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RH-regular transformation of unbounded double sequences

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Abstract. *At the Ithaca meeting in 1946 it was conjectured that it is possible to construct a two-dimensional regular summability matrix* $A = \{a_{n,k}\}$ *with the property that, for every real sequence* $\{s_k\}$, *the transformed sequence*

$$t_n = \sum_{k=0}^{\infty} a_{n,k} s_k$$

possesses at least one limit point in the finite plane. It was also counter-conjectured that, for every regular summability matrix A, *there exists a single sequence* $\{s_k\}$ *such that the transformed sequence* t_n *tends to infinity monotonically. In 1947 Erdos and Piranian presented answers to these conjectures. The goal of this paper is to present a multidimensional version of the above conjectures. The first conjecture is the following: A four-dimensional RH-regular summability matrix* $A = \{a_{m,n,k,l}\}$ *can be constructed with the property that every double sequence* $\{s_{k,l}\}$ *transformed into the double sequence*

$$t_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} s_{k,l}$$

possesses at least one Pringsheim limit point in the finite plane. The multidimensional counter-conjecture is the following. For every RH-regular summability matrix A *there exists a double sequence* $\{s_{k,l}\}$ *such that the four-dimensional transformed double sequence* $\{t_{m,n}\}$ *tends to infinity monotonically Pringsheim sense. This paper established that both multidimensional conjectures are false.*

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1. Definitions, Notations, and Preliminary Results

Definition 1.1. *[Pringsheim, [4]] A double sequence $x = [x_{k,l}]$ has a* **Pringsheim limit** *L (denoted by P-$\lim x = L$) provided that, given an $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. Such an x is described more briefly as "P-convergent".*

Definition 1.2. *[Patterson, [3]] A double sequence y is a* **double subsequence** *of x provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that, if $x_j = x_{n_j,k_j}$, then y is formed by*

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & -. \end{array}$$

In [5] Robison presented the following notion of conservative four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such notion.

Definition 1.3. *A four-dimensional matrix A is said to be* **RH-regular** *if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.*

This assumption of boundedness is made because a double sequence which is P-convergent is not necessarily bounded. Along these same lines, Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [2] and [5].

Theorem 1.4. *(Hamilton [2], Robison [5]) The four dimensional matrix A is RH-regular if and only if*

- RH_1: P-$\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*
- RH_2: P-$\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$;
- RH_3: P-$\lim_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each l;*
- RH_4: P-$\lim_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each k;*
- RH_5: $\sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent;*
- RH_6: *there exist finite positive integers Δ and Γ such that* $\sum_{k,l>\Gamma} |a_{m,n,k,l}| < \Delta$.

Definition 1.5. *Let A be a four dimensional matrix with pairwise column (m, n). Then the (i, j)-reverse L-string , denoted by, $L^{m,n}_{i,j}$ is*

$$\{a_{m,n,1,j}, a_{m,n,2,j}, a_{m,n,3,i}, \cdots, a_{m,n,i,j}, a_{m,n,i,j-1}, a_{m,n,i,j-2}, \cdots, a_{m,n,i,1}, \}.$$

Given a double sequence x the (i, j)-reverse L-string , denoted by, $L_{i,j}$ is

$$\{x_{1,j}, x_{2,j}, x_{3,i}, \cdots, x_{i,j}, x_{i,j-1}, x_{i,j-2}, \cdots, x_{i,1}, \}.$$

2. Main Results

THEOREM 2.1. *If A is a pairwise-row finite RH-regular summability matrix then there exists a double sequence $\{s_{k,l}\}$ such that the corresponding transformed double sequence $|t_{m,n}|$ tends to infinity, in the Pringsheim with arbitrary rapidity.*

Proof. Let A be a pairwise-row finite RH-regular summability matrix. If m_0 and n_0 are sufficiently large, then each pairwise index whose indices exceed m_0 and n_0, respectively, contains a non-zero element. For fixed pairwise column index (m,n) let C-string denote the last column of the pairwise row whose sum is non-zero, and R-string denote the last row of the pairwise row whose sum is non-zero. Using the terms from C-string and R-string along with Definition 1.5 we can now construct a last reverse L-string whose sum is non-zero. Therefore a terminal reverse L-string exists. Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ and $\beta_1, \beta_2, \beta_3, \cdots$ be the indices of the pairwise-columns that contain terminal reverse L-string. Without of loss of generality we may assume that $\alpha_1 < \alpha_2 < \alpha_3 < \cdots$ and $\beta_1 < \beta_2 < \beta_3 < \cdots$. Define the terms of $\{s_{k,l}\}$ such that

$$k \neq \alpha_1, \alpha_2, \alpha_3, \cdots$$

and

$$l \neq \beta_1, \beta_2, \beta_3, \cdots$$

be arbitrary. Since A is pairwise row finite, each pairwise-column contains at most a finite number of pairwise-terminal reverse L-string of elements, that is, for each pairwise column the pairwise-terminal reverse L-string of element are bounded away from zero. If $f(m,n)$ is any arbitrary real function the terms

$$\begin{matrix} s_{k_1,l_1} & s_{k_1,l_2} & \cdots \\ s_{k_2,l_1} & s_{k_2,l_2} & \cdots \\ \cdots & \cdots & \ddots \end{matrix}$$

can be chosen large enough so that $|t_{m,n}| > f(m,n)$; $m > m_0$ and $n > n_0$. $\square$

THEOREM 2.2. *If A is an RH-regular summability matrix then there exists a double sequence $\{s_{m,n}\}$ such that the transformed double sequence $\{t_{m,n}\}$ has no P-limit points in the finite plane.*

Proof. Let c be a constant such that $\sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}| < \frac{c}{5}$ for all (m,n). Such a constant exists by RH_5 of the RH-regularity conditions of A. We can choose $m_0 = n_0$ sufficiently large such that, regularity conditions RH_3, RH_4, and RH_5 of A assure us, that there exists a pair (α_1, β_1) such that

$$\sum_{\{(k,l):k>\alpha_1 \text{ or } l>\beta_1\}} |a_{m_0,n_0,k,l}| < \frac{1}{c^2}.$$

Now choose m_1 and n_1 with $m_1 > m_0$ and $n_1 > n_0$ such that

$$\sum_{\{(k,l):0\leq k\leq\alpha_1;0\leq l\leq\beta_1\}} |a_{m,n,k,l}| < \frac{1}{5}$$

for $m > m_1$ and $n > n_1$ by RH_1. Let us construct the second stage. Conditions RH_3, RH_4, and RH_5 assure us that we can choose (α_2, β_2) with $\alpha_2 > \alpha_1$ and $\beta_2 > \beta_1$ such that

$$\sum_{\{(k,l):k>\alpha_2 \text{ or } l>\beta_2\}} |a_{m,n,k,l}| < \frac{1}{c^4}$$

whenever $m, n \leq m_1, n_1$, respectively. Using RH_1, we can now choose m_2 and n_2 with $m_2 > m_1$ and $n_2 > n_1$ such that

$$\sum_{\{(k,l):0\leq k\leq\alpha_2;0\leq l\leq\beta_2\}} |a_{m,n,k,l}| < \frac{1}{5}$$

for $m > m_2$ and $n > n_2$. Using the RH-regularity conditions of A the general stage is constructed as follows. Let (α_r, β_s)be such that $\alpha_r > \alpha_{r-1}$ and $\beta_s > \beta_{s-1}$ with

$$\sum_{\{(k,l):k>\alpha_r \text{ or } l>\beta_s\}} |a_{m,n,k,l}| < \frac{1}{c^{r+s}}$$

where $m, n \leq m_{r-1}, n_{s-1}$, respectively. Now we choose m_r and n_s with $m_r > m_{r-1}$ and $n_s > n_{s-1}$ such that

$$\sum_{\{(k,l):0\leq k\leq\alpha_r;0\leq l\leq\beta_s\}} |a_{m,n,k,l}| < \frac{1}{5}$$

for $m > m_r$ and $n > n_s$, where $r, s = 1, 2, 3, \ldots$. Let us now consider following double sequence

$$s_{k,l} = \begin{cases} \left(1+\frac{1}{c}\right)^{r+s} & \text{if} \quad \alpha_{r-1} < k \leq \alpha_r \text{ and/or } \beta_{s-1} < l \leq \beta_s \\ 0 & \text{if} \quad \text{otherwise} \\ r, s = 1, 2, 3, \ldots \end{cases}.$$

Let us now partition the A transformation of $\{s_{k,l}\}$ into three parts with

$$m_{r-1} < m \leq m_r \text{ and/or } n_{s-1} < n \leq n_s.$$

The first partition satisfy the following inequality

$$\sum_{k,l=0,0}^{\alpha_{r-1},\beta_{s-1}} |a_{m,n,k,l}| < \frac{1}{5}\left(1+\frac{1}{c}\right)^{r+s-2} \text{ with } r, s = 2, 3, 4, \ldots \tag{1}$$

and the second satisfies the inequality

$$
\begin{aligned}
&\sum_{k,l=\alpha_{r+1}+1,\beta_{s+1}+1}^{\infty,\infty} |a_{m,n,k,l}| \qquad (2)\\
< \quad & \frac{1}{c^{r+s+2}}\left(1+\frac{1}{c}\right)^{r+s+4} + \frac{1}{c^{r+s+4}}\left(1+\frac{1}{c}\right)^{r+s+6} + \cdots \\
= \quad & \frac{1}{c^{r+s+2}}\left(1+\frac{1}{c}\right)^{r+s+4}\left[1+\frac{1}{c^2}\left(1+\frac{1}{c}\right)^{2}+\frac{1}{c^4}\left(1+\frac{1}{c}\right)^{4}+\cdots\right] \\
= \quad & \frac{1}{c^{r+s+2}}\left(1+\frac{1}{c}\right)^{r+s+4}\sum_{i=0}^{\infty}\frac{1}{c^{2i}}\left(1+\frac{1}{c}\right)^{2i} \\
= \quad & \frac{1}{c^{r+s+2}}\left(1+\frac{1}{c}\right)^{r+s+4}\frac{1}{1-\frac{1}{c^2}\left(1+\frac{1}{c}\right)^2} \\
\leq \quad & \frac{1}{c^{r+s+2}}\left(1+\frac{1}{c}\right)^{r+s+4}\left(\frac{1}{1-\frac{4}{25}}\right) \\
\leq \quad & \frac{1}{c^{r+s+2}}\left(1+\frac{1}{c}\right)^{r+s+4}\frac{25}{21} \\
\leq \quad & \frac{1}{21c^{r+s}}\left(1+\frac{1}{c}\right)^{r+s+4} \\
\text{with} \quad & r,s = 0,1,2,\ldots.
\end{aligned}
$$

The final partition satisfies the equality

$$
\begin{aligned}
\sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}s_{k,l} &= \sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_r,\beta_s} a_{m,n,k,l}s_{k,l} \\
&\quad + \sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}s_{k,l} \\
&= \sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_r,\beta_s} a_{m,n,k,l}\left(1+\frac{1}{c}\right)^{r+s} \\
&\quad + \sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}\left(1+\frac{1}{c}\right)^{r+s+2} \qquad (3)
\end{aligned}
$$

In addition,the final partition also satisfies the following inequality

$$
\begin{aligned}
\sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}s_{k,l} &= \left(1+\frac{1}{c}\right)^{r+s}\left[\sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_r,\beta_s} a_{m,n,k,l}\right.\\
&\qquad \left. + \left(1+\frac{1}{c}\right)^{2}\sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}\right]\\
&> \left(1+\frac{1}{c}\right)^{r+s}\left[\sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_r,\beta_s} a_{m,n,k,l}\right.\\
&\qquad + \sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}\\
&\qquad \left. + \frac{1}{c}\sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}\right].
\end{aligned}
$$

Observe that, if m and n are sufficiently large the following is true by the RH-regularity of A:

$$\frac{1}{c}\sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l} < \frac{1}{5} \tag{4}$$

and

$$\left|\sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_r,\beta_s} a_{m,n,k,l} + \sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}\right| > \frac{3}{5}. \tag{5}$$

Therefore, for m and n sufficiently large, inequalities (1) through (5) imply the following

$$\begin{aligned}
\left|\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}s_{k,l}\right| &= \left|\sum_{k,l=0,0}^{\alpha_{r-1},\beta_{s-1}} a_{m,n,k,l}s_{k,l}\right. \\
&\quad \left. + \sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}s_{k,l} + \sum_{k,l=\alpha_{r+1}+1,\beta_{s+1}+1}^{\infty,\infty} a_{m,n,k,l}s_{k,l}\right| \\
&\geq \left|\sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}s_{k,l}\right| \\
&\quad - \sum_{k,l=0,0}^{\alpha_{r-1},\beta_{s-1}} |a_{m,n,k,l}|s_{k,l} \\
&\quad - \sum_{k,l=\alpha_{r+1}+1,\beta_{s+1}+1}^{\infty,\infty} |a_{m,n,k,l}|s_{k,l} \\
&> \left(1+\frac{1}{c}\right)^{r+s}\left[\frac{2}{5} - \frac{1}{5(1+\frac{1}{c})} - \frac{1}{21c^{r+s}}\left(1\frac{1}{c}\right)^4\right] \\
&> \frac{1}{100}\left(1+\frac{1}{c}\right)^{r+s}.
\end{aligned}$$

□

THEOREM 2.3. *If A is a four-dimensional RH-regular summability matrix then there exists a double sequence $\{s_{k,l}\}$ such that $t_{m,n} = \rho_{m,n}e^{i\theta_{m,n}}$, with*

$$P - \lim_{m,n} \rho_{m,n} = \infty \text{ and } P - \lim_{m,n} \theta_{m,n} = 0.$$

If the matrix A is also real then the double sequence $s_{m,n}$ can be chosen so that the double sequence $t_{m,n}$ is real and positive.

In the proof of Theorem 2.2, replace $\frac{1}{5}$ with a Pringsheim null double sequence and replace $\{s_{k,l}\}$ with the following sequence, or a sequence similar to the following, with respect to order.

$$s'_{k,l} = \begin{cases} \left(1+\frac{1}{c}\right)^{\sqrt{r+s}} & \text{if} \quad \alpha_{r-1} < k \leq \alpha_r \text{ and/or } \beta_{s-1} < l \leq \beta_s \\ 0 & \text{if} \quad \text{otherwise} \\ r,s = 1,2,3,\ldots \end{cases}.$$

The result then follows from RH_1, RH_3, RH_4, and RH_5 of the RH-regularity conditions of A.

THEOREM 2.4. *If the double real valued function $f(m,n)$ is such that*

$$P-\lim_{m,n} f(m,n)=\infty$$

then there exists an RH-regular summability matrix A such that, for every double sequence $\{s_{m,n}\}$ to which transformation A is applicable, the inequality

$$|t_{m,n}|<f(m,n) \tag{6}$$

is satisfied for infinitely many ordered pairs (m,n).

Proof. This asserts that there exists an RH-regular transformation that transforms every double sequence to which it is summable either into a double sequence with at least one finite Pringsheim limit point or else into a double sequence whose terms tend to infinity at an arbitrary slow rate, independent of the double sequence. The following four-dimensional summability matrix satisfies the conditions of the theorem.

$$a_{m,n,k,l}=\begin{cases} 1 & \text{if both } m \text{ and } n \text{ are even with } k=\frac{m}{2} \text{ and } l=\frac{n}{2}\\ 0 & \text{if both } m \text{ and } n \text{ are even with } k\neq\frac{m}{2} \text{ and } l\neq\frac{n}{2}\\ 1 & \text{if both } m \text{ and } n \text{ are odd with } k=\frac{m-1}{2} \text{ and } l=\frac{n-1}{2}\\ 0 & \text{if both } m \text{ and } n \text{ are odd with } k<\frac{m-1}{2} \text{ and } l<\frac{n-1}{2}\\ 0 & \text{if both } m \text{ and } n \text{ are odd with } k>\frac{m-1}{2} \text{ and } l>\frac{n-1}{2}\\ & \text{except when } k=k_1,k_2,k_3,\ldots \text{ and } l=l_1,l_2,l_3,\ldots\\ 2^{-(r+s)} & \text{if both } m \text{ and } n \text{ are odd with } k<\frac{m-1}{2} \text{ and } l<\frac{n-1}{2}\\ & k=k_1,k_2,k_3,\ldots \text{ and } l=l_1,l_2,l_3,\ldots\\ r,s=1,2,3,\ldots. \end{cases}.$$

Suppose that the double sequence $\{s_{m,n}\}$ is such that inequality (6) does not hold infinitely often in the Pringsheim sense. Choose index sequences $\{k_r\}$, $\{l_s\}$ such that $f(k_r,l_s)>2^{r+s}$; and if each element of (m,n) is odd and $k_r>\frac{m-1}{2}$ and $l_s>\frac{n-1}{2}$, $a_{m,n,k_r,l_s}=\frac{1}{2^{r+s}}$.

Since A is such that its pairwise row contains only one nonzero element, then $|s_{m,n}|>f(m,n)$ for all sufficiently large m and n. Therefore, for odd m and n, the series

$$\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}s_{k,l}$$

contains infinity many terms whose absolute value is 1. Therefore the four-dimensional A transformation is not applicable to the double sequence $\{s_{m,n}\}$. □

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