Rend. Istit. Mat. Univ. Trieste
Volume 44 (2012), 423–437

Contra continuity on weak structure spaces

Ahmad Al-Omari

Abstract. *We introduce some contra continuous functions in weak structure spaces such as contra $(\mathcal{M}, w)$-continuous functions, contra $(\alpha(m), w)$-continuous functions, contra $(\sigma(m), w)$-continuous functions, contra $(\pi(m), w)$-continuous functions and contra $(\beta(m), w)$-continuous functions. We investigate their characterization and relationships among such functions.*

Keywords: weak structure, contra continuity, contra $(\mathcal{M}, w)$-continuity
MS Classification 2010: 54A05, 54C10

1. Introduction and Preliminaries

Császár [4] introduced a generalized structure called generalized topology. Recently, Császár [5] has introduced a new notion of structures called a weak structure which is weaker than both a generalized topology [4] and a minimal structure [8, 9]. Let X be a nonempty set and $w \subseteq \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X. Then w is called a weak structure (briefly WS) on X if $\emptyset \in w$. Each member of w is said to be w-open and the complement of a w-open set is said to be w-closed. Let w be a weak structure on X and $A \subseteq X$. Császár [5] defined (as in the general case) $i_w(A)$ as the union of all w-open subsets of A (e.g. $\emptyset$) and $c_w(A)$ as the intersection of all w-closed sets containing A (e.g. X). Quite recently, Al-Omari and Noiri [1, 2, 3, 7] has obtained several fundamental properties of weak structure spaces.

Let X be a nonempty set and $\mathcal{M} \subseteq \mathcal{P}(X)$. Then $\mathcal{M}$ is called a minimal structure on X if $\emptyset$, $X \in \mathcal{M}$ [8], in this case $(X, \mathcal{M})$ is called a minimal space. Each member of $\mathcal{M}$ is said to be m-open and the complement of an m-open set is said to be m-closed. Let $\mathcal{M}$, be a minimal structure on X and $A \subseteq X$. Maki, Umehara and Noiri [8] defined (as in the general case) $i_m(A)$ as the union of all m-open subsets of A and $c_m(A)$ as the intersection of all m-closed sets containing A.

We call a class $\mu \subseteq \mathcal{P}(X)$ a generalized topology [4] (briefly GT) if $\phi \in \mu$

and the arbitrary union of elements of μ belongs to μ. A set X with a GT μ on it is called a generalized topological space (briefly GTS) and is denoted by (X, μ). In this paper, We introduce some contra continuous functions in weak structure spaces such as contra $(\mathcal{M}, w)$-continuous functions, contra $(\alpha(m), w)$-continuous functions, contra $(\sigma(m), w)$-continuous functions, contra $(\pi(m), w)$-continuous functions and contra $(\beta(m), w)$-continuous functions. We investigate their characterization and relationships among such functions.

The following lemmas are useful in the sequel:

LEMMA 1.1 ([5]). *Let w be a WS on X and A, B subsets of X, then the following properties hold:*

1. $i_w(A) \subseteq A \subseteq c_w(A)$.
2. *If* $A \subseteq B$ *implies that* $i_w(A) \subseteq i_w(B)$ *and* $c_w(A) \subseteq c_w(B)$.
3. $i_w(i_w(A)) = i_w(A)$ *and* $c_w(c_w(A)) = c_w(A)$.
4. $i_w(X - A) = X - c_w(A)$ *and* $c_w(X - A) = X - i_w(A)$.

LEMMA 1.2 ([5]). *Let w be a WS on X and A a subset of X, then the following properties hold:*

1. $x \in i_w(A)$ *if and only if there is* $W \in w$ *such that* $x \in W \subseteq A$.
2. $x \in c_w(A)$ *if and only if* $W \cap A \neq \emptyset$ *whenever* $x \in W \in w$.
3. *If* $A \in w$, *then* $A = i_w(A)$ *and if A is w-closed, then* $A = c_w(A)$.

REMARK 1.3. *If* w *is a WS on X, then*

1. $i_w(\emptyset) = \emptyset$ *and* $c_w(X) = X$.
2. $i_w(X)$ *is the union of all* w*-open sets in X.*
3. $c_w(\emptyset)$ *is the intersection of all* w*-closed sets in X.*

THEOREM 1.4 ([1]). *For a WS space (X, w), the following properties are equivalent:*

1. $w = \mu$ *i.e. w is a generalized topology in the sense of Császár;*
2. $i_w(A)$ *is w-open for every subset A of X;*
3. $c_w(A)$ *is w-closed for every subset A of X.*

THEOREM 1.5 ([1]). *Let w be a WS on X and* $w^* = \{A \subset X : A = i_w(A)\}$. *Then, the following properties hold:*

1. w^* *is a GT containing w;*
2. *w is a GT if and only if* $w = w^*$.

2. Contra $(\mathcal{M}, w)$-continuity on weak structure spaces

DEFINITION 2.1. *Let $\mathcal{M}$ be minimal structure on X and w be weak structure on Y. A function $f : (X, \mathcal{M}) \to (Y, w)$ is said to be*

1. *contra $(\mathcal{M}, w)$-continuous if for each w-open set U in Y, $f^{-1}(U)$ is m-closed in X.*
2. *contra $(\mathcal{M}, w)$-continuous at some $x \in X$ if for each w-closed set V containing $f(x)$, there exists $U \in \mathcal{M}$ containing x such that $f(U) \subseteq V$.*

THEOREM 2.2. *Let $\mathcal{M}$ be minimal structure on X and w be weak structure on Y. For a function $f : (X, \mathcal{M}) \to (Y, w)$. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold. If $\mathcal{M} = \mathcal{M}^*$, then the following statements are equivalent:*

1. *f is contra $(\mathcal{M}, w)$-continuous.*
2. *f is contra $(\mathcal{M}, w)$-continuous at any $x \in X$.*
3. *$f^{-1}(F) \subseteq i_m(f^{-1}(F))$ for any w-closed F of Y.*
4. *$c_m(f^{-1}(V)) \subseteq f^{-1}(V)$ for any w-open V of Y.*

Proof. (1)$\Rightarrow$ (2). Let $x \in X$ and V be w-closed set containing $f(x)$. By (1), $f^{-1}(V) \in \mathcal{M}$. Put $U = f^{-1}(V)$. We have U is m-open containing x and $f(U) \subseteq V$.
(2)$\Rightarrow$ (3). Let F be w-closed F of Y. For each $x \in f^{-1}(F)$, $f(x) \in F$. By (2), there exists $U \in \mathcal{M}$ containing x such that $f(U) \subseteq F$. Since $x \in U \subseteq f^{-1}(F)$, we have $x \in i_m(f^{-1}(F))$. This implies $f^{-1}(F) \subseteq i_m(f^{-1}(F))$.
(3)$\Rightarrow$ (4). Let $V \in w$. Then $Y - V$ is w-closed. By (3) and Lemma 1.1, $f^{-1}(Y - V) \subseteq i_m(f^{-1}(Y - V)) = i_m(X - f^{-1}(V)) = X - c_m(f^{-1}(V))$. Thus $c_m(f^{-1}(V)) \subseteq f^{-1}(V)$.
(4)$\Rightarrow$ (1). Let $V \in w$. By (4), we have $c_m(f^{-1}(V)) \subseteq f^{-1}(V)$ and hence $c_m(f^{-1}(V)) = f^{-1}(V)$. Since $\mathcal{M} = \mathcal{M}^*$, then $f^{-1}(V)$ is m-closed. Hence f is contra $(\mathcal{M}, w)$-continuous. □

The implication (2)$\Rightarrow$ (1) of Theorem 2.2 need not be true in general.

EXAMPLE 2.3. *Let $X = \{a, b, c\}$ and $\mathcal{M} = \{\phi, \{a\}, \{b\}, \{c\}, X\}$ be a minimal structure on X. Let $f : (X, \mathcal{M}) \to (X, \mathcal{M})$ be the identity function. Then f is contra $(\mathcal{M}, \mathcal{M})$-continuous at any $x \in X$ but not contra-$(\mathcal{M}, \mathcal{M})$-continuous.*

THEOREM 2.4. *Let $\mathcal{M}$ be minimal structure on X and w be weak structure on Y. For a function $f : (X, \mathcal{M}) \to (Y, w)$. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold. If $w = w^*$, then the following statements are equivalent:*

1. *$c_m(f^{-1}(i_w(B))) \subseteq f^{-1}(i_w(B))$ for any $B \subseteq Y$.*

2. $f^{-1}(c_w(B)) \subseteq i_m(f^{-1}(c_w(B)))$ *for any* $B \subseteq Y$.

3. $c_m(f^{-1}(V)) \subseteq f^{-1}(V)$ *for any* w*-open* V *of* Y.

Proof. (1)⇒ (2). Let $B \subseteq Y$. By (1), $c_m(f^{-1}(i_w(Y-B))) \subseteq f^{-1}(i_w(Y-B))$. By Lemma 1.1, $c_m(f^{-1}(i_w(Y-B))) = c_m(f^{-1}(Y-c_w(B))) = c_m(X-f^{-1}(c_w(B))) = X - i_m(f^{-1}(c_w(B)))$ and $X - i_m(f^{-1}(c_w(B))) \subseteq X - f^{-1}(c_w(B))$. Thus $f^{-1}(c_w(B)) \subseteq i_m(f^{-1}(c_w(B)))$.
(2)⇒ (3). Let $V \in w$. Then $Y-V$ is w-closed and hence $c_w(Y-V) = Y-V$. Now by (2), we have $f^{-1}(c_w(Y-V)) \subseteq i_m(f^{-1}(c_w(Y-V)))$ and hence $f^{-1}(Y-V) \subseteq i_m(f^{-1}(Y-V)) = X - c_m(f^{-1}(V))$. Then $c_m(f^{-1}(V)) \subseteq f^{-1}(V)$.
(3)⇒ (1). Let $B \subseteq Y$. Since $w = w^*$, then $i_w(B)$ is w-open set, by (3) $c_m(f^{-1}(i_w(B))) \subseteq f^{-1}(i_w(B))$. □

Definition 2.5 ([1]). *Let* (X, w) *be a* WS *space. Then the weak kernel of* $A \subseteq X$ *is denoted by* w-*ker*(A) *and defined as* w-*ker*$(A) = \cap\{G \in w : A \subseteq G\}$.

Lemma 2.6 ([1]). *Let* A *and* B *be two subsets of a* WS *space* (X, w). *Then the following properties hold:*

1. $x \in$*w-ker*(A) *if and only if* $A \cap F \neq \phi$ *for any* w*-closed* F *containing* x.

2. $A \subseteq$*w-ker*(A) *and* $A =$*w-ker*(A) *if* $A \in w$.

3. *If* $A \subseteq B$, *then w-ker*$(A) \subseteq$*w-ker*(B).

Lemma 2.7. *Let* A *be a subset of a* WS *space* (X, w). *Then w-ker*$(A) =$ *w-ker*$(w$*-ker*$(A))$

Proof. By Lemma 2.6, we have w-ker$(A) \subseteq w$-ker$(w$-ker$(A))$. Conversely, if $x \notin w$-ker(A) there exists F which is w-closed such that $x \in F$ and $F \cap A = \phi$. Since $X - F \in w$ and $A \subseteq X - F$, and since w-ker(A) is the intersection of all w-open sets containing A, we have w-ker$(A) \subseteq X - F$ so that $F \cap w$-ker$(A) = \phi$. Since $x \in F$, we have that $x \notin w$-ker$(w$-ker$(A))$. Thus w-ker$(w$-ker$(A)) \subseteq w$-ker(A). □

Theorem 2.8. *Let* $\mathcal{M}$ *be minimal structure on* X *and* w *be weak structure on* Y. *For a function* $f : (X, \mathcal{M}) \rightarrow (Y, w)$. *The implications (1)* ⇒ *(2)* ⇒ *(3) hold. If* $\mathcal{M} = \mathcal{M}^*$, *then the following statements are equivalent:*

1. f *is contra* $(\mathcal{M}, w)$*-continuous;*

2. $f(c_m(A)) \subseteq$*w-ker*$(f(A))$ *for any* $A \subseteq X$*;*

3. $c_m(f^{-1}(B)) \subseteq f^{-1}$(*w-ker*$(B))$ *for any* $B \subseteq Y$.

Proof. (1) $\Rightarrow$ (2). Let $A \subseteq X$. Suppose that $f(c_m(A)) - w\text{-ker}(f(A)) \neq \phi$. Pick $y \in f(c_m(A)) - w\text{-ker}(f(A))$. By $y \notin w\text{-ker}(f(A))$, there exists w-closed set F containing y such that $f(A) \cap F = \phi$. Then $A \cap f^{-1}(F) = \phi$ and $c_m(A) \cap f^{-1}(F) = \phi$, since $f^{-1}(F) \in m$. This implies that $f(c_m(A)) \cap F = \phi$ and $y \notin f(c_m(A))$. Thus $f(c_m(A)) \subseteq w\text{-ker}(f(A))$.
(2) $\Rightarrow$ (3). Let $B \subseteq Y$. By (2), $f(c_m(f^{-1}(B))) \subseteq w\text{-ker}(f(f^{-1}(B))) \subseteq w$-*ker*$(B)$. Thus $c_m(f^{-1}(B)) \subseteq f^{-1}(w\text{-ker}(B))$.
(3) $\Rightarrow$ (1). Let $B \in w$. By (3) $c_m(f^{-1}(B)) \subseteq f^{-1}(w\text{-ker}(B))$. By Lemma 2.6, $B = w\text{-ker}(B))$. Thus $c_m(f^{-1}(B)) \subseteq f^{-1}(B)$. Since $\mathcal{M} = \mathcal{M}^*$ implies that $f^{-1}(B)$ is m-closed. Hence f is contra $(\mathcal{M}, w)$-continuous. □

Definition 2.9. *Let (X, w) be a WS space. X is called w-connected, if there are no nonempty disjoint w-open subsets U, V of X such that $U \cup V = X$.*

Lemma 2.10. *Let (X, w) be a WS space. If U, V are nonempty disjoint w-open subsets of X and $U \cup V = X$, then U and V are w-closed.*

Theorem 2.11. *Let $f : (X, \mathcal{M}) \to (Y, w)$ be a contra $(\mathcal{M}, w)$-continuous surjection. If X is m-connected, then Y is w-connected.*

Proof. Let $f : (X, \mathcal{M}) \to (Y, w)$ be a contra $(\mathcal{M}, w)$-continuous surjection and let X be m-connected. Suppose Y is not w-connected. Then there exists nonempty disjoint w-open subsets V_1 and V_2 of Y such that $V_1 \cup V_2 = Y$. By Lemma 2.10, V_1 and V_2 are w-closed. Since f is contra $(\mathcal{M}, w)$-continuous, then $f^{-1}(V_1)$, $f^{-1}(V_2) \in \mathcal{M}$. Note that $f^{-1}(V_1) \cap f^{-1}(V_2) \neq \phi$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Then X is not m-connected, contradiction. Thus Y is w-connected. □

Definition 2.12. *A WS space (X, w) is said to be* strongly w-closed *if every cover of X by w-closed sets of (X, w) has a finite subcover.*

Definition 2.13. *A minimal space $(X, \mathcal{M})$ is said to be m-compact if every m-open cover of X has a finite subcover.*

Theorem 2.14. *Let $f : (X, \mathcal{M}) \to (Y, w)$ be a contra-$(\mathcal{M}, w)$-continuous surjection. If $(X, \mathcal{M})$ is m-compact, then (Y, w) is strongly w-closed.*

Proof. Let $(X, \mathcal{M})$ be m-compact and $\{V_\alpha : \alpha \in \Delta\}$ any cover of Y by w-closed sets of (Y, w). Since f is contra-$(\mathcal{M}, w)$-continuous, the family $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a m-open cover of X. Since $(X, \mathcal{M})$ is m-compact, there exists a finite subset Δ_0 of Δ such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$. Therefore, $Y = f(X) = \cup\{V_\alpha : \alpha \in \Delta_0\}$. This shows that (Y, w) is strongly w-clsoed. □

3. Contra continuity on weak structure spaces

DEFINITION 3.1 ([10]). *Let $(X, \mathcal{M})$ be a minimal structure space and $A \subseteq X$. Then A is said to be*

1. *m-semi-open if $A \subseteq c_m(i_m(A))$,*
2. *m-preopen if $A \subseteq i_m(c_m(A))$,*
3. *m-α-open if $A \subseteq i_m(c_m(i_m(A)))$,*
4. *m-β-open if $A \subseteq c_m(i_m(c_m(A)))$,*
5. *mr-open if $A = i_m(c_m(A))$.*

The complement of m-semi-open (resp. m-preopen, m-α-open, m-β-open, mr-open) is said to be m-semi-closed (resp. m-preclosed, m-α-closed, m-β-closed, wr-closed). Let us denote by $\sigma(m)$ (resp. $\pi(m)$, $\alpha(m)$, $\beta(m)$) the class of all m-semi-open (resp. m-preopen, m-α-open, m-β-open) sets of $(X, \mathcal{M})$.

DEFINITION 3.2. *Let $\mathcal{M}$ be minimal structure on X and w be weak structure on Y. A function $f : (X, \mathcal{M}) \to (Y, w)$ is said to be*

1. *contra $(\alpha(m), w)$-continuous if for each w-open set U in Y, $f^{-1}(U)$ is m-α-closed in X.*
2. *contra $(\sigma(m), w)$-continuous if for each w-open set U in Y, $f^{-1}(U)$ is m-σ-closed in X.*
3. *contra $(\pi(m), w)$-continuous if for each w-open set U in Y, $f^{-1}(U)$ is m-π-closed in X.*
4. *contra $(\beta(m), w)$-continuous if for each w-open set U in Y, $f^{-1}(U)$ is m-β-closed in X.*
5. *contra $(\sigma(m), w^*)$-continuous if for each w^*-open set U in Y, $f^{-1}(U)$ is m-σ-closed in X.*
6. *contra $(\pi(m), w^*)$-continuous if for each w^*-open set U in Y, $f^{-1}(U)$ is m-π-closed in X.*

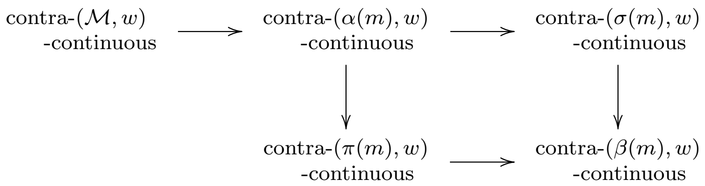
LEMMA 3.3 ([5]). *For a WS w on X, the following relations hold:*

1. *$w \subseteq \alpha(w) \subseteq \sigma(w) \subseteq \beta(w)$.*
2. *$w \subseteq \alpha(w) \subseteq \pi(w) \subseteq \beta(w)$.*

THEOREM 3.4 ([5]). *If w is a WS, each of the structures $\alpha(w)$, $\sigma(w)$, $\pi(w)$ and $\beta(w)$ is a generalized topology.*

For several functions defined above, we have the following implications.

DIAGRAM



The reverse implication may be not true in general and this can be clearly seen from the following examples.

Example 3.5. *Let* $X = \{a, b, c, d\}$ *and* $\mathcal{M} = \{\phi, \{a\}, \{b\}, \{a, b, c\}, X\}$ *be a minimal structure on* X*. Define* $f : (X, \mathcal{M}) \to (X, \mathcal{M})$ *as follows:* $f(a) = f(b) = d$ *and* $f(c) = f(d) = a$*. Then* $f^{-1}(\{a\}) = \{c, d\}$*,* $f^{-1}(\{b\}) = \phi$ *and* $f^{-1}(\{a, b, c\}) = \{c, d\}$*. We have* f *is contra-*$(\alpha(m), \mathcal{M})$*-continuous but not contra-*$(\mathcal{M}, \mathcal{M})$*-continuous.*

Example 3.6. *Let* $X = Y = \{a, b, c\}$*,* $\mathcal{M} = \{\phi, \{a\}, \{b\}, X\}$ *be a minimal structure on* X *and* $w = \{\phi, \{a\}, \{b\}\}$ *a* WS *on* Y*. Define* $f : (X, \mathcal{M}) \to (Y, w)$ *be the identity function. We have* f *is contra-*$(\sigma(m), w)$*-continuous but not contra-*$(\alpha(m), w)$*-continuous.*

Example 3.7. *Let* $X = Y = \{a, b, c\}$*,* $\mathcal{M} = \{\phi, \{a\}, \{b\}, X\}$ *be a minimal structure on* X *and* $w = \{\phi, \{a, c\}, \{b\}\}$ *a* WS *on* Y*. Define* $f : (X, \mathcal{M}) \to (Y, w)$ *as follows:* $f(a) = a$*,* $f(b) = c$ *and* $f(c) = c$*. Then* $f^{-1}(\{a, b\}) = \{a\}$ *and* $f^{-1}(\{b\}) = \phi$*. We have* f *is contra-*$(\beta(m), w)$*-continuous but not contra-*$(\pi(m), w)$*-continuous.*

Example 3.8. *Let* $X = Y = \{a, b, c\}$*,* $\mathcal{M} = \{\phi, \{a, c\}, \{b, c\}, X\}$ *be a minimal structure on* X *and* $w = \{\phi, \{a, c\}\}$ *a* WS *on* Y*. Define* $f : (X, \mathcal{M}) \to (Y, w)$ *as follows:* $f(a) = f(b) = a$ *and* $f(c) = b$*. Then* $f^{-1}(\{a, c\}) = \{a, b\}$*. We have* f *is contra-*$(\pi(m), w)$*-continuous but not contra-*$(\sigma(m), w)$*-continuous.*

Theorem 3.9. *Let* $\mathcal{M}$ *be a minimal structure on* X *and* w *be weak structures on* Y*. A function* $f : (X, \mathcal{M}) \to (Y, w)$ *is contra-*$(\alpha(m), w)$*-continuous if and only if it is both contra-*$(\pi(m), w)$*-continuous and contra-*$(\sigma(m), w)$*-continuous.*

Proof. Necessity. It is clear from the above diagram.
Sufficiency. Follows from the fact that $\alpha(w) = \pi(w) \cap \sigma(w)$. □

Definition 3.10. *Let* $\mathcal{M}$ *be a minimal structure on* X *and* w *be weak structures on* Y*. A function* $f : (X, \mathcal{M}) \to (Y, w)$ *is said to be*

1. *$(\sigma(m), w)$-continuous if $f^{-1}(V)$ is m-semi-open in X for each w-open set V of Y,*

2. *$(\pi(m), w)$-continuous if $f^{-1}(V)$ is m-preopen in X for each w-open set V of Y.*

LEMMA 3.11. *For a subset A of a WS space (X, w), the following properties are equivalent:*

1. *A is wr-closed;*

2. *A is w-preclosed and w-semi-open;*

3. *A is w-α-closed and w-β-open.*

Proof. (1)$\Rightarrow$ (2). Let A be wr-closed. Then $A = c_w(i_w(A))$ and A is w-preclosed and w-semi-open.
(2)$\Rightarrow$ (3). Let A be w-preclosed and w-semi-open. Then $A \subseteq c_w(i_w(A))$ and $c_w(i_w(A)) \subseteq A$. Therefore, we have $c_w(A) = c_w(i_w(A))$ and hence $c_w(i_w(c_w(A))) = c_w(i_w(c_w(i_w(A)))) = c_w(i_w(A)) \subseteq A$. This shows that A is w-α-closed. Since $\sigma(w) \subseteq \beta(w)$, it is obvious that A is w-β-open.
(3)$\Rightarrow$ (1). Let A be w-α-closed and w-β-open. Then $A = c_w(i_w(c_w(A)))$ and hence $c_w(i_w(A)) = c_w(i_w(c_w(i_w(c_w(A))))) = c_w(i_w(c_w(A))) = A$. Therefore, A is wr-closed. $\square$

DEFINITION 3.12. *Let $\mathcal{M}$ be a minimal structure on X and w be weak structures on Y. A function $f : (X, \mathcal{M}) \to (Y, w)$ is said to be RC-$(\mathcal{M}, w)$-continuous if $f^{-1}(V)$ is mr-closed in X for each w-open set of Y.*

As a consequence of Lemma 3.11, we have the following result:

THEOREM 3.13. *Let $\mathcal{M}$ be a minimal structure on X and w be weak structures on Y. For a function $f : (X, \mathcal{M}) \to (Y, w)$, the following statements are equivalent:*

1. *f is RC-$(\mathcal{M}, w)$-continuous;*

2. *f is contra-$(\pi(m), w)$-continuous and $(\sigma(m), w)$-continuous;*

3. *f is contra-$(\alpha(m), w)$-continuous and $(\beta(m), w)$-continuous.*

Let $\mathcal{M}$ be a minimal structure on X or w be a weak structures on X and $A \subseteq X$. The m-α-closure (resp. m-semi-closure, m-preclosure, m-β-closure, w^*-closure) of a subset A of X, denoted by $c_\alpha(A)$ (resp. $c_\sigma(A)$, $c_\pi(A)$, $c_\beta(A)$, $c_{w^*}(A)$), is the intersection of m-α-closed (resp. m-semi-closed, m-preclosed, m-β-closed, w^*-closed) sets including A. The m-α-interior (resp. m-semi-interior, m-preinterior, m-β-interior, w^*-interior) of a subset A of X, denoted by $i_\alpha(A)$ (resp. $i_\sigma(A)$, $i_\pi(A)$, $i_\beta(A)$, $i_{w^*}(A)$), is the union of m-α-open (resp. m-semi-open, m-preopen, m-β-open, w^*-open) sets contained in A.

THEOREM 3.14. *Let $\mathcal{M}$ be a minimal structure on X and w be a weak structures on Y. For a function $f : (X, \mathcal{M}) \to (Y, w)$, the following properties are equivalent:*

1. *f is contra $(\pi(m), w^*)$-continuous;*
2. *$f^{-1}(A)$ is m-preopen set in X for every w^*-closed set A in Y;*
3. *$f^{-1}(A) \subseteq i_m(c_m(f^{-1}(c_{w^*}(A))))$ for every subset A in Y;*
4. *$c_m(i_m(f^{-1}(i_{w^*}(A)))) \subseteq f^{-1}(A)$ for every subset A in Y;*
5. *$A \subseteq i_m(c_m(f^{-1}(c_{w^*}(f(A)))))$ for every subset A in X.*

Proof. (1) $\Leftrightarrow$ (2). It is obvious.
(2) $\Rightarrow$ (3). Let $A \subseteq Y$. Then $c_{w^*}(A)$ is w^*-closed set in Y. By (2) implies that $f^{-1}(c_{w^*}(A))$ is m-preopen set in X. Therefore, $f^{-1}(c_{w^*}(A)) \subseteq i_m(c_m(f^{-1}(c_{w^*}(A))))$. Hence $f^{-1}(A) \subseteq i_m(c_m(f^{-1}(c_{w^*}(A))))$.
(3) $\Leftrightarrow$ (4). It is obvious.
(3) $\Rightarrow$ (5). Let $A \subseteq X$. Then $f(A) \subseteq Y$. By (3) implies that $f^{-1}(f(A)) \subseteq i_m(c_m(f^{-1}(c_{w^*}(f(A)))))$. Therefore, $A \subseteq f^{-1}(f(A)) \subseteq i_m(c_m(f^{-1}(c_{w^*}(f(A)))))$.
(5) $\Rightarrow$ (2). Let A be w^*-closed in Y. Then $f^{-1}(A) \subseteq X$. By hypothesis

$$\begin{aligned} f^{-1}(A) &\subseteq i_m(c_m(f^{-1}(c_{w^*}(f(f^{-1}(A)))))) \\ &\subseteq i_m(c_m(f^{-1}(c_{w^*}(A)))) \\ &= i_m(c_m(f^{-1}(A))). \end{aligned}$$

Hence $f^{-1}(A)$ is m-preopen set in X.

□

REMARK 3.15. *Since every w-open set is w^*-open set in Y. Then every contra $(\pi(m), w^*)$-continuous is contra $(\pi(m), w)$-continuous.*

THEOREM 3.16. *Let $\mathcal{M}$ be a minimal structure on X and w be weak structures on Y. For a function $f : (X, \mathcal{M}) \to (Y, w)$, the following properties are equivalent:*

1. *f is contra $(\sigma(m), w^*)$-continuous;*
2. *$f^{-1}(A)$ is m-semi-open set in X for every w^*-closed set A in Y;*
3. *$f^{-1}(A) \subseteq c_m(i_m(f^{-1}(c_{w^*}(A))))$ for every subset A in Y;*
4. *$i_m(c_m(f^{-1}(i_{w^*}(A)))) \subseteq f^{-1}(A)$ for every subset A in Y;*
5. *$A \subseteq c_m(i_m(f^{-1}(c_{w^*}(f(A)))))$ for every subset A in X.*

Proof. (1) $\Leftrightarrow$ (2). It is obvious.
(2) $\Rightarrow$ (3). Let $A \subseteq Y$. Then $c_{w^*}(A)$ is w^*-closed set in Y. By (2) implies that $f^{-1}(c_{w^*}(A))$ is m-semi-open set in X. Therefore, $f^{-1}(c_{w^*}(A)) \subseteq c_m(i_m(f^{-1}(c_{w^*}(A))))$. Hence $f^{-1}(A) \subseteq f^{-1}(c_{w^*}(A)) \subseteq c_m(i_m(f^{-1}(c_{w^*}(A))))$.
(3) $\Leftrightarrow$ (4). It is obvious by taking complement.
(3) $\Rightarrow$ (5). Let $A \subseteq X$. Then $f(A) \subseteq Y$. By (3) implies that $f^{-1}(f(A)) \subseteq c_m(i_m(f^{-1}(c_{w^*}(f(A)))))$. Therefore, $A \subseteq f^{-1}(f(A)) \subseteq c_m(i_m(f^{-1}(c_{w^*}(f(A)))))$.
(5) $\Rightarrow$ (2). Let A be w^*-closed in Y. Then $f^{-1}(A) \subseteq X$. By hypothesis

$$\begin{aligned} f^{-1}(A) &\subseteq c_m(i_m(f^{-1}(c_{w^*}(f(f^{-1}(A)))))) \\ &\subseteq c_m(i_m(f^{-1}(c_{w^*}(A)))) \\ &= c_m(i_m(f^{-1}(A)))\,. \end{aligned}$$

Hence $f^{-1}(A)$ is m-semi-open set in X. □

REMARK 3.17. *Since every w-open set is w^*-open set in Y. Then every contra (σ, w^*)-continuous is contra (σ, w)-continuous.*

THEOREM 3.18. *Let $\mathcal{M}$ be a minimal structure on X and w be weak structures on Y. A function $f : (X, \mathcal{M}) \to (Y, w)$ is contra $(\beta(m), w^*)$-continuous if and only if $f^{-1}(c_\beta(B)) \subseteq i_\beta(f^{-1}(c_{w^*}(B))$ for each subset B in Y.*

Proof. Necessity. Let $B \subseteq Y$. Then $c_{w^*}(B)$ is w^*-closed in Y. By hypothesis, $f^{-1}(c_{w^*}(B)) \in \beta(m)$ and since $w^* \subseteq \beta(w)$. Therefore, $f^{-1}(c_\beta(B)) \subseteq f^{-1}(c_{w^*}(B)) = i_\beta(f^{-1}(c_{w^*}(B))$. Hence $f^{-1}(c_\beta(B)) \subseteq i_\beta(f^{-1}(c_{w^*}(B))$.
Sufficiency. Let $B \subseteq Y$ be w^*-closed. Then $c_{w^*}(B) = B$. By hypothesis, $f^{-1}(c_\beta(B)) \subseteq i_\beta(f^{-1}(c_{w^*}(B))) = i_\beta(f^{-1}(B)$. Now $f^{-1}(B) \subseteq f^{-1}(c_\beta(B)) \subseteq i_\beta(f^{-1}(B)) \subseteq f^{-1}(B)$. This implies that $i_\beta(f^{-1}(B)) = f^{-1}(B)$ and by Theorem 3.4. Hence $f^{-1}(B) \in \beta(m)$ and hence f is contra $(\beta(m), w^*)$-continuous. □

REMARK 3.19. *Since every w-open set is w^*-open set in Y. Then every contra $(\beta(m), w^*)$-continuous is contra $(\beta(m), w)$-continuous.*

THEOREM 3.20. *Let $\mathcal{M}$ be a minimal structure on X and w be weak structures on Y. A function $f : (X, \mathcal{M}) \to (Y, w)$ is contra $(\alpha(m), w^*)$-continuous if and only if $f^{-1}(c_\alpha(B)) \subseteq i_\alpha(f^{-1}(c_{w^*}(B))$ for each subset B in Y.*

Proof. Similar as in Theorem 3.18. □

THEOREM 3.21. *Let $\mathcal{M}$ be a minimal structure on X and w be weak structures on Y. For a function $f : (X, \mathcal{M}) \to (Y, w)$. Suppose that one of the following conditions holds:*

1. $f^{-1}(c_w(B)) \subseteq i_m(c_\beta(f^{-1}(B)))$ *for each subset* B *in* Y*;*
2. $c_m(i_\beta(f^{-1}(B))) \subseteq f^{-1}(i_w(B))$ *for each subset* B *in* Y*;*
3. $f(c_m(i_\beta(A))) \subseteq i_w(f(A))$ *for each subset* A *in* X*;*
4. $f(c_m(A)) \subseteq i_w(f(A))$ *for each* m*-*β*-open set* A *in* X*.*

Then f *is contra* $(\beta(m), w)$*-continuous.*

Proof. (1) $\Rightarrow$ (2). It is obvious by taking complement.
(2) $\Rightarrow$ (3). Let $A \subseteq X$, then $f(A) \subseteq Y$. By (2) implies that $c_m(i_\beta(f^{-1}(f(A)))) \subseteq f^{-1}(i_w(f(A)))$. That is $c_m(i_\beta(A)) \subseteq c_m(i_\beta(f^{-1}(f(A)))) \subseteq f^{-1}(i_w(f(A)))$. Hence $f(c_m(i_\beta(A))) \subseteq f(f^{-1}(i_w(f(A)))) \subseteq i_w(f(A))$.
(3) $\Rightarrow$ (4). Let $A \subseteq X$ be m-β-open. Then $f(c_m(i_\beta(A))) \subseteq i_w(f(A))$. That is $f(c_w(A)) = f(c_m(i_\beta(A))) \subseteq i_w(f(A))$, since $i_\beta(A) = A$. Hence $f(c_m(A)) \subseteq i_w(f(A))$.
Suppose (4) holds: Let $A \subseteq Y$ be w-open. Then $f^{-1}(A) \subseteq X$ and $i_\beta(f^{-1}(A))$ is m-β-open in X, by Theorem 3.4. By (4) implies that $f(c_m(i_\beta(f^{-1}(A)))) \subseteq i_w(f(i_\beta(f^{-1}(A)))) \subseteq i_w(f(f^{-1}(A))) \subseteq i_w(A) = A$. Now $c_m(i_\beta(f^{-1}(A))) \subseteq f^{-1}(f(c_m(i_\beta(f^{-1}(A))))) \subseteq f^{-1}(A)$. We have $c_m(i_m(f^{-1}(A))) \subseteq f^{-1}(A)$. Therefore, $f^{-1}(A)$ is a m-preclosed set and hence a m-β-closed set. Thus f is contra $(\beta(m), w)$-continuous. □

THEOREM 3.22. *Let* $\mathcal{M}$ *be a minimal structure on* X *and* w *be weak structures on* Y*. For a function* $f : (X, \mathcal{M}) \to (Y, w)$*. Suppose that one of the following conditions holds:*

1. $f^{-1}(c_w(B)) \subseteq i_m(c_\alpha(f^{-1}(B)))$ *for each subset* B *in* Y*;*
2. $c_m(i_\alpha(f^{-1}(B))) \subseteq f^{-1}(i_w(B))$ *for each subset* B *in* Y*;*
3. $f(c_m(i_\alpha(A))) \subseteq i_w(f(A))$ *for each subset* A *in* X*;*
4. $f(c_m(A)) \subseteq i_w(f(A))$ *for each* m*-*α*-open set* A *in* X*.*

Then f *is contra* $(\alpha(m), w)$*-continuous.*

Proof. Similar as in Theorem 3.21. □

THEOREM 3.23. *Let* $\mathcal{M}$ *be a minimal structure on* X *and* w *be weak structures on* Y*. For a function* $f : (X, \mathcal{M}) \to (Y, w)$*. Suppose that one of the following conditions holds:*

1. $f(c_\beta(A)) \subseteq i_w(f(A))$ *for each subset* A *in* X*;*
2. $c_\beta(f^{-1}(B))) \subseteq f^{-1}(i_w(B))$ *for each subset* B *in* Y*;*

3. $f^{-1}(c_w(B)) \subseteq i_\beta(f^{-1}(B)))$ *for each subset* B *in* Y*.*

Then f *is contra* $(\beta(m), w)$*-continuous*

Proof. (1) $\Rightarrow$ (2). Let $B \subseteq Y$. Then $f^{-1}(B) \subseteq X$. By (1) implies that $f(c_\beta(f^{-1}(B))) \subseteq i_w(f(f^{-1}(B))) \subseteq i_w(B)$. Therefore$f^{-1}(f(c_\beta(f^{-1}(B)))) \subseteq f^{-1}(i_w(B))$. So that $c_\beta(f^{-1}(B)) \subseteq f^{-1}(f(c_\beta(f^{-1}(B)))) \subseteq f^{-1}(i_w(B))$. Hence $c_\beta(f^{-1}(B)) \subseteq f^{-1}(i_w(B))$.
(2) $\Rightarrow$ (3). It is obvious by taking complement in (2).
Suppose (3) holds: Let $B \subseteq Y$ be w-closed. Then, by hypothesis, $f^{-1}(c_w(B)) \subseteq i_\beta(f^{-1}(B)))$. That is $f^{-1}(B) = f^{-1}(c_w(B)) \subseteq i_\beta(f^{-1}(B)) \subseteq f^{-1}(B)$ and by Theorem 3.4. Therefore, $f^{-1}(B)$ is m-β-open in X. Hence f is contra $(\beta(m), w)$-continuous.

□

THEOREM 3.24. *Let* $\mathcal{M}$ *be a minimal structure on* X *and* w *be weak structures on* Y*. For a function* $f : (X, \mathcal{M}) \to (Y, w)$*. Suppose that one of the following conditions holds:*

1. $f(c_\alpha(A)) \subseteq i_w(f(A))$ *for each subset* A *in* X*;*

2. $c_\alpha(f^{-1}(B))) \subseteq f^{-1}(i_w(B))$ *for each subset* B *in* Y*;*

3. $f^{-1}(c_w(B)) \subseteq i_\alpha(f^{-1}(B)))$ *for each subset* B *in* Y*.*

Then f *is contra* $(\alpha(m), w)$*-continuous.*

Proof. Similar as in Theorem 3.23. □

THEOREM 3.25. *Let* $\mathcal{M}$ *be a minimal structure on* X *and* w *be weak structures on* Y*. A function* $f : (X, \mathcal{M}) \to (Y, w)$ *is contra* $(\beta(m), w)$*-continuous if* $c_w(f(A)) \subseteq f(i_\beta(A))$ *for each subset* A *of* X *and* f *is bijective.*

Proof. Let $B \subseteq Y$ be w-closed. Then $f^{-1}(B) \subseteq X$. By hypothesis $c_w(f(f^{-1}(B))) \subseteq f(i_\beta(f^{-1}(B)))$. Now $B = c_w(B) = c_w(f(f^{-1}(B))) \subseteq f(i_\beta(f^{-1}(B)))$. Therefore, $f^{-1}(B) \subseteq f^{-1}(f(i_\beta(f^{-1}(B)))) = i_\beta(f^{-1}(B)) \subseteq f^{-1}(B)$ and by Theorem 3.4. Hence $f^{-1}(B) \in \beta(m)$ and hence f is contra $(\beta(m), w)$-continuous.

□

THEOREM 3.26. *Let* $\mathcal{M}$ *be a minimal structure on* X *and* w *be weak structures on* Y*. Let* $f : (X, \mathcal{M}) \to (Y, w)$ *be a contra* $(\beta(m), w)$*-continuous. Then the following properties hold:*

1. $c_\beta(f^{-1}(B)) \subseteq f^{-1}(i_w(c_\beta(B)))$ *for each* w*-open set* B *in* Y*.*

2. $f^{-1}(c_w(i_\beta(B))) \subseteq i_\beta(f^{-1}(B))$ *for each* w*-closed set* B *in* Y*.*

Proof. (1). Let $B \subseteq Y$ be w-open. By hypothesis, $f^{-1}(B)$ is m-β-closed in X. Then $c_\beta(f^{-1}(B)) = f^{-1}(B) = f^{-1}(i_w(B)) \subseteq f^{-1}(i_w(c_\beta(B)))$. Hence
$c_\beta(f^{-1}(B)) \subseteq f^{-1}(i_w(c_\beta(B)))$.
(2). It is obvious by taking complement in (1). □

THEOREM 3.27. *Let $\mathcal{M}$ be a minimal structure on X and w be weak structures on Y. For a function $f : (X, \mathcal{M}) \to (Y, w)$. The following conditions are equivalent:*

1. *f is contra $(\beta(m), w)$-continuous;*
2. *for each $x \in X$ and each w-closed set B containing $f(x)$, there exists $A \in \beta(m)$ and $x \in A$ such that $A \subseteq f^{-1}(B)$;*
3. *for each $x \in X$ and each w-closed set B containing $f(x)$, there exists $A \in \beta(m)$ and $x \in A$ such that $f(A) \subseteq B$.*

Proof. (1) $\Rightarrow$ (2). Let $B \subseteq Y$ be w-closed and $f(x) \in B$. By hypothesis $f^{-1}(B) \in \beta(m)$. Therefore, $i_\beta(f^{-1}(B)) = f^{-1}(B)$. Put $A = i_\beta(f^{-1}(B))$. Then $A \in \beta(w)$ and $A \subseteq f^{-1}(B)$.
(2) $\Rightarrow$ (3). Let $B \subseteq Y$ be w-closed and $f(x) \in B$. By hypothesis there exists $A \in \beta(m)$ and $x \in A$ such that $A \subseteq f^{-1}(B)$. Therefore, $f(A) \subseteq f(f^{-1}(B)) \subseteq B$. Thus $f(A) \subseteq B$.
(3) $\Rightarrow$ (1). Let B be w-closed in Y. Let $x \in X$ and $f(x) \in B$. By hypothesis there exists $A \in \beta(m)$ and $x \in A$ such that $f(A) \subseteq B$. This implies that $x \in A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(B)$. That is $x \in f^{-1}(B)$. Since $A \in \beta(m)$, $A = i_\beta(A) \subseteq i_\beta(f^{-1}(B))$. Hence $x \in i_\beta(f^{-1}(B))$. Therefore, $f^{-1}(B) = \cup\{x : x \in f^{-1}(B)\} \subseteq i_\beta(f^{-1}(B)) \subseteq f^{-1}(B)$. Thus $i_\beta(f^{-1}(B)) = f^{-1}(B)$ and by Theorem 3.4 we have $f^{-1}(B) \in \beta(m)$. Hence f is contra $(\beta(m), w)$-continuous. □

THEOREM 3.28. *Let $\mathcal{M}$ be a minimal structure on X and w be weak structures on Y. For a function $f : (X, \mathcal{M}) \to (Y, w)$. The following conditions are equivalent:*

1. *f is contra $(\pi(m), w)$-continuous;*
2. *$f^{-1}(A) \in \pi(m)$ for every w-closed set A in Y;*
3. *for each $x \in X$ and each w-closed set A containing $f(x)$, there exists $B \in \pi(m)$ containing x such that $f(B) \subseteq A$;*
4. *$f(c_\pi(A)) \subseteq w$-ker$(f(A))$ for every subset A of X;*
5. *$c_\pi(f^{-1}(B)) \subseteq f^{-1}(w$-ker$(B))$ for every subset B of Y.*

Proof. (1) $\Leftrightarrow$ (2). It is obvious.
(2) $\Rightarrow$ (3). Let $x \in X$ and A be w-closed set containing $f(x)$. By hypothesis, $f^{-1}(A) \in \pi(m)$. Now put $B = f^{-1}(A)$, then $f(B) = f(f^{-1}(A)) \subseteq A$. Thus $f(B) \subseteq A$.
(3) $\Rightarrow$ (2). Let A be a w-closed set in Y and $x \in f^{-1}(A)$. Then $f(x) \in A$. By (3) there exists $B_x \in \pi(m)$ containing x such that $f(B_x) \subseteq A$. This implies that $B_x \subseteq f^{-1}(f(B_x)) \subseteq f^{-1}(A)$. Now $f^{-1}(A) = \cup\{B_x : x \in f^{-1}(A)\}$ and since $\pi(m)$ is a generalized topology, $f^{-1}(A) \in \pi(m)$.
(2) $\Rightarrow$ (4). Let A be any subset of X. Suppose $y \notin w\text{-ker}(f(A))$, then by Lemma 2.6 there exists w-closed set B containing y such that $f(A) \cap B = \phi$. thus we have $A \cap f^{-1}(B) = \phi$ and $c_\pi(A) \cap f^{-1}(B) = \phi$. Therefore, $f(c_\pi(A)) \cap B = \phi$ and $y \notin f(c_\pi(A))$. This implies $f(c_\pi(A)) \subseteq w\text{-ker}(f(A))$.
(4) $\Rightarrow$ (5). Let B be any subset of Y. By (4) and Lemma 2.6, we have $f(c_\pi(f^{-1}(B))) \subseteq w\text{-ker}(f(f^{-1}(B))) \subseteq w\text{-ker}(B)$ and $c_\pi(f^{-1}(B)) \subseteq f^{-1}(w\text{-ker}(B))$.
(5) $\Rightarrow$ (1). Let B be any w-open set in Y. By Lemma 2.6, we have $c_\pi(f^{-1}(B)) \subseteq f^{-1}(w\text{-ker}(B)) = f^{-1}(B)$ and $c_\pi(f^{-1}(B)) = f^{-1}(B)$ and by Theorem 3.4. Hence $f^{-1}(B) \in \pi(m)$. □

Acknowledgements

The author wishes to thank the referees for their useful comments and suggestions.

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Author's address:

Ahmad Al-Omari
Department of Mathematics
Faculty of Science
Al al-Bayt University
P.O. Box 130095
Mafraq 25113, Jordan
E-mail: omarimutah1@yahoo.com

Received July 10, 2012
Revised August 25, 2012