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Semilinear evolution equations in abstract spaces and applications[1]

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Dedicated to professor Fabio Zanolin on the occasion of his 60th birthday

Abstract. *The existence of mild solutions is obtained, for a semilinear multivalued equation in a reflexive Banach space. Weakly compact valued nonlinear terms are considered, combined with strongly continuous evolution operators generated by the linear part. A continuation principle or a fixed point theorem are used, according to the various regularity and growth conditions assumed. Applications to the study of parabolic and hyperbolic partial differential equations are given.*

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1. Introduction

The paper deals with the initial value problem associated to a semilinear multivalued evolution equation

$$\begin{cases} x'(t) \in A(t)x(t) + F(t, x(t)), & \text{for a.a. } t \in [a, b], \\ x(0) = x_0 \in E \end{cases} \tag{1}$$

in a reflexive Banach space $(E, \|\cdot\|)$ where

(A) $\{A(t)\}_{t\in[a,b]}$ is a family of linear, not necessarily bounded, operators with $A(t) : D(A) \subset E \to E$, $D(A)$ dense in E, which generates a strongly continuous evolution operator $U : \Delta \to \mathcal{L}(E)$ (see Section 2 for details);

(F1) $F(\cdot, x) : [a, b] \multimap E$ has a measurable selection for any $x \in E$ and $F(t, x)$ is nonempty, convex and weakly compact for any $t \in [a, b]$ and $x \in E$.

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When E is a separable Banach space, the measurability of $F(\cdot, x)$ for any $x \in E$ implies the existence of a selection as in (F1) (see the Theorem of Kuratowski-Ryll-Nardzewski [6, Theorem A]). Sufficient conditions are given in [6] in order to obtain the existence of a strongly measurable selection for the multivalued map (multimap for short) $F(\cdot, x)$ in a not necessarily separable Banach space.

Two different sets of regularity and growth assumptions on F are assumed, which cause the use of different techniques for studying (1). In Section 3 we treat the case when the evolution operator $U(t, s)$ is compact for $t > s$ and we assume that

(F2) $F(t, \cdot) : E \multimap E_\sigma$ is upper semicontinuous (u.s.c. for short) for a.a. $t \in [a, b]$.

We denote with X_σ the topological space obtained when $X \subseteq E$ is equipped with the weak topology.
If we further impose the growth condition

(F3) $\sup_{x \in \Omega} \|F(t, x)\| \leq \eta_\Omega(t)$ for a.a. $t \in [a, b]$, with $\Omega \subset E$ bounded and $\eta_\Omega \in L^1([a, b]; \mathbb{R})$,

which allows the nonlinearity F to have a superlinear growth, we make use of a classical continuation principle for compact multivalued fields (see Theorem 2.3).

In Section 4 we allow $U(t, s)$ to be non-compact, but we replace (F2) with the stronger regularity condition

(F2′) $F(t, \cdot) : E_\sigma \to E_\sigma$ is u.s.c. for a.a. $t \in [a, b]$

and we use a recent continuation principle in Frechét spaces due to the same authors (see Theorem 2.4). To this aim we also need the following condition

(F2″) $F(t, \cdot)$ is locally compact for a.a. $t \in [a, b]$.

Moreover, in Sections 3 and 4 we also show that, if we restrict the growth condition on F to

(F3′) $\|F(t, x)\| \leq \alpha(t)(1 + \|x\|)$ for a.a. $t \in [a, b]$, every $x \in E$ and some $\alpha \in L^1([a, b]; \mathbb{R})$,

then Ky Fan fixed point Theorem (see Theorem 2.5) can be used in both regularity assets and the solution set is compact in the appropriate topology.

We always investigate the existence of mild solutions of problem (1).

Definition 1.1. *A continuous function $x : [a, b] \to E$ is said to be a mild solution of the problem (1) if there exists a function $f \in L^1([a, b]; E)$ such that $f(t) \in F(t, x(t))$ for a.a. $t \in [a, b]$ and*

$$x(t) = U(t, a)x_0 + \int_a^t U(t, s) f(s)\, ds, \quad \forall\, t \in [a, b].$$

We refer to [5, 10] for the study of problem (1) when $F(t,\cdot)\colon E \multimap E$ is u.s.c. for a.a. $t \in [a,b]$ and it has compact values. Instead, the case when the linear part $A(t)$ is defined and bounded on all the space E was treated in [2, 12] under different regularity conditions. Nonlocal boundary value problems associated to the evolution equation in (1) are investigated in [4, 13] respectively in the case when F satisfies (F2$'$) and (F2). Many differential operators satisfy condition (A) and frequently they generate a compact evolution operator (see e.g. [14, 16]; see also Example 2.1). The introduction of a multivalued equation is often motivated by the study of a control problem. In Sections 5 we propose an application of our theory to the study of a parabolic partial differential inclusion, hence generating a compact evolution operator. In Section 6 we investigate a feedback control problem associated to an hyperbolic partial differential equation, and thus with a non-compact associated evolution operator. Section 2 contains some preliminary results.

2. Preliminary results

This part contains some preliminary results, of different types, which are useful in the sequel.
Throughout the paper we denote with B the closed unit ball of E centered at 0. Given the measure space (S,Σ,μ) and the Banach space X, we denote with $\|\cdot\|_p$ the norm of the Lebesgue space $L^p(S;X)$.

Let $\Delta = \{(t,s) \in [a,b]\times[a,b] : a \le s \le t \le b\}$. A two parameter family $\{U(t,s)\}_{(t,s)\in\Delta}$, where $U(t,s) : E \to E$ is a bounded linear operator and $(t,s)\in\Delta$, is called an *evolution system* if the following conditions are satisfied:

1. $U(s,s) = I,\ a \le s \le b$; $\quad U(t,r)U(r,s) = U(t,s),\ a \le s \le r \le t \le b$;
2. $(t,s) \longmapsto U(t,s)$ is strongly continuous on Δ, i.e. the map $(t,s) \to U(t,s)x$ is continuous on Δ for every $x \in E$.

For every evolution system, we can consider the respective *evolution operator* $U : \Delta \to \mathcal{L}(E)$, where $\mathcal{L}(E)$ is the space of all bounded linear operators in E. Since the evolution operator U is strongly continuous on the compact set Δ, by the uniform boundedness theorem there exists a constant $D = D_\Delta > 0$ such that

$$\|U(t,s)\|_{\mathcal{L}(E)} \le D\,, \quad (t,s)\in\Delta. \tag{2}$$

An evolution operator is said to be *compact* when $U(t,s)$ is a compact operator for all $t - s > 0$, i.e. $U(t,s)$ sends bounded sets into relatively compact sets. We refer to [14] for details on this topic.

EXAMPLE 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and consider the linear elliptic partial differential operator in divergence form $A\colon W^{2,2}(\Omega;\mathbb{R}) \cap W_0^{1,2}(\Omega;\mathbb{R}) \to L^2(\Omega;\mathbb{R})$ given by*

$$(A\ell)(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial \ell(x)}{\partial x_i}\right),$$

under the following conditions

(i) $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$ *for* $i,j = 1,2,..,n$*;*

(ii) $c\|\xi\|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$ *a.e. for every* $\xi \in \mathbb{R}^n$ *with* $c > 0$*.*

It is known that A (see e.g. [16]) generates a strongly continuous semigroup of contractions $S(t)$ with $S(t)$ compact for $t > 0$. Notice that, whenever $a_{ij} = 0$ for $i \neq j$ and $a_{ii} = 1$ for $i = 1,2,...n$, then $A\ell = \Delta\ell$.

Given $q \in C([a,b];E)$, let us denote with

$$S_q = \{f \in L^1([a,b];E) \;:\; f(t) \in F(t,q(t)) \text{ a.a. } t \in [a,b]\}.$$

PROPOSITION 2.2. *For a multimap $F : [a,b] \times E \multimap E$ satisfying properties (F1), (F2) and (F3), the set S_q is nonempty for any $q \in C([a,b];E)$.*

Proof. Let $q \in C([a,b];E)$; by the uniform continuity of q there exists a sequence $\{q_n\}$ of step functions, $q_n : [a,b] \to E$ such that

$$\sup_{t\in[a,b]} \|q_n(t) - q(t)\| \to 0, \quad \text{for } n \to \infty. \tag{3}$$

Hence, by (F1), there exists a sequence of functions $\{w_n\}$ such that $w_n(t) \in F(t,q_n(t))$ for a.a. $t \in [a,b]$ and $w_n : [a,b] \to E$ is measurable for any $n \in \mathbb{N}$. From (3) there exists a bounded set $\Omega \subset E$ such that $q_n(t), q(t) \in \Omega$ for any $t \in [a,b]$ and $n \in \mathbb{N}$ and by (F3) there exists $\eta_\Omega \in L^1([a,b];\mathbb{R})$ such that

$$\|w_n(t)\| \leq \|F(t,q_n(t))\| \leq \eta_\Omega(t) \quad \forall\, n \in \mathbb{N}, \text{ and a.a. } t \in [a,b].$$

Hence $\{w_n\} \subset L^1([a,b];E)$, $\{w_n\}$ is bounded and uniformly integrable and $\{w_n(t)\}$ is bounded in E for a.a. $t \in [a,b]$. According to the reflexivity of the space E and by the Dunford-Pettis Theorem (see [7, p. 294]), we have the existence of a subsequence, denoted as the sequence, such that

$$w_n \rightharpoonup w \in L^1([a,b];E).$$

By Mazur's convexity Theorem we obtain a sequence

$$\tilde{w}_n = \sum_{i=0}^{k_n} \lambda_{n,i} w_{n+i}, \qquad \lambda_{n,i} \geq 0, \ \sum_{i=0}^{k_n} \lambda_{n,i} = 1$$

such that $\tilde{w}_n \to w$ in $L^1([a,b];E)$ and, up to a subsequence, $\tilde{w}_n(t) \to w(t)$ for a.a. $t \in [a,b]$.
To conclude we have only to prove that $w(t) \in F(t,q(t))$ for a.a. $t \in [a,b]$. Indeed, let N_0 with Lebesgue measure zero be such that $F(t,\cdot) : E \multimap E_\sigma$ is u.s.c., $w_n(t) \in F(t,q_n(t))$ and $\tilde{w}_n(t) \to w(t)$ for all $t \in [a,b] \setminus N_0$ and $n \in \mathbb{N}$. Fix $t_0 \notin N_0$ and assume by contradiction that $w(t_0) \notin F(t_0,q(t_0))$.
Since $F(t_0,q(t_0))$ is closed and convex, from the Hahn Banach Theorem there is a weakly open convex set $V \supset F(t_0,q(t_0))$ satisfying $w(t_0) \notin \overline{V}$. Since $F(t_0,\cdot) : E \multimap E_\sigma$ is u.s.c., we can find a neighborhood U of $q(t_0)$ such that $F(t_0,x) \subset V$ for all $x \in U$. The convergence $q_n(t_0) \to q(t_0)$ implies the existence of $n_0 \in \mathbb{N}$ such that $q_n(t_0) \in U$ for all $n > n_0$. Therefore $w_n(t_0) \in F(t_0,q_n(t_0)) \subset V$ for all $n > n_0$. Since V is convex we also have that $\tilde{w}_n(t_0) \in V$ for all $n > n_0$ and, by the convergence, we arrive to the contradictory conclusion that $w(t_0) \in \overline{V}$. We conclude that $w(t) \in F(t,q(t))$ for a.a. $t \in [a,b]$. □

We propose now the two continuation principles (see Theorems 2.3 and 2.4) that we use, respectively in Sections 3 and 4, and recall Ky Fan fixed point Theorem (see Theorem 2.5).

THEOREM 2.3 ([1]). *Let Q be a closed, convex subset of a Banach space Y with nonempty interior and $H\colon Q \times [0,1] \multimap Y$ be such that*

(a) H is nonempty convex valued and it has closed graph;

(b) H is compact;

(c) $H(Q,0) \subset Q$;

(d) $H(\cdot,\lambda)$ is fixed points free on the boundary of Q for all $\lambda \in [0,1)$.

Then there exists $y \in Q$ such that $y \in H(y,1)$.

A metric space X is *contractible* if the identity map on it, i.e. $\mathrm{id}_X\colon X \to X$ is homotopic to a constant map. A compact nonempty metric space X is called an *R_δ-set* if there exists a decreasing sequence $\{X_n\}$ of compact, contractible sets X_n such that $X = \cap\{X_n : n \in \mathbb{N}\}$. Every convex compact subset of a metric space is an R_δ-set (see e.g. [1] for details).

Theorem 2.4 ([3, Theorem 2.1]). *Let F be a Hausdorff locally convex topological vector space, $X \subset F$ be a convex metrizable set, $Z \subset X$ be an open set in X and $H : Z \times [0,1] \multimap F$ be a compact u.s.c. multimap with R_δ values satisfying*

$$\begin{array}{l} \text{if } \{x_n\} \subset Z \text{ converges to } x \in H(x,\lambda), \text{ for some } \lambda \in [0,1), \text{ there is } n_0 \\ \text{such that } H(\{x_n\} \times [0,1]) \subset X, \text{ for all } n \geq n_0 \end{array} \tag{4}$$

and such that

(1) *$H(\cdot,0)(Z) \subset X$;*

(2) *there exists a compact u.s.c. multimap with R_δ values $H' \colon X \multimap X$ such that $H'\lfloor_Z = H(\cdot,0)$ and $Fix(H') \cap X \setminus Z = \emptyset$.*

Then there exists $x \in Z$ such that $x \in H(x,1)$.

When making use of a continuation principle it is often very delicate to show the so called transversality condition, i.e. condition *(d)* in Theorem 2.3 and condition (4) in Theorem 2.4. In both cases we assume here, to this aim, the existence of $R > \|x_0\|$ satisfying

$$D\left[\|x_0\| + \|\eta_{RB \setminus \|x_0\| B}\|_1\right] \leq R \tag{5}$$

with D given in (2) and η appearing in (F3).

Theorem 2.5. *Let X be a Hausdorff locally convex topological vector space, V be a compact convex subset of X and $G : V \multimap V$ an u.s.c. multimap with closed, convex values. Then G has a fixed point.*

We finally propose a useful compactness result for semicompact sequences (see Theorem 2.7).

Definition 2.6. *We say that a sequence $\{f_n\} \subset L^1([a,b];E)$ is* semicompact *if it is integrably bounded and the set $\{f_n(t)\}$ is relatively compact for a.a. $t \in [a,b]$.*

Theorem 2.7 ([10, Theorem 5.1.1]). *Let $S \colon L^1([a,b];E) \to C([a,b];E)$ be an operator satisfying the following conditions*

(i) *there is $L>0$ such that $\|Sf - Sg\|_C \leq L\|f-g\|_1$ for all $f, g \in L^1([a,b];E)$;*

(ii) *for any compact $K \subset E$ and sequence $\{f_n\} \subset L^1([a,b];E)$ such that $\{f_n(t)\} \subset K$ for a.a. $t \in [a,b]$ the weak convergence $f_n \rightharpoonup g$ implies $Sf_n \to Sg$.*

Then for every semicompact sequence $\{f_n\} \subset L^1([a,b];E)$ the sequence $\{Sf_n\}$ is relatively compact in $C([a,b];E)$ and, moreover, if $f_n \rightharpoonup f_0$ then $Sf_n \to Sf_0$.

3. The case of a compact evolution operator

In this Section we assume that the family $\{A(t)\}$ generates a compact evolution operator and that the nonlinear term F satisfies the regularity condition (F2) and, when not explicitly mentioned, the growth condition (F3).

First we introduce the solution multioperator $T : C([a,b];E) \times [0,1] \multimap C([a,b];E)$ defined as

$$T(q,\lambda) = \left\{ \begin{array}{r} x \in C([a,b];E) \,:\, x(t) = U(t,a)x_0 + \lambda \displaystyle\int_a^t U(t,s)f(s)\,ds, \\ \text{for all } t \in [a,b] \text{ and } f \in S_q \end{array} \right\} \tag{6}$$

which is well-defined according to Proposition 2.2 and we investigate its regularity properties. Notice that the fixed points of $T(\cdot,1)$ are mild solutions of the problem (1).

Proposition 3.1. *The multioperator T has a closed graph.*

Proof. Since $C([a,b];E)$ is a metric space, it is sufficient to prove the sequential closure of the graph. Let $\{q_n\}$, $\{x_n\} \subset C([a,b];E)$ and $\{\lambda_n\} \subset [0,1]$ satisfying $x_n \in T(q_n,\lambda_n)$ for all n and $q_n \to q$, $x_n \to x$ in $C([a,b];E)$, $\lambda_n \to \lambda$ in $[0,1]$. We prove that $x \in T(q,\lambda)$.

The fact that $x_n \in T(q_n,\lambda_n)$ means that there exists a sequence $\{f_n\}$, $f_n \in S_{q_n}$, such that

$$x_n(t) = U(t,a)x_0 + \lambda_n \int_a^t U(t,s)f_n(s)\,ds, \quad \forall\, t \in [a,b]. \tag{7}$$

Let $\Omega \subset E$ be such that $q_n(t)$, $q(t) \in \Omega$ for all $t \in [a,b]$ and $n \in \mathbb{N}$. Since $q_n \to q$ in $C([a,b];E)$, it follows that Ω is bounded and according to (F3) there is $\eta_\Omega \in L^1([a,b];\mathbb{R})$ satisfying $\|f_n(t)\| \le \eta_\Omega(t)$ for a.a. t and every n, implying that $\{f_n\}$ is bounded and uniformly integrable in $L^1([a,b];E)$ and $\{f_n(t)\}$ is bounded in E for a.a. $t \in [a,b]$. Hence, by the reflexivity of the space E and by the Dunford-Pettis Theorem (see [7, p. 294]), we have the existence of a subsequence, denoted as the sequence, and a function g such that $f_n \rightharpoonup g$ in $L^1([a,b];E)$. It is also easy to show that $U(t,\cdot)f_n \rightharpoonup U(t,\cdot)g$ in $L^1([a,t];E)$ for all $t \in [a,b]$. Since $\lambda_n \to \lambda$, we obtain that

$$x_n(t) \rightharpoonup x_0(t) := U(t,a)x_0 + \lambda \int_a^t U(t,s)g(s)\,ds \tag{8}$$

for all $t \in [a,b]$. By the uniqueness of the weak limit in E, we get that $x_0(t) = x(t)$ for all $t \in [a,b]$. Finally, reasoning as in the second part of the proof of Proposition 2.2 it is possible to show that $g(t) \in F(t,q(t))$ for a.a. $t \in [a,b]$. □

Proposition 3.2. *$T(Q \times [0,1])$ is relatively compact, for every bounded $Q \subset C([a,b];E)$.*

Proof. Let $Q \subset C([a,b];E)$ be bounded. Since $C([a,b];E)$ is a metric space it is sufficient to prove the relative sequential compactness of $T(Q \times [0,1])$. Consider $\{q_n\} \subset Q$, $\{x_n\} \subset C([a,b];E)$ and $\{\lambda_n\} \subset [0,1]$ satisfying $x_n \in T(q_n, \lambda_n)$ for all n. By the definition of the multioperator T, there exist a sequence $\{f_n\}$, $f_n \in S_{q_n}$, such that x_n satisfies (7). Let $\Omega \subset E$ be such that $q_n(t) \in \Omega$ for all t and n. Since Q is bounded, we have that Ω is bounded too and according to (F3) there exists $\eta_\Omega \in L^1([a,b];\mathbb{R})$ such that $\|f_n(t)\| \le \eta_\Omega(t)$ for a.a. $t \in [a,b]$ and all n.

According to (2) and the compactness of the evolution operator U, the sequence $\{U(t,\cdot)f_n\}$ is semicompact in $[a,t]$ for every fixed $t \in (a,b]$ (see Definition 2.6). Since the operator $S\colon L^1([a,t];E) \to C([a,t];E)$ defined by $Sf(\tau) = \int_a^\tau f(s)\,ds$ for $\tau \in [a,t]$ satisfies conditions *(i)* and *(ii)* in Theorem 2.7 we obtain that the sequence

$$\tau \mapsto \int_a^\tau U(t,s) f_n(s)\,ds, \qquad \tau \in [0,t],\ n \in \mathbb{N}$$

is relatively compact in $C([a,t];E)$; in particular $\left\{\int_a^t U(t,s)f_n(s)\,ds\right\}$ is a relatively compact set in E for all $t \in [a,b]$.

Now consider $a < t_0 < t \le b$. For every $\sigma \in (0, t_0 - a)$ we have that

$$\begin{aligned}
&\left\| \int_a^t U(t,s) f_n(s)\,ds - \int_a^{t_0} U(t_0,s) f_n(s)\,ds \right\| \\
&\quad \le \left\| \int_a^{t_0-\sigma} \left[U(t,s) - U(t_0,s)\right] f_n(s)\,ds \right\| \\
&\qquad + \left\| \int_{t_0-\sigma}^{t_0} \left[U(t,s) - U(t_0,s)\right] f_n(s)\,ds \right\| + \left\| \int_{t_0}^{t} U(t,s) f_n(s)\,ds \right\|.
\end{aligned} \tag{9}$$

Since it is known that $t \to U(t,s)$ is continuous in the operator norm topology, uniformly with respect to s such that $t - s$ is bounded away from zero (see e.g. [13]), for each $\epsilon > 0$ there is $\delta \in (0, t_0 - a)$ satisfying

$$\left\| \int_a^{t_0-\delta} \left[U(t,s) - U(t_0,s)\right] f_n(s)\,ds \right\| \le \epsilon \int_a^{t_0-\delta} \eta_\Omega(s)\,ds;$$

whenever $t - t_0 < \delta$; hence, according to (9), we obtain that

$$\left\| \int_a^t U(t,s) f_n(s)\,ds - \int_a^{t_0} U(t_0,s) f_n(s)\,ds \right\| \le \epsilon \int_a^{t_0-\delta} \eta_\Omega(s)\,ds + 2D \int_{t_0-\delta}^{t} \eta_\Omega(s)\,ds.$$

Thanks to the absolute continuity of the integral function, it implies that the sequence $\{\int_a^t U(t,s)f_n(s)\,ds\}$ is equicontinuous in $[a,b]$. Consequently, passing to a subsequence, denoted as the sequence, such that $\lambda_n \to \lambda \in [0,1]$ and using Arzelá-Ascoli theorem, we obtain that $\{x_n\}$ is relatively compact in $C([a,b];E)$ and the proof is complete. □

PROPOSITION 3.3. *The multioperator T has convex and compact values.*

Proof. Fix $q \in C([a,b];E)$ and $\lambda \in [0,1]$, since F is convex valued, the set $T(q,\lambda)$ is convex from the linearity of the integral and of the operator $U(t,s)$ for all $(t,s) \in \Delta$. The compactness of $T(q,\lambda)$ follows by Propositions 3.1 and 3.2. □

THEOREM 3.4. *Problem* (1) *under conditions (A) (F1), (F2), (F3),* (5) *and with $\{A(t)\}_{t\in[a,b]}$ generating a compact evolution operator has at least one solution.*

Proof. Consider the set $Q = C([a,b];RB)$ with R defined in (5). We show that the solution multioperator T defined in (6), when restricted to Q, satisfies the assumptions of Theorem 2.3. In fact Q is closed, convex, bounded and with a nonempty interior. According to Propositions 3.1, 3.2 and 3.3, T satisfies conditions *(a)* and *(b)* in Theorem 2.3.
Notice that $T(Q\times\{0\}) \subset D\|x_0\|B \subset \operatorname{int} Q$, hence condition *(c)* in Theorem 2.3 holds and $T(\cdot,0)$ is fixed point free on ∂Q. Let us now prove that T satisfies condition *(d)* also for $\lambda \in (0,1)$. Let $q \in Q$ and $\lambda \in (0,1)$ be such that $q \in T(q,\lambda)$ and assume, by contradiction, the existence of $t_0 \in (a,b]$ such that $q(t_0) \in \partial Q$ which is equivalent to $\|q(t_0)\| = R$. Since q is continuous and $q \in T(q,\lambda)$, from $\|x_0\| < R$ it follows that there exist $\hat{t}_0, \hat{t}_1 \in (a,t_0]$ with $\hat{t}_0 < \hat{t}_1$ such that $\|q(\hat{t}_0)\| = \|x_0\|$, $\|x_0\| < \|q(t)\| < R$ for $t \in (\hat{t}_0,\hat{t}_1)$ and $\|q(\hat{t}_1)\| = R$. Moreover there exists $f \in S_q$ such that $q(t) = U(t,\hat{t}_0)q(\hat{t}_0) + \lambda\int_{\hat{t}_0}^t U(t,s)f(s)\,ds$ for $t \in [\hat{t}_0,\hat{t}_1]$. According to (F3), $\|f(t)\| \le \eta_{RB\setminus\|x_0\|B}(t)$ for $t \in (\hat{t}_0,\hat{t}_1)$; so we arrive to the contradiction $R = \|q(\hat{t}_1)\| \le D[\|x_0\| + \lambda\|\eta_{RB\setminus\|x_0\|B}\|_1] < R$, and also condition *(d)* in Theorem 2.3 is satisfied.
Hence $T(\cdot,1)$ has a fixed point in Q which is a mild solution of problem (1). □

When the nonlinear term F has an at most linear growth, i.e. when it satisfies (F3′) instead of condition (F3), then the transversality condition (5) can be eliminated and the compactness of the solution set can be obtained too.

THEOREM 3.5. *Under conditions (A), (F1), (F2), (F3′) and with $\{A(t)\}_{t\in[a,b]}$ generating a compact evolution operator, the solution set of problem* (1) *is nonempty and compact.*

Proof. Consider the set Q defined as

$$Q = \left\{q \in C([a,b];E) \ : \ \|q(t)\| \le Re^{Lt} \text{ a.a. } t \in [a,b]\right\}$$

where L and R are such that

$$\max_{t\in[a,b]} D\int_a^t e^{L(s-t)}\alpha(s)ds := \overline{\beta} < 1,$$

$$R \geq e^{-La}D(\|x_0\| + \|\alpha\|_1)(1-\overline{\beta})^{-1}$$

and α was given in (F3′). Define the operator $\Gamma := T(\cdot, 1)$. According to Propositions 3.1, 3.2 and 3.3, it is easy to see that Γ is locally compact, with nonempty convex compact values and it has a closed graph. Hence it is also u.s.c. (see e.g. [10, Theorem 1.1.5]). We prove now that Γ maps the set Q into itself.

Indeed if $q \in Q$ and $x \in \Gamma(q)$ there exists a function $f \in S_q$ such that

$$x(t) = U(t,a)x_0 + \int_a^t U(t,s)f(s)\,ds.$$

By hypothesis $(F3')$ we have that

$$\begin{aligned}\|x(t)\| &= \left\| U(t,a)x_0 + \int_a^b U(t,s)f(s)\,ds \right\| \leq D\left(\|x_0\| + \int_a^t \alpha(s)(1+Re^{Ls})ds \right) \\ &\leq D\left(\|x_0\| + \|\alpha\|_1\right) + D\int_a^t \alpha(s)Re^{Ls}ds \leq D\left(\|x_0\| + \|\alpha\|_1\right) + Re^{Lt}\overline{\beta} \\ &\leq Re^{La}(1-\overline{\beta}) + Re^{Lt}\overline{\beta} \leq Re^{Lt}.\end{aligned}$$

Then $\Gamma(Q) \subseteq Q$. Let $V = \Gamma(Q)$ and $W = \overline{\mathrm{co}}(V)$, where $\overline{\mathrm{co}}(V)$ denotes the closed convex hull of V. Since $\overline{V}$ is a compact set, W is compact too. Moreover from the fact that $\Gamma(Q) \subset Q$ and that Q is a convex closed set we have that $W \subset Q$ and hence

$$\Gamma(W) = \Gamma(\overline{\mathrm{co}}(\Gamma(Q))) \subseteq \Gamma(Q) = V \subset W.$$

Hence, according to Theorem 2.5, Γ has a fixed point, which is a solution of (1).

We prove now that the solution set is compact. Indeed a solution of the problem (1) is a fixed point of the operator Γ. If $x \in \Gamma(x)$, by the definition of Γ and $(F3')$ we have the existence of $f \in S_x$ and reasoning as above

$$\begin{aligned}\|x(t)\| &\leq \|U(t,s)x_0\| + \int_0^t \|U(t,s)f(s)\|\,ds \\ &\leq D\left(\|x_0\| + \|\alpha\|_1 + \int_0^t \alpha(s)\|x(s)\|\,ds \right).\end{aligned}$$

By the Gronwall's inequality it holds

$$\|x(t)\| \leq D(\|x_0\| + \|\alpha\|_1)e^{D\|\alpha\|_1} := \overline{n}.$$

Hence $\mathrm{Fix}\,\Gamma$ is a bounded set and so $\Gamma(\mathrm{Fix}\,\Gamma)$ is relatively compact. Since $\mathrm{Fix}\,\Gamma \subset \Gamma(\mathrm{Fix}\,\Gamma)$, then $\mathrm{Fix}\,\Gamma$ is relatively compact too. Finally, according to the closure of the graph of Γ, $\mathrm{Fix}\,\Gamma$ is also closed and hence compact. □

4. The case of a non-compact evolution operator

If we drop the assumption that the family $\{A(t)\}$ generates a compact evolution operator, we need stronger regularity hypotheses on F to consider the richer class of evolution operators which we discuss now. We take, precisely, F satisfying $(F2')$; moreover, when not explicitly mentioned, we always assume the growth restriction (F3).

Since an u.s.c. multimap from E_σ to E_σ is u.s.c. from E to E_σ, the Proposition 2.2 is still true under the condition $(F2')$. Hence the set $S_q \neq \emptyset$ for any $q \in C([a,b];E)$ and the solution operator $T : C([a,b];E) \times [0,1] \multimap C([a,b];E)$ can be defined as in (6) and it has nonempty convex values. With a similar reasoning as in Proposition 3.1 it is also possible to prove that T has a weakly sequentially closed graph. Now we show that T is locally weakly compact.

PROPOSITION 4.1. $T(Q \times [0,1])$ *is weakly relatively compact for every bounded* $Q \subset C([a,b];E)$.

Proof. Let $Q \subset C([a,b];E)$ be bounded. We first prove that $T(Q \times [0,1])$ is weakly relatively sequentially compact.

Consider $\{q_n\} \subset Q$, $\{x_n\} \subset C([a,b];E)$ and $\{\lambda_n\} \subset [0,1]$ satisfying $x_n \in T(q_n, \lambda_n)$ for all n. By the definition of T, there exist a sequence $\{f_n\}$, $f_n \in S_{q_n}$ such that x_n satisfies (7). Passing to a subsequence, denoted as the sequence, we have that $\lambda_n \to \lambda \in [0,1]$. Moreover, reasoning as in the proof of Proposition 3.1, we obtain that there exists a subsequence, denoted as the sequence, and a function g such that $f_n \rightharpoonup g$ in $L^1([a,b];E)$, implying that $x_n(t)$ satisfies (8) for all $t \in [a,b]$. Furthermore, by (2) and the weak convergence of $\{f_n\}$ we have

$$\|x_n(t)\| \leq D\|x_0\| + D\|f_n\|_1 \leq N$$

for all $n \in \mathbb{N}$, $t \in [a,b]$, and for some $N > 0$. Hence $x_n \rightharpoonup x_0$ in $C([a,b];E)$. Thus $T(Q \times [0,1])$ is weakly relatively sequentially compact, hence weakly relatively compact by the Eberlein-Smulian Theorem (see [11, Theorem 1, p. 219]). □

REMARK 4.2. *Notice that, since T has weakly sequentially closed graph and according to Proposition 4.1, T has also weakly compact values.*

THEOREM 4.3. *Assume conditions (A), (F1),* $(F2')$, $(F2'')$, *(F3) and* (5). *If E is separable, then problem* (1) *has at least one solution.*

Proof. Put $\hat{R} := D\|x_0\| + \|\eta_{RB}\|_1 + 1$ with R defined in (5) and η in (F3) and define $Q = C([a,b];\hat{R}B)$. The set Q is closed, convex and bounded. Since E is separable, $C([a,b];E)$ is separable too and then Q is also metrizable. Consider the solution operator T defined in (6). Now we prove that it satisfies Theorem 2.4 with $F = (C([a,b];E))_\sigma$ and $X = Z = Q_\sigma$. According to Proposition 4.1, $T(Q \times [0,1])$ is weakly relatively compact so, in particular, $T(Q \times [0,1])$ is bounded and then $(T(Q \times [0,1]))_\sigma$ is metrizable. Since $T\colon Q\times[0,1] \multimap C([a,b];E)$ is weakly sequentially closed then it has weakly compact values and hence it is R_δ-valued. Moreover, according to Eberlein-Smulian Theorem and [10, Theorem 1.1.5], T is u.s.c. when both Q and $C([a,b];E)$ are endowed with the weak topology. Reasoning as in the proof of Theorem 3.4, it is also possible to show that condition *(1)* in Theorem 2.4 is satisfied; while condition *(2)* is trivially true. It remains to prove (4). So take $q_n \rightharpoonup q \in T(q,\lambda_0)$ for some $\lambda_0 \in [0,1)$. Let $x_n \in T(q_n,\lambda_n)$ for some $\lambda_n \in [0,1]$ and all n; then x_n satisfies (7) for some $f_n \in S_{q_n}$ and according to (F3) $\|f_n(t)\| \le \eta_{\hat{R}B}(t)$ for a.a. $t \in [a,b]$. Reasoning as in the proof of Proposition 3.1 we obtain a subsequence, denoted as the sequence, such that $f_n \rightharpoonup g \in L^1([a,b];E)$. Up to a subsequence we also have that $\lambda_n \to \lambda \in [0,1]$. Moreover, since $\{f_n(t)\} \subset F(t,\hat{R}B)$, according to $(F2'')$ we have that $\{f_n(t)\}$ is relatively compact for a.a. t. Let $G\colon L^1([a,b];E) \to C([a,b];E)$ be the generalized Cauchy operator associated to U, i.e. let $Gf(t) = \int_a^t U(t,s)f(s)\,ds$ for $t \in [a,b]$. It satisfies condition *(i)* in Theorem 2.7 and according to [5, Theorem 2], it also satisfies condition *(ii)* in Theorem 2.7. Hence $x_n \to x$ in $C([a,b];E)$ where $x(t) := U(t,a)x_0 + \lambda\int_a^t U(t,s)g(s)\,ds$ for $t \in [a,b]$. Since T has sequentially weakly closed graph, we obtain that $x \in T(q,\lambda)$. According to (5) and with a similar reasoning as in the proof of Theorem 3.4, we can show that $\|q(t)\| < R$ for all $t \in [a,b]$. Condition (F3) then implies that $\|g(t)\| \le \eta_{RB}(t)$ a.e. in $[a,b]$ and hence $\|x(t)\| \le D\|x_0\| + D\|\eta_{RB}\|_1 < \hat{R}$ for all $t \in [a,b]$ and we can find n_0 such that $x_n \in Q$ for every $n \ge n_0$. All the assumptions of Theorem 2.4 are then satisfied and hence $T(\cdot,1)$ has a fixed point which is a solution of problem (1) thus the proof is complete. □

If we assume, as in the previous section, the stronger growth condition $(F3')$, instead of $(F3)$, we can remove conditions (5) and $(F2'')$ as well as the requirement of the separability of the space E. Indeed, recalling that by the Krein Smulian Theorem (see e.g. [7, p. 434]) the convex closure of a weakly compact set is weakly compact, it is possible to reason exactly as in the proof of Theorem 3.2 to obtain the following result.

THEOREM 4.4. *Under assumptions (A), (F1),(F2′) and (F3′) the solution set of problem* (1) *is nonempty and weakly compact.*

5. Application to a parabolic partial differential inclusion

Let $t \in [0,T]$ and $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a sufficiently regular boundary. Consider the initial value problem

$$\begin{cases} u_t \in \Delta u + \left[p_1\left(t,x,\int_\Omega k(x,y)u(t,y)dy\right), p_2\left(t,x,\int_\Omega k(x,y)u(t,y)dy\right)\right] f(t,u(t,x)), \\ \hfill t \in [0,T]\, x \in \Omega \\ u(t,x) = 0 \quad t \in [0,T],\, x \in \partial\Omega \\ u(0,x) = u_0(x), \quad x \in \Omega \end{cases} \tag{10}$$

under the following hypotheses:

(a) $k\colon \Omega \times \Omega \to \mathbb{R}$ is measurable with $k(x,\cdot) \in L^2(\Omega;\mathbb{R})$ and $\|k(x,\cdot)\|_2 \leq 1$ for all $x \in \Omega$;

(b) $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(t,\cdot)$ L-Lipschitzian and $f(t,0) = 0$ for a.a. $t \in [0,T]$;

(c) $u_0 \in L^2(\Omega;\mathbb{R})$;

(d) $p_1, p_2\colon [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

  (i) $p_i(\cdot,\cdot,r)$ is measurable for $i = 1,2$ and all $r \in \mathbb{R}$;

  (ii) $-p_1(t,x,\cdot)$ and $p_2(t,x,\cdot)$ are u.s.c. for a.a. $t \in [0,T]$ and all $x \in \Omega$;

  (iii) $p_1(t,x,r) \leq p_2(t,x,r)$ in $[0,T] \times \Omega \times \mathbb{R}$;

  (iv) there exist $\psi \in L^1([0,T];\mathbb{R})$, $M\colon [0,\infty) \to \mathbb{R}$ increasing and $R > \|u_0\|_2$ such that $|p_i(t,x,r)| \leq \psi(t)M(|r|)$ for $i = 1,2$ and all x and

$$\|u_0\|_2 + \|\psi\|_1 LRM(R) \leq R. \tag{11}$$

We search for solutions $u \in C([a,b];L^2(\Omega;\mathbb{R}))$ of the initial value problem (10). Namely the following abstract formulation

$$\begin{cases} y'(t) \in Ay(t) + F(t,y(t)), & t \in [0,T] \\ y(0) = y_0, \end{cases} \tag{12}$$

should be satisfied, with $y(t) = u(t,\cdot) \in L^2(\Omega;\mathbb{R})$ for any $t \in [0,T]$. $A : W^{2,2}(\Omega;\mathbb{R}) \cap W_0^{1,2}(\Omega;\mathbb{R}) \to L^2(\Omega;\mathbb{R})$ is the linear operator defined as $Ay = \Delta y$ and $y_0 = u_0(\cdot)$. Given $\alpha \in L^2(\Omega;\mathbb{R})$, let $I_\alpha\colon \Omega \to \mathbb{R}$ be the function defined by $I_\alpha(x) = \int_\Omega k(x,y)\alpha(y)\,dy$. I_α is well-defined and measurable, according to (a), and it satisfies $|I_\alpha(x)| \leq \|\alpha\|_2$ for all $x \in \Omega$. Given $(t,\alpha) \in [0,T] \times L^2(\Omega;\mathbb{R})$, we define the multimap $F : [0,T] \times L^2(\Omega;\mathbb{R}) \multimap L^2(\Omega;\mathbb{R})$ as $y \in F(t,\alpha)$ if and only if there is a measurable function $\beta\colon \Omega \to \mathbb{R}$ satisfying $p_1(t,x,I_\alpha(x)) \le \beta(x) \le p_2(t,x,I_\alpha(x))$ for all $x \in \Omega$ such that $y(x) = \beta(x)f(t,\alpha(x))$ for all $x \in \Omega$.

Notice that, given $(t,\alpha) \in [0,T] \times L^2(\Omega;\mathbb{R})$ and according to *(d)(i)(ii)*, the maps $x \longmapsto p_i(t,x,I_\alpha(x))$, $i = 1,2$ are measurable in Ω; hence F has nonempty values and it is easy to see that they are also convex. Moreover $\|y\|_2 \le LM(\|\alpha\|_2)\|\alpha\|_2\psi(t)$, for all $y \in F(t,\alpha)$. Consequently, if $W \subset L^2(\Omega;\mathbb{R})$ is bounded, that is if $\|w\|_2 \le \mu$ for some $\mu > 0$ and all $w \in W$ we have that

$$\|F(t,W)\|_2 \le L\mu M(\mu)\psi(t) \tag{13}$$

implying (F3).

Now we investigate (F2) and hence we fix $t \in [a,b]$ and consider two sequences $\{\alpha_n\},\{y_n\} \subset L^2(\Omega;\mathbb{R})$ satisfying $\alpha_n \to \alpha$, $y_n \rightharpoonup y$ in $L^2(\Omega;\mathbb{R})$ and $y_n \in F(t,\alpha_n)$ for all $n \in \mathbb{N}$. Notice that $I_{\alpha_n}(x) \to I_\alpha(x)$ for all x. Since $\{\alpha_n\}$ is bounded, there is $\sigma > 0$ such that $\|\alpha_n\|_2 \le \sigma$ for all n. According to *(b)* the sequence $f(t,\alpha_n(\cdot)) \to f(t,\alpha(\cdot))$ in $L^2(\Omega;\mathbb{R})$ and then, passing to a subsequence denoted as usual as the sequence, we obtain that $f(t,\alpha_n(x)) \to f(t,\alpha(x))$ for a.a. $x \in \Omega$. By Mazur's convexity Theorem we have the existence of a sequence

$$\tilde{y}_n = \sum_{i=0}^{k_n} \delta_{n,i}y_{n+i}, \qquad \delta_{n,i} \ge 0, \ \sum_{i=0}^{k_n} \delta_{n,i} = 1$$

such that $\tilde{y}_n \to y$ in $L^2(\Omega;\mathbb{R})$ and up to a subsequence, denoted as the sequence, $\tilde{y}_n(x) \to y(x)$ for a.a. $x \in \Omega$. We prove now that $y \in F(t,\alpha)$. In fact, if $f(t,\alpha(x)) > 0$ then also $f(t,\alpha_n(x)) > 0$ for n sufficiently large, and it implies that $p_1(t,x,I_{\alpha_n}(x))f(t,\alpha_n(x)) \le y_n(x) \le p_2(t,x,I_{\alpha_n}(x))f(t,\alpha_n(x))$ for a.a. x. Consequently

$$\sum_{i=0}^{k_n} \delta_{n,i}p_1(t,x,I_{\alpha_{n+i}})f(t,\alpha_{n+i}(x)) \le \tilde{y}_n(x) \le \sum_{i=0}^{k_n} \delta_{n,i}p_2(t,x,I_{\alpha_{n+i}})f(t,\alpha_{n+i}(x)).$$

Passing to the limit as $n \to \infty$ and according to *(d)(ii)*, we obtain that $p_1(t,x,I_\alpha(x))f(t,\alpha(x)) \le y(x) \le p_2(t,x,I_\alpha(x))f(t,\alpha(x))$. With a similar reasoning we arrive to the estimate

$$p_2(t,x,I_\alpha(x))f(t,\alpha(x)) \le y(x) \le p_1(t,x,I_\alpha(x))f(t,\alpha(x))$$

when $f(t,\alpha(x)) < 0$. So, it remains to consider $\Omega_0 = \{x \in \Omega : f(t,\alpha(x)) = 0\}$. Notice that $f(t,\alpha_n(x)) \to 0$ in Ω_0. Since $y_n(\cdot) = \beta_n(\cdot)f(t,\alpha_n(\cdot))$ for some bounded and measurable $\beta_n\colon \Omega \to \mathbb{R}$ satisfying $p_1(t,x,I_{\alpha_n}(x)) \le \beta_n(x) \le p_2(t,x,I_{\alpha_n}(x))$ a.e. in Ω, it follows that $y_n(x) \to 0$ and then also $\tilde{y}_n(x) \to 0$, implying $y(x) \equiv 0$ in Ω_0. Therefore, it is possible to define a measurable function $\beta\colon \Omega \to \mathbb{R}$ such that $p_1(t,x,I_\alpha(x)) \le \beta(x) \le p_2(t,x,I_\alpha(x))$ and $y(x) =$

$\beta(x)f(t,\alpha(x))$ a.e. in Ω. We have showed that F has closed graph. Then by (13) $F(t,\cdot)$ has weakly compact values and it is locally weakly compact, since $L^2(\Omega;\mathbb{R})$ is reflexive, thus it satisfies (F2) (see e.g. [10, Theorem 1.1.5]). Moreover, according to Pettis measurability Theorem (see [15, p. 278]) it is possible to see that, for all $\alpha \in L^2(\Omega;\mathbb{R})$, the map $t \mapsto p_1\,(t,\cdot,I_\alpha(\cdot))\,f(t,\alpha(\cdot))$ is a measurable selection of $F(\cdot,\alpha)$, hence condition (F1) is satisfied. According to (13), for $\Theta = RB \setminus \|u_0\|B$ we can define η_Θ in (F3) as $\eta_\Theta(t) = LRM(R)\psi(t)$ and hence, according to *(d)(iv)* also condition (5) is satisfied.
All the assumptions of Theorem 3.4 are then satisfied and hence problem (12) is solvable, implying that (10) has at least one solution $u \in C([a,b];L^2(\Omega;\mathbb{R}))$.

6. Applications to an hyperbolic partial differential inclusion

Let Ω be a bounded domain in $\mathbb{R}^n$ with a sufficiently regular boundary. Consider the feedback control problem associated to a partial differential equation

$$\begin{cases} u_{tt} = \Delta u + p\left(t,x,\displaystyle\int_\Omega u(t,\xi)\,d\xi\right)u(t,x) + a(t,x)w(t,x) + b(t,x), \text{ in } [0,d]\times\Omega \\ w(t,x) \in W(u(t,x)) \\ u(t,x) = 0 \;\; t\in[0,d],\; x\in\partial\Omega \\ u(0,x) = u_0(x); u_t(0,x) = u_1(x)\;,\; x\in\Omega \end{cases} \tag{14}$$

where $W(r) = \{s \in \mathbb{R} \;:\; \ell r + m_1 \le s \le \ell r + m_2\}$, with $\ell > 0$ and $m_1 < m_2$. Assume the following hypotheses:

(i) a and b are globally measurable in $[0,d]\times\Omega$ and there exist two functions $\varphi_1,\varphi_2 \in L^1([0,d];\mathbb{R})$ such that

$$|a(t,x)| \le \varphi^1(t) \text{ for a.a. } x\in\Omega \text{ and } \forall\, t\in[0,d];$$

$$|b(t,x)| \le \varphi^2(t) \text{ for a.a. } x\in\Omega \text{ and } \forall\, t\in[0,d];$$

the map $p : [0,d]\times\Omega\times\mathbb{R} \to \mathbb{R}$ satisfies the following conditions

(ii) $p(\cdot,\cdot,r) : [0,d]\times\Omega \to \mathbb{R}$ is measurable, for all $r\in\mathbb{R}$;

(iii) $p(t,x,\cdot) : \mathbb{R}\to\mathbb{R}$ is continuous, for a.a. $(t,x)\in[0,d]\times\Omega$;

(iv) there exists $\varphi^3 \in L^1([0,d];\mathbb{R})$ such that

$$|p(t,x,r)| \le \varphi^3(t) \text{ for a.e. } x\in\Omega,\; \forall\, t\in[0,d] \text{ and } \forall\, r\in\mathbb{R}.$$

Let $y : [0,d] \to L^2(\Omega;\mathbb{R})$, $v : [0,d] \to L^2(\Omega;\mathbb{R})$, $f : [0,d] \times L^2(\Omega;\mathbb{R}) \times L^2(\Omega;\mathbb{R}) \to L^2(\Omega;\mathbb{R})$, and $V : L^2(\Omega;\mathbb{R}) \multimap L^2(\Omega;\mathbb{R})$ be the maps defined by

$$\begin{aligned}
&y(t) = u(t,\cdot);\\
&v(t) = w(t,\cdot);\\
&f(t,\alpha,\beta) : \Omega \to \mathbb{R},\ f(t,\alpha,\beta)(x) = p\left(t,x,\int_\Omega \alpha(\xi)\,d\xi\right)\alpha(x) + a(t,x)\beta(x) + b(t,x);\\
&V(z) = \{v \in L^2(\Omega;\mathbb{R}) \ :\ \ell z(x) + m_1 \le v(x) \le \ell z(x) + m_2,\ \text{a.a. } x \in \Omega\}.
\end{aligned}$$

In the Hilbert space $L^2(\Omega;\mathbb{R})$ problem (14) can be rewritten as a second order inclusion of the following form

$$\begin{cases} y''(t) \in Ay(t) + F(t,y(t)), & t \in [0,d],\ y(t) \in L^2(\Omega;\mathbb{R})\\ y(0) = y_0;\ y'(0) = y_1 \end{cases} \tag{15}$$

where $F(t,y(t)) = f(t,y(t),V(y(t)))$, $y_0 = u_0(\cdot)$, $y_1 = u_1(\cdot)$ and $A : D(A) = W^{2,2}(\Omega;\mathbb{R}) \cap W_0^{1,2}(\Omega;\mathbb{R}) \to L^2(\Omega;\mathbb{R})$ is the linear operator defined as $Ay = \Delta y$.

From the fact that $-A$ is a self-adjoint and positive definite operator on $L^2(\Omega;\mathbb{R})$ with a compact inverse, we have that there exists a unique positive definite square root $(-A)^{1/2}$ with domain $D((-A)^{1/2}) = W_0^{1,2}(\Omega;\mathbb{R})$. Introduce the Hilbert space $\mathcal{E} = W_0^{1,2}(\Omega;\mathbb{R}) \times L^2(\Omega;\mathbb{R})$ with the inner product

$$\left\langle \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \cdot \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} \right\rangle = \int_\Omega \nabla p_0\, \nabla q_0\, dx + \int_\Omega p_0\, q_0\, dx + \int_\Omega p_1\, q_1\, dx.$$

Since the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix},\quad D(\mathcal{A}) = D(A) \times W_0^{1,2}(\Omega;\mathbb{R})$$

generates a strongly continuous semigroup (see e.g. [8]), we can treat (15) as a first order semilinear differential inclusion in $\mathcal{E}$

$$\begin{cases} z'(t) \in \mathcal{A}z(t) + \mathcal{F}(t,z(t)),\ t \in [0,d]\\ z(0) = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \end{cases} \tag{16}$$

where $\mathcal{F} : [0,d] \times \mathcal{E} \multimap \mathcal{E}$ is defined as

$$\mathcal{F}\left(t, \begin{pmatrix} c^0 \\ c^1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ F(t,c^0) \end{pmatrix}.$$

Observe that the semigroup generated by $\mathcal{A}$ is noncompact.

Denoted $I_\alpha = \int_\Omega \alpha(y)dy$, by the separability of the space $L^2(\Omega;\mathbb{R})$ and the

Pettis measurability Theorem [15], we have that the map $t \to p(t,\cdot,I_\alpha)\alpha(\cdot)+a(t,\cdot)(\ell\alpha(\cdot)+m_1)+b(t,\cdot)$ is a measurable selection of $F(\cdot,\alpha)$. We prove, now, that the map F satisfies condition (F2$'$). Reasoning like in Section 5 it is possible to prove that the multimap V is weakly sequentially closed. Let, now, $t\in[0,d]$ be fixed, let $\{\alpha_n\}\subset L^2(\Omega;\mathbb{R})$, be weakly convergent to $\alpha\in L^2(\Omega;\mathbb{R})$ and let $\{w_n\}\subset L^2(\Omega;\mathbb{R})$ with $w_n\in F(t,\alpha_n)$ for any $n\in\mathbb{N}$, be weakly convergent to $w\in L^2(\Omega;\mathbb{R})$. By the definition of the multimap F we have

$$w_n = f_1(t,\alpha_n)+f_2(t,\beta_n), \quad \text{with } \beta_n\in V(\alpha_n) \text{ for any } n\in\mathbb{N},$$

where $f_1(t,\alpha)(x)=p(t,x,I_\alpha)\alpha(x)$ and $f_2(t,\beta)(x)=a(t,x)\beta(x)+b(t,x)$. By the definition of the multimap V and the weak convergence of $\{\alpha_n\}$ we have that the sequence $\{\beta_n\}$ is norm bounded. Hence, by the reflexivity of the space $L^2(\Omega;\mathbb{R})$, up to subsequence, $\{\beta_n\}$ weakly converges to $\beta\in L^2(\Omega;\mathbb{R})$ and the weak closure of the multimap V implies $\beta\in V(\alpha)$. Moreover by the continuity of the map p we have that $\{f_1(t,\alpha_n)\}$ converges weakly to $f_1(t,\alpha)$ and it is easy to see that $\{f_2(t,\beta_n)\}$ converges weakly to $f_2(t,\beta)$. In conclusion we have obtained

$$w=f_1(t,\alpha)+f_2(t,\beta)\in f(t,\alpha,V(\alpha))=F(t,\alpha).$$

Furthermore, easily, V has convex and closed values, thus, by the linearity of the map f_2 and following the same reasonings above, F is convex closed valued as well.

Finally (see e.g. [4])

$$\|F(t,\alpha)\|_2\le\left(\varphi^3(t)+2\ell\varphi^1(t)\right)\|\alpha\|_2+|\Omega|^{1/2}\left[(m_1+m_2)\varphi^1(t)+\varphi^2(t)\right],$$

obtaining both that for any $t\in[0,d]$ and $\alpha\in L^2(\Omega;\mathbb{R})$ the set $F(t,\alpha)$ is bounded (hence relatively compact by the reflexivity of $L^2(\Omega;\mathbb{R})$), and that condition $(F3')$ is satisfied.

Let $z=(y_0,y_1)$ be a solution of (16). Applying the Implicit Function Theorem of Filippov's type (see [9, Theorem 7.2]) we have that there exists $v:[0,d]\to L^2(\Omega;\mathbb{R})$ such that $v(t)\in V(y_0(t))$ and $g(t)=f(t,y_0(t),v(t))$, $t\in[0,d]$. Hence the feedback control problem (14) admits a weakly compact set of solutions.

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