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# On the supports for cohomology classes of complex manifolds[1]

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*Dedicated to Fabio Zanolin on the occasion of his sixtieth birthday*

Abstract. *Let $X$ be a compact, connected complex manifold, and let $\xi \in H^i(X, \mathbb{Q})$ be a non-trivial class. The paper deals with the possibility to construct a topological cycle $\Gamma$ on $X$, whose class is the Poincaré dual of $\xi$, which is closely related in a precise sense to the complex structure of $X$. The desired properties of $\Gamma$ allow to define a differentiable relation into a suitable space of 1-jets. This relation shows that there is a preliminary topological obstruction to construct such a $\Gamma$. The main result of the paper is that, in a relevant particular case, this obstruction disappears.*

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## 1. Introduction

Throughout the paper $X$ will denote a compact, connected complex manifold of dimension $n$.

Let $\xi \in H^i(X, \mathbb{Q})$ be non zero. By a classical theorem of Thom [5] there is an integer $N > 0$ such that the Poincaré dual $PD(N\,\xi) \in H_k(X, \mathbb{Q})$ is the fundamental class of an oriented differentiable submanifold $\Gamma \subset X$, of dimension $k = 2n - i$ ( by the way, the symbol $\subset$ will denote nonstrict inclusion throughout the paper ). The set $\Gamma$ is closed in $X$, hence compact. For our purposes the relevant property is

$$\xi|_{X-\Gamma} = 0 . \tag{1}$$

To prove this, let $T$ be an open tubolar neighborhood of $\Gamma$ inside $X$. Then $Z := X - T$ is a deformation retract of $X - \Gamma$, and it is sufficient to prove that

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$\xi|_Z = 0$. Denote the inclusion $Z \subset X$ by $h$, and assume that $h^*(\xi) = \xi|_Z \neq 0$. Therefore, since the Kronecker pairing

$$\langle\ ,\ \rangle : H^i(Z, \mathbb{Q}) \times H_i(Z, \mathbb{Q}) \to \mathbb{Q}$$

is non degenerate, there is $u \in H_i(Z, \mathbb{Q})$ such that $\langle h^*(\xi), u \rangle \neq 0$. But

$$\langle h^*(\xi), u \rangle = \langle \xi, h_*(u) \rangle$$

and it is well known that the right hand side agrees with the intersection number of the $k$-cycles $PD(\xi) = [\Gamma]$ and $h_*(u)$ on $X$. Since these cycles can be represented by disjoint chains, we conclude $\langle \xi, h_*(u) \rangle = 0$, contradiction.

Relation (1) implies also that for any subset $S$ of $X$ containing $\Gamma$ we have

$$\xi|_{X-S} = 0.$$

We will say that such a subset of $X$ is a *support* for $\xi$. Actually, we are interested to the possibility that $\Gamma$ is contained into a *complex subspace* $Y \subset X$, i.e. that $\xi$ has supports which are of some interest from the point of view of the Complex Geometry. Let us give a necessary condition for this.

By restricting the scalars, the complex $n$-dimensional vector space $T_P X$ can be thought as a real $2n$-dimensional vector space. This real vector space is nothing but the tangent space at $P$ of the differentiable manifold underlying $X$. Recall that multiplication by $i = \sqrt{-1}$ defines on $T_P X$ a complex structure $J : T_P X \to T_P X$, and a real subspace of $T_P X$ corresponds to a *complex* subspace of the complex space $T_P X$ if and only if it is left invariant by $J$.

Now assume that $\Gamma$ is contained into some complex subspace $Y \subsetneq X$. For any point $P \in \Gamma$ which is smooth for $Y$ there is a chain of real tangent vector spaces

$$T_P\Gamma \subset T_P Y \subsetneq T_P X.$$

But $T_P Y$ is a complex subspace of $T_P X$, hence

$$T_P\Gamma + J(T_P\Gamma) \subset T_P Y \subsetneq T_P X.$$

Note that $T_P\Gamma + J(T_P\Gamma)$ is in any case the *smallest* complex subspace of $T_P X$ containing $T_P\Gamma$. If the codimension of $Y$ into $X$ is assumed to be $\geq p$, then at any point $P \in \Gamma \cap Y_{\text{sm}}$ we have

$$dim_{\mathbb{C}}\big(T_P\Gamma + J(T_P\Gamma)\big) \leq n - p. \tag{2}$$

Notice that by semi-continuity this relation is actually satisfied at every point of $\Gamma$.

To try to construct such a support $Y$ for $\xi$, the idea is to start from a $\Gamma$ obtained by Thom's theorem, and then to deform somehow the inclusion $i : \Gamma \to X$ to get, say, an immersion $f : \Gamma \to X$, which satisfies condition (2) at any point, and moreover satisfies

$$f_*\mu_\Gamma \;=\; i_*\mu_\Gamma \;=\; [\Gamma] \;=\; PD(N\xi) \;\in\; H_k(X,\mathbb{Q})\,, \tag{3}$$

where $\mu_\Gamma \in H_k(\Gamma,\mathbb{Q})$ is the fundamental class of $\Gamma$ ( recall that $\Gamma$ is oriented, see [5, p. 28], where, however, this assumption is implicit ). Since (2) involves tangent spaces to $\Gamma$ and $X$, the natural ambient to study how to deform the inclusion $\Gamma \subset X$ is the space $\mathscr{J}^1(\Gamma, X)$ of 1-jets of germs of maps $\Gamma \to X$, of class $\mathscr{C}^1$ at least. This space consists of all linear maps $L : T_c\Gamma \to T_xX$ for all possible choices of $c \in \Gamma$ and of $x \in X$. There are canonical maps $s : \mathscr{J}^1(\Gamma, X) \to \Gamma$ and $b : \mathscr{J}^1(\Gamma, X) \to X$ defined respectively by

$$s(L) \,:=\, c \qquad \text{and} \qquad b(L) \,:=\, x\,.$$

Moreover, every map $f : \Gamma \to X$, of class $\mathscr{C}^k$ with $k \geq 1$, lifts to the map

$$\begin{array}{c} \mathscr{J}^1_f : \Gamma \longrightarrow \mathscr{J}^1(\Gamma, X) \\ c \mapsto d\,f_c \end{array} \qquad\qquad \begin{array}{ccc} & & \mathscr{J}^1(\Gamma, X) \\ & \overset{\mathscr{J}^1_f}{\nearrow} & \downarrow{\scriptstyle b} \\ \Gamma & \xrightarrow[f]{} & X \end{array} \tag{4}$$

of class $\mathscr{C}^{k-1}$, which makes the diagram on the right commutative. Note that $\mathscr{J}^1_f$ is always an embedding when $k \geq 2$, even if $f$ is not. We set

$$\mathscr{R} \,:=\, \{\, L \in \mathscr{J}^1(\Gamma, X) \,|\, dim_{\mathbb{C}}\big(\, L(T_c\Gamma) + J(\, L(T_c\Gamma))\,\big) \,\leq\, n - p \,\}\,. \tag{5}$$

In Gromov language ( see e.g. [3] ) such a $\mathscr{R}$ is called a *differential relation*. Condition (2) translates nicely into this new set-up, because, if $f : \Gamma \to X$ is an immersion, it amounts to require that $\mathscr{J}^1_f(\Gamma) \subset \mathscr{R}$.

All this makes apparent that *there is a priori a topological obstruction in order to find a deformation* $f : \Gamma \to X$ *of the inclusion* $i : \Gamma \to X$ *which satisfies* (2) *and* (3). In fact, assume that there is such a $f$, and let us simply denote by $\varphi$ its lifting to $\mathscr{J}^1(\Gamma, X)$ ; then $\varphi(\Gamma) \subset \mathscr{R}$. Hence, formally the map $\varphi$ factorizes through the inclusion $u : \mathscr{R} \subset \mathscr{J}^1(\Gamma, X)$, namely we have the commutative diagram of topological spaces and continuous maps

$$\begin{array}{ccc} \mathscr{R} & \overset{u}{\hookrightarrow} & \mathscr{J}^1(\Gamma, X) \\ {\scriptstyle \psi}\uparrow & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle b} \\ \Gamma & \xrightarrow[f]{} & X \end{array}$$

which yields in homology ( the fundamental class $\mu_\Gamma$ of $\Gamma$ was already introduced above )

$$PD(N\,\xi) \;=\; [\Gamma] \;=\; f_*\,\mu_\Gamma \;=\; b_*(\varphi_*\,\mu_\Gamma) \;=\; b_*u_*\psi_*\,\mu_\Gamma \;=\; b_*\circ[u_*(\psi_*\,\mu_\Gamma)]\,.$$

Therefore, *in order that the inclusion* $\Gamma\subset X$ *can be deformed to satisfy* (2), *a necessary condition is that the class* $[\Gamma]$ *is the image via* $b_*$ *of a class supported on* $\mathscr{R}$ .

In this paper we discuss this topological obstruction in the simplest possible case, namely when $p=1$ ( recall that $p$ was introduced as the codimension into $X$ of a complex subspace $Y$ of $X$ containing $\Gamma$ ). In this case condition (2) specializes to

$$dim_{\mathbb{C}}\big(\,L(T_c\Gamma)+J(\,L(T_c\Gamma)\,)\big)\;\le\;n-1 \tag{6}$$

and the differential relation $\mathscr{R}$ involved becomes

$$\mathscr{R}\;=\;\{\,L\in\mathscr{J}^1(\Gamma,X)\,|\,dim_{\mathbb{C}}\big(\,L(T_c\Gamma)+J(\,L(T_c\Gamma))\big)\;\le\;n-1\,\}\,.$$

To justify a further restriction in the statement of the main theorem below, let me say that the paper arose from an attempt to understand from a differential geometric point of view some aspects of the Hodge Conjecture. It is well known that Hodge $(p,p)$-conjecture can be reduced to the case when $dim(X)=2p$. Therefore, it was natural for a first exploration to consider only the case when $i=dim(X)=k$.

The main result of the paper is that in the particular case when $p=1$ and $i=dim(X)=k$, the topological obstruction mentioned above disappears. More precisely, we have

THEOREM 1.1. *For* $X$ *of arbitrary dimension* $n$ , *let* $\mathscr{R}\subset\mathscr{J}^1(\Gamma,X)$ *be defined by (6) in the particular case* $i=dim(X)=k$ . *Then* $\mathscr{R}$ *is a deformation retract of* $\mathscr{J}^1(\Gamma,X)$ .

Following some pioneering work of Thom [6], Gromov, Eliashberg and several other people developed the theory of differential relations ( see e.g. [3] ). This theory provides technical tools which should allow, in principle, to decide whether the inclusion $\Gamma\subset X$ can be deformed as desired, or not.

However, it is well known that on a general smooth, projective hypersurface $X\subset\mathbb{P}^4$, of degree 5 , there are non-trivial $\xi\in H^3(X,\mathbb{Q})$ which are not supported by a divisor of $X$ ( see e.g. [7], Ch. 18 ). It would be of the highest interest to understand from the point of view of the differential relations why a 3-cycle $\Gamma$ corresponding to such a class $\xi$ cannot be deformed in the desired way in this case.

Theorem 1.1 is proved in § 4. The few, elementary facts about jets we will need are recalled for the reader' s convenience in the second section. The study of the basic properties of $\mathscr{R}$ used to prove Theorem 1.1 is the content of § 3. Finally, the last section contains some details on the restriction of $\mathscr{R}$ to the fibres of $(s, b) : \mathscr{J}^1(\Gamma, X) \to \Gamma \times X$, which perhaps are of independent interest.

From now on we will assume without further mention that $k = dim(X) = n$.

## 2. Some basic fact on 1-jets

We will consider only 1-jets, so we will always write in the sequel $\mathscr{J}$ for $\mathscr{J}^1(\Gamma, X)$, and $\mathscr{J}(U, V)$ for $\mathscr{J}^1(U, V)$. For the basic definitions and properties of the spaces of jets the interested reader is referred e.g. to [2].

Let $\Gamma$ and $X$ be differentiable varieties of class $\mathscr{C}^r$, where $r \geq 1$ is an integer, or $r = \omega$, namely $\Gamma$ and $X$ are real analytic varieties; we will mantain this convention about $r$ throughout the paper.

A structure of differential variety on the set $\mathscr{J}(\Gamma, X)$ is given by the following atlas. Let $(U, u^1, u^2, \dots, u^n)$ and $(V, x^1, x^2, \dots, x^{2n})$ be as above; then we can represent $L$ by a $2n \times n$ matrix with respect to the bases

$$\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \dots, \frac{\partial}{\partial u^n} \quad \text{of } T_c\Gamma \qquad \text{and} \qquad \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{2n}} \quad \text{of } T_x X$$

canonically associated to the given coordinate charts. To represent the entries of this matrix we introduce new coordinates $p_{ij}$, where $1 \leq i \leq 2n$ and $1 \leq j \leq n$. Therefore, if we consider the canonical map

$$(s, b) : \mathscr{J}(\Gamma, X) \to \Gamma \times X \tag{7}$$

on the subset $\mathscr{J}(U, V) := (s, b)^{-1}(U \times V)$ of $\mathscr{J}(\Gamma, X)$ we have the local coordinates

$$u^1, u^2, \dots, u^n, x^1, x^2, \dots, x^{2n}, p_{ij}, \ 1 \leq i \leq 2n, \ 1 \leq j \leq n. \tag{8}$$

We will need in the sequel the explicit expression for the change of local coordinates in $\mathscr{J}$. For this, consider coordinate charts $(U', v^1, v^2, \dots, v^n)$ on $\Gamma$ and $(V', y^1, \dots, y^{2n})$ on $X$, such that $U \cap U' \neq \varnothing$ and $V \cap V' \neq \varnothing$. It is clear than that

$$\mathscr{J}(U, V) \cap \mathscr{J}(U', V') = \mathscr{J}(U \cap U', V \cap V') \neq \varnothing.$$

On $\mathscr{J}(U', V')$ the local coordinates are

$$v^1, \dots, v^n, y^1, \dots, y^{2n}, q_{hk}, \ 1 \leq h \leq 2n, \ 1 \leq k \leq n,$$

and the change of local coordinates is given by the maps

$$v^k = v^k(u^1, \dots, u^n), \qquad 1 \le k \le n, \tag{9}$$

$$y^h = y^h(x^1, \dots, x^{2n}), \qquad 1 \le h \le 2n, \tag{10}$$

$$q_{hk} = \sum_{\substack{1 \le i \le 2n \\ 1 \le j \le n}} \frac{\partial y^h}{\partial x^i} \frac{\partial u^j}{\partial v^k} p_{ij}, \qquad 1 \le h \le 2n, 1 \le k \le n. \tag{11}$$

In particular, notice that, for fixed $c \in U \cap U'$ and $x \in V \cap V'$, relations (11) define a linear map. This implies that the map (7) realizes $\mathscr{J}(\Gamma, X)$ as a real vector bundle over $\Gamma \times X$, of rank $2n^2$ ( by the way, if we consider higher order jets, i.e. $\mathscr{J}^r(\Gamma, X)$ with $r > 1$, we can only say that $(s, b) : \mathscr{J}^r(\Gamma, X) \to \Gamma \times X$ is an *affine* bundle ). It is clear how this vector bundle trivializes; in fact, if $M$ denotes the real vector space of $2n \times n$ matrices, then $\mathscr{J}(U, V)$ can be identified with $U \times V \times M$, and then $(s, b) : \mathscr{J}(U, V) \to U \times V$ corresponds to the projection $U \times V \times M \to U \times V$.

Define the rank of the 1-jet $(c, x, L)$ as the rank of $L$. The map $\rho$ which associates to every 1-jet its rank is easily seen to be lower semicontinuous. Hence, for any integer $r$, with $0 \le r \le n$, the set $\mathscr{J}_r := \{ j \in \mathscr{J} \mid \rho(j) \le r \}$ is closed in $\mathscr{J}$. We will mostly *restrict in the sequel to work on the open subset* $\mathscr{Y}$ *of* $\mathscr{J}$ *of the jets of rank* $n$.

## 3. The differential relation $\mathscr{R}$

Let us now introduce some more standard notation which will be used freely throughout the paper.

Consider coordinate charts $(U, u^1, u^2, \dots, u^n)$ for $\Gamma$ and $(V, x^1, x^2, \dots, x^{2n})$ for $X$. More precisely, we will always assume that $V$ is a domain of holomorphic coordinates $(z^1, \dots, z^n) \in \mathbb{C}^n$ on $X$, and that $z^h = x^h + ix^{n+h}$ is the decomposition of $z^h$ into its real and imaginary parts. Then the complex structure $J$ is given by

$$J : (x^1, \dots, x^n, x^{n+1}, \dots, x^{2n}) \mapsto (-x^{n+1}, -x^{n+2}, \dots, -x^{2n}, x^1, \dots, x^n). \tag{12}$$

Now assume that we have an immersion $f : U \to V$; we can write it in coordinates. For any $c \in U$ the image $T_c := d f_c(T_c\Gamma)$ of the differential map $d f_c$ is generated inside $T_{f(c)}X$ by the columns of the jacobian matrix

$$J_c = \frac{\partial ( x^1, x^2, \dots, x^{2n} )}{\partial ( u^1, u^2, \dots, u^n )}(c),$$

which is a $2n \times n$ matrix. We write $J_c$ in block form

$$J_c = \begin{pmatrix} A \\ B \end{pmatrix}, \tag{13}$$

where both $A, B$ are $n \times n$ real matrices, whose entries depend on $c$. Then by (12) the subspace $T_c + J(T_c)$ of $T_P X$ is generated by the columns of the matrix

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

and relation (6) is verified at all points of $U$ if and only if on $U$

$$\det \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \equiv 0 . \tag{14}$$

(13) and (14) suggest to organize the matrix $(p_{ij})$ in block form

$$(p_{ij})_{i,j} = \begin{pmatrix} A \\ B \end{pmatrix} \tag{15}$$

and to set

$$\mathscr{M} := \begin{pmatrix} A & -B \\ B & A \end{pmatrix} . \tag{16}$$

The determinant $D_{UV}$ of $\mathscr{M}$ is a homogeneous polynomial, with coefficients in $\mathbb{Z}$, in the indeterminates $p_{ij}$, of degree $2n$.

We will check now that the loci defined on the various charts $\mathscr{J}(U, V)$ by the corresponding equations $D_{UV} = 0$ patch toghether to define a closed subset of $\mathscr{J}(\Gamma, X)$, which is the differential relation $\mathscr{R}$.

The key point is to understand how the various maps $D_{UV}$ behave under a change of coordinates. So, let $U' \subseteq \Gamma$ and $V' \subseteq X$ denote as usual coordinate charts such that $U \cap U' \neq \varnothing$ and $V \cap V' \neq \varnothing$. Then on $\mathscr{J}(U \cap U', V \cap V')$ we have the restrictions of both $D_{UV}$ and $D_{U'V'}$.
To simplify notations we will denote the jacobian matrices involved by

$$\mathscr{U} = \frac{\partial (y^1, \ldots, y^{2n})}{\partial (x^1, \ldots, x^{2n})} \qquad \text{and} \qquad \mathscr{V} = \frac{\partial (u^1, \ldots, u^n)}{\partial (v^1, \ldots, v^n)} .$$

Moreover, let us write the matrix $\mathscr{M}$ in block form as

$$\mathscr{M} = (\mathscr{P} \,|\, S\,\mathscr{P}) , \tag{17}$$

where the size of each block is $2n \times n$, and

$$S := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

is the matrix of the complex structure $J$ (note that this matrix is the same on every chart of $X$). Finally, arrange the various $q_{hk}$ appearing in (11) in a $2n \times n$ matrix $\mathscr{Q}$. Equations (11) tell us that $\mathscr{P}$ and $\mathscr{Q}$ are related by

$$\mathscr{Q} = \mathscr{U}\,\mathscr{P}\,\mathscr{V} .$$

Then

$$\big(\,\mathscr{Q}\,|\,S\,\mathscr{Q}\,\big) \;=\; \big(\,\mathscr{U}\,\mathscr{P}\,\mathscr{V}\,|\,S\,\mathscr{U}\,\mathscr{P}\,\mathscr{V}\,\big)\,.$$

Since $X$ is a complex manifold, we can restrict to the case when all the changes of coordinates (10) are holomorphic, hence their differentials are $\mathbb{C}$-linear. In matrix terms this is $S\,\mathscr{U} \;=\; \mathscr{U}\,S$ , which yields

$$\big(\,\mathscr{U}\,\mathscr{P}\,\mathscr{V}\,|\,S\,\mathscr{U}\,\mathscr{P}\,\mathscr{V}\,\big) \;=\; \big(\,\mathscr{U}\,\mathscr{P}\,\mathscr{V}\,|\,\mathscr{U}\,S\,\mathscr{P}\,\mathscr{V}\,\big) \;=\; \mathscr{U}\,\big(\,\mathscr{P}\,|\,S\,\mathscr{P}\,\big)\begin{pmatrix}\mathscr{V} & 0\\ 0 & \mathscr{V}\end{pmatrix}.$$

Taking determinants we get

$$\det\big(\,\mathscr{Q}\,|\,S\,\mathscr{Q}\,\big) \;=\; \det(\mathscr{U}\,)\det(\mathscr{M})\det(\mathscr{V}\,)^2\,.$$

In terms of the functions $D$ this relation becomes

$$D_{U'V'} \;=\; \lambda\,D_{UV}\,, \tag{18}$$

where

$$\lambda \;:=\; \det(\mathscr{U}\,)\det(\mathscr{V}\,)^2 : \mathscr{J}(U\cap U', V\cap V') \to \mathbb{R}_{>}\,. \tag{19}$$

In fact, $\det(\mathscr{U}\,) > 0$ because $X$ is canonically oriented. It is a simple exercise to check that *the functions* $\lambda$ *satisfy the cocycle condition.*

Therefore $\mathscr{R}$ *can be defined coherently by the vanishing of the functions* $D$ *on the coordinate charts of* $\mathscr{J}(\Gamma, X)$ .

Let us analyze more closely the functions $D$. Elementary operations on the matrix $\mathscr{M}$ in the block form (16) transform it into

$$\begin{pmatrix}\mathscr{A}+i\mathscr{B} & 0\\ \frac{i}{2}(\mathscr{A}-i\mathscr{B}) & \mathscr{A}-i\mathscr{B}\end{pmatrix} \qquad \text{and finally into} \qquad \begin{pmatrix}\mathscr{A}+i\mathscr{B} & 0\\ 0 & \mathscr{A}-i\mathscr{B}\end{pmatrix}.$$

Note that the rank of the first $n$ columns in the above matrices changes only when the last group of elementary operations is performed. Note also that $\mathscr{A}+i\mathscr{B}$ and $\mathscr{A}-i\mathscr{B}$ have the same rank, hence

$$rk(\mathscr{M}) \;=\; 2\;rk(\mathscr{A}+i\mathscr{B})\,. \tag{20}$$

Moreover, $\det(\,\mathscr{A}+i\mathscr{B}\,)$ is a homogeneous polynomial in the indeterminates $p_{ij}$ , with complex coefficients, of degree $n$ . It is convenient to write it in the form

$$E \;:=\; \det(\,\mathscr{A}+i\mathscr{B}\,) \;=\; R+iI\,, \tag{21}$$

where $R$ *and* $I$ *are both homogeneous polynomials with real coefficients, of degree* $n$ . Therefore

$$D_{UV} \;=\; \det\begin{pmatrix}\mathscr{A}+i\mathscr{B} & 0\\ 0 & \mathscr{A}-i\mathscr{B}\end{pmatrix} \;=\; (\,R+iI\,)\,(\,R-iI\,) \;=\; R^2+I^2\,. \tag{22}$$

Corollary 3.1. *$D_{UV}$ is a homogeneous polynomial with real coefficients, in the indeterminates $p_{ij}$, of degree $2n$. Moreover, as a function, $D_{UV} \geq 0$.*

For future use we have also to analyze the behaviour of the maps $D$ outside $\mathscr{R}$. For this, set

$$\mathscr{F} := \mathscr{J}(U,V) - \mathscr{R}.$$

The restriction of $D = D_{UV}$ to $\mathscr{F}$ is a *smooth* map ( actually, an algebraic one, hence real-analytic ) $\mathscr{F} \to \mathbb{R}_>$. It is elementary to check that *such a $D$ is a surjective submersion.*

Corollary 3.2. *For any $a > 0$ the set $D^{-1}(a)$ is a smooth hypersurface of $\mathscr{F}$.*

Assume now that $U, U'$ and $V, V'$ are domains of coordinate charts for $\Gamma$ and $X$ respectively, such that $U \cap U' \neq \varnothing$ and $V \cap V' \neq \varnothing$. We have the restrictions of both $D_{UV}$ and $D_{U'V'}$ on $\mathscr{J}(U \cap U', V \cap V')$. But the map $\lambda : \mathscr{J}(U \cap U', V \cap V') \to \mathbb{R}_>$ defined in (19) is not constant in general, hence the hypersurfaces $D^{-1}_{UV}(a)$ and $D^{-1}_{U'V'}(a')$ of $\mathscr{F}$ *do not glue*, however $a, a'$ are choosen.

The throubles with $\lambda$ disappear if we restrict to a fiber of $(s,b)$. In fact, take any $c \in U \cap U'$ and $x \in V \cap V'$, and set

$$\Phi := (s,b)^{-1}(c,x).$$

Because of (18) we then have

$$\Phi \cap D^{-1}_{U'V'}\big(\lambda(c,x)\,a\big) = \Phi \cap D^{-1}_{UV}(a). \tag{23}$$

Hence *these hypersurfaces of $\Phi \simeq \mathbb{R}^{2n^2}$ are independent from the system of local coordinates on $\mathscr{J}$ used to define them.* They will play an important role in the sequel, mainly bacause of the following proposition, quite similar to Corollary 3.2. From now on we will denote by $D$ both the restriction to $\Phi$ of the map $D_{UV}$, and the homogeneous polynomial which is the determinant of the matrix (16).

Proposition 3.3. *For any $a > 0$ the subsets*

$$\mathscr{D}_a := D^{-1}(a)$$

*of $\Phi$ are smooth hypersurfaces, and $\Phi - \mathscr{R}$ is foliated by them when $a$ runs into $\mathbb{R}_>$.*

*Proof.* Take any $P \in \Phi$, such that $D(P) > 0$. Since $D$ is homogeneous, of degree $2n$, by Euler formula we have

$$\sum_{i,j} p_{ij}(P)\,\frac{\partial D}{\partial p_{ij}}(P) \;=\; 2n\,D(P) \;>\; 0\,.$$

Hence the various

$$\frac{\partial D}{\partial p_{ij}}(P)$$

cannot be all zero. □

## 4. Proof of Theorem 1.1

The outline of the construction of a retraction map $r : \mathscr{J} \to \mathscr{R}$ is rather simple. In fact, recall that $\mathscr{J}$ has a structure of real vector bundle over $\Gamma \times X$, given by the map $(s,b)$, as was already remarked in § 2. Hence $r$ can be constructed fiberwise. In any fiber $\Phi$ there are the level hypersurfaces of the maps $D$. Though the "levels" actually depend on the function $D$, hence on the local coordinates used to define it, the hypersurfaces themselves do not because of (18), and we can therefore consider the corresponding normal directions field, with respect to some metric on $\Phi$. This metric will be supplyied by a Riemannian structure on $\mathscr{J}$, namely a smoothly varying positive definite symmetric bilinear form on each fiber. It is well known that any vector bundle over a smooth base can be endowed with such a structure.

The directions field mentioned above corresponds to several (nowhere vanishing) vector fields, e.g. the gradient of $D$. The integral curves of any of these vector fields foliate $\Phi - \mathscr{R}$, and the key point is that every integral curve "ends" on $\mathscr{R}$. Then, given any $P \in \Phi - \mathscr{R}$, there is exactly one integral curve containing it, and we can define $r(P)$ to be the limit point of this curve into $\mathscr{R}$.

Let us fix on $\mathscr{J}$ a Riemannian structure $\mathcal{M}$. On $\Phi = (s,b)^{-1}(c,x)$ we fix an ortonormal basis with respect to the metric $\mathcal{M}(c,x)$. On $\Phi$ we will use the coordinates $q_{ij}$ given by the dual basis, instead of the $p_{ij}$ introduced previously, to simplify somewhat the computations. In the new coordinates the function $D$ has still the form (22), namely

$$D \;=\; \widetilde{R}^{\,2} + \widetilde{I}^{\,2}\,, \tag{24}$$

where $\widetilde{R}$ and $\widetilde{I}$ are both homogeneous polynomials of degree $n$ in the variables $q_{ij}$. Therefore, $D$ is homogeneous, of degree $2n$. Moreover, the set $\mathscr{C} = \Phi \cap \mathscr{R}$ is defined into $\Phi$ by the equation $D = 0$.

We are interested to the family of ortogonal curves to the level hypersurfaces of the function $D$. Hence, by definition, the more general system of differential equations with integral curves the family of curves we want is

$$\frac{\mathrm{d}\, q_{ij}}{\mathrm{d}\, t} = \nu \, \nabla D \,, \tag{25}$$

where $\nabla D$ denotes the gradient vector field of $D$, and $\nu$ is a nowhere vanishing real function defined in a suitable open set of $\Phi$, to be determined in order that any solution of (25) satisfies some desired property.

Notice that $\nabla D$ vanishes exactly along $\mathscr{C}$. In fact, (24) implies that $\nabla D$ vanishes along $\mathscr{C}$, and at any point where $\nabla D$ vanishes, $D$ vanishes as well by Euler formula. This allows us to consider the following specialization of (25) on $\Phi - \mathscr{C}$

$$\frac{\mathrm{d}\, q_{ij}}{\mathrm{d}\, t} = \frac{\nabla D}{\| \nabla D \|^2} \,. \tag{26}$$

The reason for (26) is that the relation of our integral curves with the level hypersurfaces of $D$ makes reasonable to try to parametrize the integral curves, at least locally, by the "level" itself. More precisely, if $\varphi(t)$ is a function $\mathbb{R} \to \Phi$ whose image is an integral curve, then we want the following relation to be identically satisfied

$$D(\, \varphi(t)\,) \equiv t \,. \tag{27}$$

To determine the function $\nu$ in (25) such that (27) will be satisfied, we differentiate (27), where $\varphi(t)$ is assumed to be a solution of (25), thus getting

$$\nu \, \| \nabla D \|^2 \equiv 1 \,.$$

Conversely, let $\varphi(t)$ be a solution of (26). Then,

$$\frac{\mathrm{d}}{\mathrm{d}\, t} D(\, \varphi(t)\,) = \sum_{i,j} \frac{\partial D}{\partial\, q_{ij}} (\, \varphi(t)\,)\, \varphi'_{ij}(t) \equiv 1$$

and there is a real constant $C$ such that

$$D(\, \varphi(t)\,) \equiv t + C \,.$$

But the system (26) is autonomous, and we can safely assume that $C = 0$.

Lemma 4.1. *Every solution $\varphi$ of (26) is maximally defined on* $(0, +\infty)$. *Moreover, the function $t \mapsto \| \varphi(t) \|^2$ is strictly increasing, and*

$$\lim_{t \to +\infty} \| \varphi(t) \| = \infty \,. \tag{28}$$

*Proof.* Take any $P \in \Phi$ not in $\mathscr{C}$, and set $t_0 = D(P)$. Moreover, let $\varphi(t)$ be the solution of (26) such that $\varphi(t_0) = P$. It is customary to consider $\nabla D$ as a column vector; if $P$ is considered as a row vector, then by Euler formula we get

$$P \cdot \nabla D(P) \;=\; 2n\, D(P) \;=\; 2n\, t_0 \;>\; 0$$

and Schwarz inequality yields

$$2n\, t_0 \;=\; |\, P \cdot \nabla D(P)\,| \;\leq\; \|\, P\,\|\, \|\, \nabla D(P)\,\| \;.$$

Hence

$$\left\| \frac{\nabla D(P)}{\|\, \nabla D(P)\,\|^2} \right\| \;=\; \frac{1}{\|\, \nabla D(P)\,\|} \;\leq\; \frac{1}{2n\, t_0}\; \|\, P\,\| \;.$$

Therefore, if $a$ is any real number such that $0 < a < t_0$, then for every $P' \in \Phi - \mathscr{C}$ such that $D(P') \geq a$, the following inequality is satisfied

$$\left\| \frac{\nabla D(P')}{\|\, \nabla D(P')\,\|^2} \right\| \;\leq\; \frac{1}{2n\, a}\; \|\, P'\,\| \;.$$

This shows that $\varphi(t)$ is defined on any $[t_1, t_2] \subseteq \mathbb{R}$, where $a < t_1 < t_0 < t_2$, hence on $[t_1, \infty)$. Since $a > 0$ is arbitrary, we conclude that every solution of (26) is defined on $(0, +\infty)$.

Moreover, we have by Euler formula and (26) (here ${}^{\mathbf{t}}\varphi(t)$ denotes the transposed of the column vector $\varphi(t)$)

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\; \|\, \varphi(t)\,\|^2 = \; 2\, {}^{\mathbf{t}}\varphi(t) \cdot \frac{\nabla D(\varphi(t))}{\|\, \nabla D(\varphi(t))\,\|^2} \;=\; 4n\, \frac{D(\varphi(t))}{\|\, \nabla D(\varphi(t))\,\|^2} \;>\; 0\,,$$

for every $t$, hence $t \mapsto \|\, \varphi(t)\,\|^2$ is a strictly increasing function.

Finally, set $\mathscr{D}_a := D^{-1}(a)$ for every $a > 0$. Note that, if $b > 0$ is another real number, then the ubiquitous Euler formula yields also the diffeomorphism

$$\mathscr{D}_a \to \mathscr{D}_b \qquad \text{given by} \qquad P \mapsto \left(\frac{b}{a}\right)^{\frac{1}{2n}} P\,.$$

Therefore, if we set $\mu_a := \inf\{\, \|\, P\,\| \;\; |\, P \in \mathscr{D}_a\,\}$ (clearly $\mu_a > 0$), then $\mu_a$ and $\mu_b$ are related by

$$\mu_b \;=\; \left(\frac{b}{a}\right)^{\frac{1}{2n}} \mu_a$$

and (28) follows because $\varphi(t) \in \mathscr{D}_t$ for any $t > 0$ by (27), hence

$$\|\, \varphi(t)\,\| \geq \; \mu_t\,.$$

□

It remains to analyze the behaviour of the solutions of (26) when $t \to 0^+$. The key point is disposed by the following result ( for the proof see [4] ).

THEOREM 4.2. *Let $D : \Phi \to \mathbb{R}$ be a real-analytic function $\geq 0$. Then, for every $P \in \mathscr{C}$ there is a neighborhood $W_P$ of $P$ inside $\Phi$ such that for every $Q \in W_P$ the solution $q_Q$ of the Cauchy problem $q_Q(0) = Q$ for the system of first order ODE*

$$\left( q_{ij} \right)' = -\nabla D \tag{29}$$

*is defined in $[0, \infty)$, has finite length, and converges uniformly to a point of $\mathscr{C}$ when $t \to \infty$. Moreover, if $Q \in W_P$ then $q_Q(t) \in W_P$ for every $t \geq 0$.*

REMARK 4.3. *To keep close to [4] we stated the above theorem with the orientation of the integral curves* reversed *with respect to our conventions. Moreover, notice for future use that this result is local, namely it is sufficient to consider the restriction of $D$ to any neighborhood $L$ of a given $P \in \mathscr{C}$. In this case the solution $q_Q$ will converge to a point of $\mathscr{C} \cap L$ when $t \to \infty$.*

Consider, now, an arbitrary solution $\psi$ of (26). Since $t \mapsto \| \psi(t) \|$ is a strictly increasing function, for every fixed $b > 0$ we have $\| \psi(t) \| \leq \| \psi(b) \|$ whenever $t \leq b$. Let $K$ denote the intersection of $\mathscr{C}$ with the closed ball $B$ of vectors with norm $\leq \| \psi(b) \|$; then $K$ is compact and there are finitely many points $P_1, \ldots, P_s \in K$ such that

$$K \subset W_{P_1} \cup \ldots \cup W_{P_s},$$

where any $W_{P_i}$ is an open neighborhood of $P_i$ like in Theorem 4.2.

The function $D$ has a minimum on $B - ( W_{P_1} \cup \ldots \cup W_{P_s} )$, and this minimum is $> 0$, because this set is compact and disjoint from $\mathscr{C}$. Then, for $a > 0$ sufficiently small ( and $a < b$ ), we get

$$\mathscr{D}_a \cap B \subset W_{P_1} \cup \ldots \cup W_{P_s}. \tag{30}$$

But every hypersurface $\mathscr{D}_a$ of $\Phi$ can be used to assign the initial condition for the solutions of (26), uniformly with respect to the time $t$. In fact, we have the straightforward consequence of (27).

COROLLARY 4.4. *For any fixed real number $a > 0$, every solution $\varphi$ of (26) intersects $\mathscr{D}_a$ in exactly one point.*

Therefore (30) implies that $\psi(a) \in W_{P_i}$ for a suitable $i$. Then Theorem 4.2 applied to $\psi$ ( cum grano salis! ) yields

$$\lim_{t \to 0^+} \psi(t) \in \mathscr{C}.$$

We are in position now to define a map $\rho : \Phi - \mathscr{C} \to \mathscr{C}$, the first step toward the retraction $r : \mathscr{J} \to \mathscr{R}$. In fact, if $Q \in \Phi - \mathscr{C}$ is arbitrary, let $\varphi$ be the unique solution of (26) such that $\varphi(D(Q)) = Q$. We set

$$\rho(Q) := \lim_{t \to 0^+} \varphi(t) .$$

Lemma 4.5. *The map $\rho$ is continuous.*

*Proof.* For the proof we need another lemma. To state it, let us introduce a small piece of notation. If $P$ is any point of $\Phi - \mathscr{C}$, and $D(P) = a$, we will denote by $\varphi_{P}$ the unique solution of (26) such that $\varphi_{P}(a) = P$.

Lemma 4.6. *For any fixed real number $c > 0$*

$$\chi : (0, +\infty) \times \mathscr{D}_c \to \Phi - \mathscr{C} \qquad \textit{given by} \qquad \chi(t, P) = \varphi_{P}(t)$$

*is a homeomorphism. It follows, in particular, that for any two strictly positive real numbers $a, b$, the hypersurfaces $\mathscr{D}_a$ and $\mathscr{D}_b$ of $\Phi$ are homeomorphic via*

$$P \mapsto \varphi_{P}(b) \qquad \textit{for every} \qquad P \in \mathscr{D}_a .$$

*Proof.* Corollary 4.4 implies that $\chi$ is bijective. Moreover, $\chi$ is the restriction to $(0, +\infty) \times \mathscr{D}_c$ of

$$(0, +\infty) \times (\, \Phi - \mathscr{C}\,) \to (\, \Phi - \mathscr{C}\,) , \qquad \text{defined by } (t, P) \mapsto \varphi_{P}(t) , \tag{31}$$

which gives the flow of the vector field at the R.H.S. of (26), and it is well known that this map is continuous. Finally, $\chi^{-1} : \Phi - \mathscr{C} \to (0, +\infty) \times \mathscr{D}_c$ is given by

$$P \mapsto (\, D(P) \, , \, \varphi_{P}(c) \,)$$

and to show that it is continuous it is sufficient to check that $P \mapsto \varphi_{P}(c)$ is such. But this is a standard consequence of the theorem of the continuous dependence of solutions on initial data. □

To conclude the proof of Lemma 4.5, for an arbitrary $P \in \Phi - \mathscr{C}$, set $Q = \rho(P)$. Here we use the fact that Theorem 4.2 is of local nature. In fact, for any neighborhood $L$ of $Q$, we can consider the neighborhood $W_{Q} \subset L$ as in the statement of Theorem 4.2, referred now to $D|_{L}$. Then, for $b > 0$ sufficiently small we have $\varphi_{P}(b) \in W_{Q}$. Fix one of such $b$.

Let $M$ denote an open neighborhood of $\varphi_{P}(b)$ into $\mathscr{D}_b$, such that

$$M \subset W_{Q} . \tag{32}$$

If $D(P) = a$ and $0 < \eta < a$ is real, then Lemma 4.6 tells us that

$$\mathscr{L} = \{\, R \in \Phi - \mathscr{C} \mid a - \eta < D(R) < a + \eta \quad \text{and} \quad \varphi_{R}(b) \in M \,\}$$

is an open neighborhood of $P$ inside $\Phi - \mathscr{C}$. Then $\rho(\mathscr{L}) \subseteq L$ by Theorem 4.2 because of (32), and the proof of Lemma 4.5 is complete. □

REMARK 4.7. *I believe that $\rho : \Phi - \mathscr{C} \to \mathscr{C}$ is surjective, but I don't know how to prove this. Notice however that, as a straightforward consequence of Theorem 4.2, the set $\rho(\Phi - \mathscr{C})$ is dense inside $\mathscr{C}$.*

LEMMA 4.8. *The map $\rho : \Phi - \mathscr{C} \to \mathscr{C}$ can be extended to a continuous map $\rho_0 : \Phi \to \mathscr{C}$ by setting $\rho_0(P) = P$ when $P \in \mathscr{C}$.*

*Proof.* It remains to check the continuity at the points of $\mathscr{C}$. But this follows immediately from Theorem 4.2. □

The next step is the extension of $\rho_0$ to a coordinate neighborhood of $\mathscr{J}$. For this, let $U$ and $V$ be the usual coordinate neighborhoods for $\Gamma$ and $X$ respectively. Then we can define

$$\rho_1 : \mathscr{J}(U,V) \to \mathscr{J}(U,V) \cap \mathscr{R}$$

by assuming that it acts fiberwise (the fibres are those of $(s,b)$) like the map $\rho_0$ defined above. Since the restriction of $\mathscr{J}(\Gamma, X)$ to $U \times V$ is a trivial vector bundle, $\rho_1$ is continuous.

To extend $\rho_1$ to the desired map $r : \mathscr{J} \to \mathscr{R}$, the only delicate point is the following verification. Assume that $U'$ and $V'$ are other coordinate neighborhoods for $\Gamma$ and $X$ such that $U \cap U' \neq \varnothing$ and $V \cap V' \neq \varnothing$. Then we have also

$$\rho_1' : \mathscr{J}(U',V') \to \mathscr{J}(U',V') \cap \mathscr{R}$$

and we have to check that

$$\rho_1|_{\mathscr{J}(U\cap U', V\cap V')} = \rho_1'|_{\mathscr{J}(U\cap U', V\cap V')} . \tag{33}$$

Here we exploit the fact that both $\rho_1$ and $\rho_1'$ are defined fiberwise. So, let $\Phi = (s,b)^{-1}(c,x)$ be an arbitrary fiber contained into $\mathscr{J}(U \cap U', V \cap V')$. The two coordinate neighborhoods of $\mathscr{J}$ containing $\Phi$ give us the two maps $D, D' : \Phi \to \mathbb{R}$ related by

$$D' = \lambda_0 D$$

because of (18), where $\lambda_0 = \lambda(c,x)$ (see (19)). Therefore

$$\nabla D' = \lambda_0 \nabla D \qquad \text{and} \qquad \frac{\nabla D'}{\| \nabla D' \|^2} = \frac{1}{\lambda_0} \frac{\nabla D}{\| \nabla D \|^2} . \tag{34}$$

The system of ODE (26) for the local coordinates corresponding to $U'$ and $V'$ is then

$$\frac{\mathrm{d}\, q_{ij}'}{\mathrm{d}\, t} = \frac{1}{\lambda_0} \frac{\nabla D}{\| \nabla D \|^2} . \tag{35}$$

Now, let $Q \in \Phi - \mathscr{C}$, ad assume that $D(Q) = a$, hence $D'(Q) = \lambda_0\, a$. With the notation introduced in the proof of Lemma 4.5, let $\varphi'_Q$ be the solution of (35) such that $\varphi'_Q(\lambda_0\, a) = Q$. It is easily checked that the map

$$\varphi(t) := \varphi'_Q(\lambda_0\, t) : (0, +\infty) \to \Phi - \mathscr{C}$$

satisfies identically (26) thanks to the (34). Moreover, since $\varphi(a) = Q$, we can conclude

$$\varphi_Q(t) = \varphi'_Q(\lambda_0\, t) \qquad \text{for every } t > 0\,. \tag{36}$$

Hence,

$$\rho_1(Q) = \lim_{t \to 0^+} \varphi_Q(t) = \lim_{t \to 0^+} \varphi'_Q(t) = \rho'_1(Q)$$

and the equality (33) is completely proved.

Therefore, by (33) we can define a map $r : \mathscr{J} \to \mathscr{R}$ by just requiring that its restriction to any coordinate neighborhood $\mathscr{J}(U, V)$ of $\mathscr{J}$ is the corresponding $\rho_1$. It is clear that such an $r$ is continuous, and that, if the inclusion $\mathscr{R} \subset \mathscr{J}$ is denoted by $u$, then $r \circ u = id_{\mathscr{R}}$.

To complete the proof of Theorem 1.1 it remains to show that $u \circ r$ is homotopic to $id_{\mathscr{J}}$. Since $r$ was substantially defined fiberwise, it seems reasonable to try to construct in this way also an homotopy

$$H : [0, 1] \times \mathscr{J} \to \mathscr{J} \tag{37}$$

between $u \circ r$ and $id_{\mathscr{J}}$.

Then, let $\Phi$, $\mathscr{C}$ and $\rho_0$ be as usual, and denote by $i$ the inclusion $\mathscr{C} \subset \Phi$. For every $P \in \Phi - \mathscr{C}$ we have $\varphi_P : (0, +\infty) \to \Phi - \mathscr{C}$. This map can be extended to a continuous map

$$\widetilde{\varphi}_P : [0, +\infty) \to \Phi \qquad \text{by setting} \quad \widetilde{\varphi}_P(0) = \rho(P)\,.$$

Moreover, if $P \in \mathscr{C}$ we will define $\widetilde{\varphi}_P : [0, +\infty) \to \Phi$ to be the constant map with value $P$. After these preparations, we set

$$h : [0, 1] \times \Phi \to \Phi \qquad \text{where} \quad h(\tau, P) := \widetilde{\varphi}_P(\, \tau\, D(P)\,)\,. \tag{38}$$

The relations

$$h(1, \_\,) = id_{\mathbb{P}hi}\,, \qquad h(0, \_\,) = i \circ \rho_0\,,$$

follow from the definition. It remains to check that $h$ is continuous. Only the continuity at a point $(\tau_0, P)$ where $\tau_0 > 0$ and $P \in \mathscr{C}$ deserves some comment. In this case $h(\tau_0, P) = P$, so let $U$ be an arbitrary neighborhood of $P$. As usual, we will consider a neighborhood $W_P$ of $P$ like in the statement of Theorem 4.2, and such that $W_P \subset U$. Moreover, let $a > 0$ be such that $\tau_0 - a > 0$. Finally, let $b > 0$ such that $L := W_P \cap \mathscr{D}_b \neq \varnothing$. We set

$$V := \left\{ Q \in \Phi \mid Q \in W_P \ , \ D(Q) < \frac{b}{\tau_0 + a} \ , \ \text{ if } \ Q \notin \mathscr{C} \ \text{ then } \ \varphi_Q(b) \in L \right\}.$$

Thanks to Lemma 4.6, $V$ is an open neighborhood of $P$. Assume, now, that $\tau \in (\tau_0 - a, \tau_0 + a)$, and $Q \in V$. If $Q \in \mathscr{C}$, then

$$h(\tau, Q) \ = \ Q \ \in \ W_P \subset U \, .$$

If $Q \notin \mathscr{C}$, then $h(\tau, Q) = \varphi_Q(\tau\, D(Q))$. Therefore, the definition of $V$ yields both the relations $\tau\, D(Q) < b$ and $\varphi_Q(b) \in L \subset W_P$. Hence $h(\tau, Q) \in W_P \subset U$ by the last sentence of Theorem 4.2, and we conclude that the map $h$ in (38) is continuous.

As with the definition of the retraction $r$, the key point to define the homotopy (37) is the verification that the map (38) actually does not depend on the choice of the local coordinate system $\mathscr{J}(U, V)$ of $\mathscr{J}$ containing the fiber $\mathbb{P}hi$. In fact, with the usual notations,

$$h'(\tau, P) \ = \ h(\tau, P)$$

holds trivially true if $\tau = 0$ or $P \in \mathscr{C}$. Otherwise, by (36),

$$h'(\tau, P) \ = \ \varphi'_Q(\tau\, D'(P)) \ = \ \varphi'_Q(\tau\, \lambda_0\, D(P)) \ = \ \varphi_Q(\tau\, D(P)) \ = \ h(\tau, P)\, .$$

Therefore we can define fiberwise the map (37), and it is continuous.

The proof of Theorem 1.1 is now complete.

## 5. Some geometric property of $\mathscr{R}$

To understand $\mathscr{R}$ it is useful to first focus on the geometry of

$$\mathscr{C} \ := \ \Phi \cap \mathscr{R}$$

where, as usual, $\mathbb{P}hi$ is any fibre of the map $(s, b) : \mathscr{J}^1(\Gamma, X) \to \Gamma \times X$. In particular, we are interested in the dimension of $\mathscr{C}$, and in the structure of its singular locus. To this aim, it is easier to first study the affine variety $\mathscr{C}_{\mathbb{C}}$ defined in $\mathbb{C}^{2\, n^2}$ by the same equations than $\mathscr{C}$, namely

$$R \ = \ 0 \qquad I \ = \ 0 \tag{39}$$

because of (21). Then one can investigate the set of real points of $\mathscr{C}_{\mathbb{C}}$, which is in fact $\mathscr{C}$.

The geometry of $\mathscr{C}_{\mathbb{C}}$ becomes perfectly clear if we replace the equations (39) used to define it, by those we get from the following change of variables in the ring of polynomials $B := \mathbb{C}[\, p_{ij} \,|\, 1 \le i \le 2n \ , \ 1 \le j \le n \,]$. For every pair of integers $h, k$ such that $1 \le h, k \le n$, set

$$Z_{hk} := p_{hk} + i p_{h+n,k} \,, \quad W_{hk} := i p_{hk} + p_{h+n,k} = i(p_{hk} - i p_{h+n,k}) = i\,\overline{Z}_{hk} \,. \tag{40}$$

Under this change of variables $B$ becomes $\mathbb{C}[\, Z_{11}, \ldots, Z_{nn}, W_{11}, \ldots, W_{nn}]$. By (40) the *generic* $n \times n$ matrices

$$\mathscr{Z} := (\, Z_{ij} \,) \quad \text{and} \quad \mathscr{W} := (\, W_{ij} \,)$$

are related to the matrices $\mathscr{A}$, $\mathscr{B}$ introduced in (16) by the obvious relations

$$\mathscr{Z} \;=\; \mathscr{A} + i\,\mathscr{B} \quad \text{and} \quad -\,i\,\mathscr{W} \;=\; \mathscr{A} - i\,\mathscr{B}\,.$$

Hence by (21) ( possibly up to a constant factor $\neq 0$ for the second case )

$$\det(\, \mathscr{Z} \,) \;=\; \det(\, \mathscr{A} + i\mathscr{B} \,) \;=\; E \quad \text{and} \quad \det(\, \mathscr{W} \,) \;=\; \overline{E}\,.$$

The meaning of these relations is as follows. The change of variables (40) induces a change of coordinates

$$\omega : \mathbb{C}^{2\,n^2}_{p_{ij}} \longrightarrow \mathbb{C}^{2\,n^2}_{zw} \;. \tag{41}$$

Let $\omega(P) \;=\; \big((z), (w)\big)$. Then the coordinates $(p_{ij})$ of $P \in \mathbb{C}^{2\,n^2}$ satisfy the equation $E = 0$ if and only if

$$rk(\mathscr{Z}(z)) < n\,.$$

Therefore, if we set

$$\begin{aligned} Y \;&:=\; \big\{\, (z) \in \mathbb{C}^{n^2}_z \;\big|\; rk(\mathscr{Z}(z)) < n \,\big\} \\ Y' \;&:=\; \big\{\, (z) \in \mathbb{C}^{n^2}_w \;\big|\; rk(\mathscr{W}(w)) < n \,\big\} \end{aligned} \tag{42}$$

we can conclude that

$$\mathscr{C}_{\mathbb{C}} \;=\; Y \times Y'\,. \tag{43}$$

In fact, $D_{UV} = E \cdot \overline{E}$ because of (22). Moreover, if $P \in \mathscr{J}(U, V)$ annihilates $E$, i.e. if $E(P) = 0$, then we have also $\overline{E}(P) = 0$, and conversely.

Moreover, $Y$ and $Y'$ are generic determinantal varieties by (42), so that they are irreducible and reduced ( see e.g. [1], Ch. II, § § 2 and 3 ). Hence $\mathscr{C}_{\mathbb{C}}$ is also *irreducible and reduced, of dimension* $2n^2 - 2$ because $Y$, $Y'$ are both hypersurfaces of $\mathbb{C}^{n^2}$.

Finally, from (43) it is also easily seen that

$$Sing(\mathscr{C}_{\mathbb{C}}) = Sing(Y) \times Y' \cup Y \times Sing(Y'), \tag{44}$$

where ( see e.g. [1] )

$$Sing(Y) = \left\{ (z) \in \mathbb{C}_z^{n^2} \;\middle|\; rk(\mathscr{Z}(z)) < n-1 \right\} \tag{45}$$

and similarly for $Y'$. We can summarize all this as

THEOREM 5.1. *The variety* $\mathscr{C}_{\mathbb{C}}$ *is irreducible and reduced, of dimension* $2n^2-2$. *Its singular locus is given by (44), and has codimension* 2 *inside* $\mathscr{C}_{\mathbb{C}}$.

We are ready to start the study of the set $\mathscr{C}$ of real points of $\mathscr{C}_{\mathbb{C}}$. We will use (39) as equations for both $\mathscr{C}$ and $\mathscr{C}_{\mathbb{C}}$, inside $\mathbb{R}^{2n^2}$ and $\mathbb{C}^{2n^2}$ respectively. Then, the jacobian criterion yields

$$Sing(\mathscr{C}) = \mathscr{C} \cap Sing(\mathscr{C}_{\mathbb{C}}) \quad \text{or, equivalently} \quad \mathscr{C}_{\mathrm{sm}} = \mathscr{C} \cap \left(\mathscr{C}_{\mathbb{C}}\right)_{\mathrm{sm}}. \tag{46}$$

To get a better understanding of the above relations, and to exploit them, we have to be able to detect real points of $\mathscr{C}_{\mathbb{C}}$ when they are given in the coordinates $z, w$. For this, consider the following set-up, where $\gamma$ is the conjugation map, and $\omega$ was defined in (41)

$$\begin{array}{ccccccc} \mathbb{R}^{n^2} & \subseteq & \mathbb{C}^{n^2}_{p_{ij}} & \xrightarrow{\ \omega\ } & \mathbb{C}^{n^2}_{zw} & \supseteq & \mathscr{C}_{\mathbb{C}} \\ & & \downarrow{\scriptstyle \gamma} & & & & \\ \mathbb{R}^{n^2} & \subseteq & \mathbb{C}^{n^2}_{p_{ij}} & \xrightarrow[\ \omega\ ]{} & \mathbb{C}^{n^2}_{zw} & \supseteq & \mathscr{C}_{\mathbb{C}}. \end{array}$$

Then set

$$\delta := \omega \circ \gamma \circ \omega^{-1} : \mathbb{C}^{n^2}_{zw} \longrightarrow \mathbb{C}^{n^2}_{zw}.$$

It is clear that, for every $P \in \mathbb{C}^{n^2}_{p_{ij}}$, we have

$$P = \overline{P} \iff \delta\big(\omega(P)\big) = \omega(P). \tag{47}$$

It is easily checked that the map $\delta$ is given in coordinates by

$$\delta : (z_{11}, \dots, z_{nn}, w_{11}, \dots, w_{nn}) \mapsto (i\,\overline{w}_{11}, \dots, i\,\overline{w}_{nn}, i\,\overline{z}_{11}, \dots, i\,\overline{z}_{nn}). \tag{48}$$

This allows us to write condition (47) explicitly, namely a point $Q = \omega(P) = (z_{11}, \dots, z_{nn}, w_{11}, \dots, w_{nn})$ is such that $Q = \delta(Q)$ if and only if all the following conditions are satisfied

$$\begin{cases} z_{11} = i\,\overline{w}_{11} \\ \quad\vdots \\ z_{nn} = i\,\overline{w}_{nn}, \end{cases} \qquad \begin{cases} w_{11} = i\,\overline{z}_{11} \\ \quad\vdots \\ w_{nn} = i\,\overline{z}_{nn}. \end{cases} \tag{49}$$

Note that the conditions of one block are equivalent to those of the other block.

At this point we are able to describe explicitly the points of $\mathscr{C}$ by means of the map

$$u : Y \to \mathscr{C} \quad \text{given by} \quad (z) \mapsto (\,(z)\,|\,i\,(\overline{z})\,)\,. \tag{50}$$

In fact, by (49) the matrix $(\,(z)\,|\,i\,(\overline{z})\,)$ represents a real point of $\mathscr{C}_{\mathbb{C}}$, hence a point of $\mathscr{C}$. Notice that the restriction $p$ to $\mathscr{C}$ of the canonical projection $\mathscr{C}_{\mathbb{C}} = Y \times Y' \to Y$ is such that

$$p \circ u = id_Y\,. \tag{51}$$

Now, if $(z) \in Y_{\mathrm{sm}}$, i.e. by (42) and (45), if $rk(\mathscr{Z}(z)) = n-1$, then $u((z)) \in (\mathscr{C}_{\mathbb{C}})_{\mathrm{sm}}$. Hence $u((z)) \in \mathscr{C}_{\mathrm{sm}}$ because of (46).

On the other hand, if $P = (\,(z)\,|\,i\,(\overline{z})\,) \in \mathscr{C}_{\mathrm{sm}}$ then it is also a point of $(\mathscr{C}_{\mathbb{C}})_{\mathrm{sm}}$, hence $rk(\mathscr{Z}(z)) = n-1$ and $p(\,(\,(z)\,|\,i\,(\overline{z})\,)\,) \in Y_{\mathrm{sm}}$. By (51) the point $P$ of $\mathscr{C}_{\mathrm{sm}}$ then comes via $u$ from a smooth point of $Y$.

To summarize, we have constructed a real-analytic, bijective map

$$u : Y_{\mathrm{sm}} \to \mathscr{C}_{\mathrm{sm}}$$

with real-analytic inverse. Since $Y$ is an integral variety over $\mathbb{C}$, of dimension $n^2 - 1$, we can conclude

Proposition 5.2. *$\mathscr{C}_{\mathrm{sm}}$ is a real-analytic variety, of dimension $2\,(n^2-1)$.*

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