bijective, $\xi_j(M) \subset B(\mathcal{A}_r^m, q, 2b)$,
$$\inf\{|x - y| : x \in \partial\xi_j(E_j), y \in \partial\xi_j(H_j)\} > b/2,$$
where $\bigcup_j E_j = M$, $cl_M(E)$ denotes the closure of E in M, $\partial V := cl_{\mathcal{A}_r^m}(V) \setminus Int_{\mathcal{A}_r^m}(V)$ for a subset V in $\mathcal{A}_r^m$.

The function of the form

$$f_j(z) = \exp\left(\sum_{k=1}^m b_{k,j}\left[(_kz - _kw_j)\sum_{p=0}^{2^r-1}\left(i_p(_kz - _kw_j)\right)i_p\right]\right) \qquad (7)$$

with positive constants $b_{k,j}$ and a marked point $w_j \in \mathcal{A}_r^m$ is positive $\mathcal{A}_r$-differentiable bounded on $\mathcal{A}_r^m$ and tending to zero when $|z|$ tends to the infinity, see (5). For each bounded canonical closed subset W in $\mathcal{A}_r^m$ and its open covering $\mathcal{W}$ it is possible to choose a finite open covering $\{W_j : j = 1, ..., l\}$ of W which refines $\mathcal{W}$, since W is compact. We take W_j being intersections of open balls in $\mathcal{A}_r^m$ with W. There exist constants $c_j > 0$ and $b_{k,j} > 0$ and $w_j \in \mathcal{A}_r^m$ such that

$$g_j(z) = \frac{c_j f_j(z)}{\sum_{j=1}^l c_j f_j(z)} \qquad (8)$$

is positive and $\mathcal{A}_r$-differentiable on W and $g_j(z) < g_j(y)$ for each $z \in W \setminus W_j$

and $y \in W_j$. We can choose constants so that

$$c_1 g_j(z) > c_2 g_j(y) \qquad (9)$$

for each $z \in \xi_j(E_j)$ and $y \in \xi_j(M \setminus H_j)$, where $c_1 = \inf\{|x| : x \in \xi_j(E_j)\}$ and $c_2 = \sup\{|x| : x \in \xi_j(M \setminus H_j)\}$.

Evidently, $g(z) = \sum_{j=1}^{l} g_j(z)$ is identically unit on $\mathcal{A}_r^m$. The product of $\mathcal{A}_r$-differentiable functions is $\mathcal{A}_r$-differentiable.

Using charts (E_j, ξ_j) and of the atlas of M, the open covering $\{H_j : j\}$ of M as above and such functions g_j one can choose $\mathcal{A}_r$-differentiable mappings ψ_j for each j so that $\psi_j(M) \subset V_k^m$, where either $k = 1$ or $k = 2$, U_j and A_j are open subsets in M with $U_j \subset A_j$ for each $j = 1, ..., n$, $\bigcup_{j=1}^{n} U_j = M$, $\psi_j|_{A_j}$ is bijective for each j, and

$$|\bar{\phi}_k \circ \psi_j(y)| < |\bar{\phi}_k \circ \psi_j(z)| \qquad (10)$$

for each $z \in U_j$ and $y \in M \setminus A_j$, where $\bar{\phi}_k = (\phi_k, ..., \phi_k) : V_k^m \to \mathcal{A}_r^m$, while $\phi_k : V_k \to \mathcal{A}_r^m$ is given above.

The family of such component mappings ψ_j induces an $\mathcal{A}_r$-differentiable diffeomorphism: $\psi : M \to (S^{2^r m})^n$ with n equal to the number of charts, where $\psi(z) := (\psi_1(z), ..., \psi_n(z))$ for each $z \in M$.

Then the mapping $\psi(z)$ is the embedding into $(S^{2^r m})^n$ and hence into $\mathcal{A}_r^{n(m+1)}$, since the rank is $rank[d_z \psi(z)] = 2^r m$ at each point $z \in M$. Indeed, the rank is $rank[d_z \psi_j(z)] = 2^r m$ for each $z \in U_j$ and the dimension is bounded from above $dim_{\mathcal{A}_r} \psi(U_j) \leq dim_{\mathcal{A}_r} M = m$. If y and z are two distinct points in M, then there exists j so that $z \in U_j$. If $y \in A_j$, then $\psi_j(z) \neq \psi(y)$, since $\psi_j|_{A_j}$ is bijective. If $y \in M \setminus A_j$, then from inequality (10) it follows, that $\psi_j(z) \neq \psi_j(y)$. Therefore, $\psi(z) \neq \psi(y)$ for each two distinct points z and y in M, since a natural number j exists so that $\psi_j(z) \neq \psi_j(y)$.

Let $M \hookrightarrow \mathcal{A}_r^N$ be the $\mathcal{A}_r$- differentiable embedding as above. There is also the $\mathcal{A}_r$- differentiable embedding of M into $(S^{2^r m})^n$ as it is shown above, where $(S^{2^r m})^n$ is the $\mathcal{A}_r$- differentiable manifold as the product of $\mathcal{A}_r$- differentiable manifolds.

Let $P\mathbf{R}^n$ denote the real projective space formed from the Euclidean space $\mathbf{R}^{n+1}$, denote by $\phi : \mathbf{R}^{n+1} \setminus \{0\} \to P\mathbf{R}^n$ the corresponding projective mapping. Geometrically $P\mathbf{R}^n$ is considered as S^n/τ, where $S^n := \{y \in \mathbf{R}^{n+1} : \|y\| = 1\}$ is the unit sphere in $\mathbf{R}^{n+1}$, while τ is the equivalence relation making identical two spherically symmetric points, i.e. points belonging to the same straight line containing zero and intersecting the unit sphere.

We consider $\mathcal{A}_r^n$ as the algebra of all $n \times n$ diagonal matrices

$$A = diag(a_1, ..., a_n)$$

with entries $a_1, ..., a_n \in \mathcal{A}_r$. It naturally has the structure of the left- and right- $\mathcal{A}_r$-module. Then $\mathcal{A}_r^n$ is isomorphic with the tensor product of algebras

$\mathcal{A}_r^n = \mathcal{A}_r \otimes_{\mathbf{R}} \mathbf{R}^n$ over the real field, where $\mathbf{R}^n$ is considered as the algebra of all diagonal $n \times n$ matrices $C = diag(b_1, .., b_n)$ with entries $b_1, ..., b_n \in \mathbf{R}$. Using this realization of $\mathcal{A}_r^n$ we get an extension of ϕ from $\mathbf{R}^{n+1}$ onto $\mathcal{A}_r \otimes_{\mathbf{R}} \mathbf{R}^{n+1}$ by the formulas:

$$\phi(ax) = a\phi(x) \tag{11}$$

and

$$\phi(xa) = \phi(x)a \tag{12}$$

for each $a \in \mathcal{A}_r$ with $|a| = 1$ and every $x \in \mathbf{R}^{n+1} \setminus \{0\}$, also

$$\phi(x_0 i_0 + ... + x_{2^r-1} i_{2^r-1}) = \phi(x_0) i_0 \alpha_0 + ... + \phi(x_{2^r-1}) i_{2^r-1} \alpha_{2^r-1} \tag{13}$$

for each non-zero vector $x = x_0 i_0 + ... + x_{2^r-1} i_{2^r-1} \in \mathcal{A}_r^{n+1}$, where $\alpha_j := \|x_j\|/\|x\|$, $x_j \in \mathbf{R}^{n+1}$ for each j, the norm is given by the usual formula

$$\|x\|^2 = \|x_0\|^2 + ... + \|x_{2^r-1}\|^2. \tag{14}$$

Then we put by our definition $P\mathcal{A}_r^n = \phi([\mathcal{A}_r \otimes_{\mathbf{R}} \mathbf{R}^{n+1}] \setminus \{0\})$ to be the Cayley-Dickson projective space.

If $z \in P\mathcal{A}_r^n$, then by our definition $\phi^{-1}(z)$ is the $\mathcal{A}_r$ straight line in $\mathcal{A}_r^{n+1}$. To each element $x \in \mathcal{A}_r^{n+1}$ we pose an $\mathcal{A}_r$ straight line $< \mathcal{A}_r, x \} := \phi^{-1}(\phi(x))$. That is the bundle of all $\mathcal{A}_r$ straight lines $< \mathcal{A}_r, x \}$ in $\mathcal{A}_r^{n+1}$ is considered, where $x \in \mathcal{A}_r^{n+1}$, $x \neq 0$. Then $< \mathcal{A}_r, x \}$ is the $\mathcal{A}_r$ vector space of dimension 1 over $\mathcal{A}_r$ due to formulas (11)-(14) above. Therefore, $< \mathcal{A}_r, x \}$ has the real shadow isomorphic with $\mathbf{R}^{2^r}$, since the standard generators $i_0, i_1, ..., i_{2^r-1}$ are linearly independent over the real field $\mathbf{R}$.

Fix the standard orthonormal base $\{e_1, ..., e_N\}$ in $\mathcal{A}_r^N$ and projections on $\mathcal{A}_r$-vector subspaces relative to this base

$$P^L(x) := \sum_{e_j \in L} x_j e_j \tag{15}$$

for the $\mathcal{A}_r$ vector span $L = span_{\mathcal{A}_r}\{e_i : i \in \Lambda_L\}$, $\Lambda_L \subset \{1, ..., N\}$, where

$$x = \sum_{j=1}^N x_j e_j, \tag{16}$$

$x_j \in \mathcal{A}_r$ for each j, $e_j = (0, ..., 0, 1, 0, ..., 0)$ with 1 at j-th place. This means in particular that the projective space $P\mathcal{A}_r^n$ has the dimension $n - 1$ over the Cayley-Dickson algebra $\mathcal{A}_r$. In this base consider the $\mathcal{A}_r$-Hermitian scalar product

$$< x, y >:= \sum_{j=1}^N x_j^* y_j. \tag{17}$$

Let $l \in P\mathcal{A}_r^{N-1}$, take an $\mathcal{A}_r$-hyperplane denoted by $(\mathcal{A}_r^{N-1})_l$ and given by the condition:

$$< x, y >= 0 \text{ for each } x \in (\mathcal{A}_r^{N-1})_l \text{ and } y \in l\,. \tag{18}$$

We take a vector $0 \neq [l] \in \mathcal{A}_r^N$ as a representative characterizes the equivalence class $l = < \mathcal{A}_r, [l]\}$ of unit norm $\|[l]\| = 1$. Then the orthonormal base $\{q_1, ..., q_{N-1}\}$ in $(\mathcal{A}_r^{N-1})_l$ and the vector $[l] =: q_N$ compose the orthonormal base $\{q_1, ..., q_N\}$ in $\mathcal{A}_r^N$. This provides the $\mathcal{A}_r$- differentiable projection $\pi_l : \mathcal{A}_r^N \to (\mathcal{A}_r^{N-1})_l$ relative to the orthonormal base $\{q_1, ..., q_N\}$. Indeed, the operator π_l is $\mathcal{A}_r$ left $\pi_l(bx_0) = b\pi_l(x_0)$ and also right $\pi_l(x_0 b) = \pi_l(x_0)b$ linear for each $x_0 \in X_0$ and $b \in \mathcal{A}_r$, but certainly non-linear relative to $\mathcal{A}_r$. Therefore the mapping π_l is $\mathcal{A}_r$- differentiable.

To construct an immersion it is sufficient, that each projection $\pi_l : T_x M \to (\mathcal{A}_r^{N-1})_l$ has $ker[d(\pi_l(x))] = \{0\}$ for each $x \in M$. The set of all points $x \in M$ for which $ker[d(\pi_l(x))] \neq \{0\}$ is called the set of forbidden directions of the first kind. Forbidden are those and only those directions $l \in P\mathcal{A}_r^{N-1}$ for which there exists a point $x \in M$ such that $l' \subset T_x M$, where $l' = [l] + z$, $z \in \mathcal{A}_r^N$. The set of all forbidden directions of the first kind forms the $\mathcal{A}_r$- differentiable manifold Q due to formulas (11)-(18) and (1)-(3).

This manifold Q consists of points (x, l) with $x \in M$ and $l \in P\mathcal{A}_r^{N-1}$ so that $[l] \in T_x M$. The manifold M is m-dimensional over the Cayley-Dickson algebra $\mathcal{A}_r$. The tangent bundle TM has the structure of an $\mathcal{A}_r$- differentiable manifold of dimension $2m$ over the Cayley-Dickson algebra $\mathcal{A}_r$ in accordance with Proposition 4 above. Each point x in the manifold M has an open neighborhood locally homeomorphic with an open neighborhood of zero in $T_x M$. Then $dim_{\mathcal{A}_r} T_x M = m$ and hence $P(T_x M)^2$ is isomorphic with $P\mathcal{A}_r^{2m-1}$. On the other hand, the dimension of the projective space $P\mathcal{A}_r^{2m-1}$ over the Cayley-Dickson algebra $\mathcal{A}_r$ is $2m-1$ (see also Formulas $(1-4)$). Therefore, the manifold Q has the $\mathcal{A}_r$ dimension $(2m-1)$. Take the mapping $g : Q \to P\mathcal{A}_r^{N-1}$ given by $g(x, l) := l$. Then this mapping g is $\mathcal{A}_r$- differentiable in view of Proposition 2.4 and formulas (11)-(18) and (1)-(3).

Each paracompact manifold A modeled on $\mathcal{A}_r^p$ can be supplied with the Riemann manifold structure also. Therefore, on a manifold A there exists a Riemann volume element. In view of the Morse theorem $\mu(g(Q)) = 0$, if $N - 1 > 2m - 1$, that is, $2m < N$, where μ is the Riemann volume element in $P\mathcal{A}_r^{N-1}$. In particular, $g(Q)$ is not equal to the whole $P\mathcal{A}_r^{N-1}$ and there exists $l_0 \notin g(Q)$, consequently, there exists $\pi_{l_0} : M \to (\mathcal{A}_r^{N-1})_{l_0}$. This procedure can be prolonged, when $2m < N - k$, where k is the number of the step of projection. Hence M can be immersed into $\mathcal{A}_r^{2m}$.

Consider now the forbidden directions of the second type: $l \in P\mathcal{A}_r^{N-1}$, for which there exist $x \neq y \in M$ simultaneously belonging to l after suitable parallel translation $[l] \mapsto [l] + z$, $z \in \mathcal{A}_r^N$. The set of the forbidden directions of the second type forms the manifold $\Phi := M^2 \setminus \Delta$, where $\Delta := \{(x, x) :$

$x \in M\}$. Consider $\psi : \Phi \to P\mathcal{A}_r^{N-1}$, where $\psi(x,y)$ is the straight $\mathcal{A}_r$-line with the direction vector $[x,y]$ in the orthonormal base. Then $\mu(\psi(\Phi)) = 0$ in $P\mathcal{A}_r^{N-1}$, if $2m+1 < N$. Then the closure $cl(\psi(\Phi))$ coincides with $\psi(\Phi) \cup g(Q)$ in $P\mathcal{A}_r^{N-1}$. Hence there exists $l_0 \notin cl(\psi(\Phi))$. Then consider $\pi_{l_0} : M \to (\mathcal{A}_r)_{l_0}^{N-1}$. This procedure can be prolonged, when $2m+1 < N-k$, where k is the number of the step of projection. Hence M can be embedded into $\mathcal{A}_r^{2m+1}$.

(II). Let now M be a paracompact $\mathcal{A}_r$- differentiable manifold with countable atlas on $l_2(\lambda, \mathbf{K})$. Spaces $l_2(\lambda, \mathcal{A}_r) \oplus \mathcal{A}_r^m$ and $l_2(\lambda, \mathcal{A}_r) \oplus l_2(\lambda, \mathcal{A}_r)$ are isomorphic as $\mathcal{A}_r$ Hilbert spaces with $l_2(\lambda, \mathcal{A}_r)$, since $card(\lambda) \geq \aleph_0$. Take an additional variable $z \in \mathcal{A}_r$, when $z = j \in \mathbf{N}$. Then it gives a number of a chart. Each TU_j is $\mathcal{A}_r$- differentiably diffeomorphic with $U_j \times l_2(\lambda, \mathcal{A}_r)$. Consider $\mathcal{A}_r$- differentiable functions ψ on domains in $l_2(\lambda, \mathcal{A}_r) \oplus l_2(\lambda, \mathcal{A}_r) \oplus \mathcal{A}_r$. Then there exists an $\mathcal{A}_r$- differentiable mapping $\psi_j : M \to l_2(\lambda, \mathcal{A}_r)$ such that $\psi_j : U_j \to \psi_j(U_j) \subset l_2(\lambda, \mathcal{A}_r)$ is an $\mathcal{A}_r$- differentiable diffeomorphism. Then the mapping $(\psi_1, \psi_2, ...)$ provides the $\mathcal{A}_r$- differentiable embedding of M into $l_2(\lambda, \mathcal{A}_r)$. $\qquad\square$

REMARK 2.7. *Theorem 2.6 is the extension of the immersion and embedding Whitney theorems to $\mathcal{A}_r$-differentiable manifolds (see also Theorems 1, 2 and Footnote 4 in [37]; or Theorems 1.3.4, 1.3.5 and Proposition 2.1.0 in [18]; or Theorem in §11 Chapter II.2 [4]).*

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Author's address:

Sergey V. Ludkowski
Dept. Appl. Mathematics,
Moscow State Technical University MIREA,
av. Vernadsky 78, Moscow, 119454, Russia
E-mail: Ludkowski@mirea.ru