

# A global bifurcation result for a second order singular equation<sup>1</sup>

ANNA CAPIETTO, WALTER DAMBROSIO  
AND DUCCIO PAPINI

*Dedicated, with gratefulness and friendship, to Professor Fabio Zanolin  
on the occasion of his 60th birthday*

**ABSTRACT.** *We deal with a boundary value problem associated to a second order singular equation in the open interval  $(0, 1]$ . We first study the eigenvalue problem in the linear case and discuss the nodal properties of the eigenfunctions. We then give a global bifurcation result for nonlinear problems.*

**Keywords:** self-adjoint singular operator, spectrum, nodal properties, global bifurcation  
**MS Classification 2010:** 34C23, 34B09, 35P05

## 1. Introduction

We are concerned with a second order ODE of the form

$$-u'' + q(x)u = \lambda u + g(x, u)u, \quad \lambda \in \mathbb{R}, \quad x \in (0, 1], \quad (1)$$

where  $q \in C((0, 1])$  satisfies

$$\lim_{x \rightarrow 0^+} \frac{q(x)}{l/x^\alpha} = 1, \quad (2)$$

for some  $l > 0$  and  $\alpha \in (0, 5/4)$ , and  $g \in C([0, 1] \times \mathbb{R})$  is such that

$$\lim_{u \rightarrow 0} g(x, u) = 0, \quad \text{uniformly in } x \in (0, 1]. \quad (3)$$

The constant  $5/4$  arises in a rather straightforward manner in the study of the differential operator in the left-hand side of (1) (cf. [17, p. 287-288]); details are given in Remark 2.3 below.

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<sup>1</sup>Under the auspices of GNAMPA-I.N.d.A.M., Italy. The work has been performed in the frame of the M.I.U.R. Projects ‘Topological and Variational Methods in the Study of Nonlinear Phenomena’ and ‘Nonlinear Control: Geometrical Methods and Applications’.

We will look for solutions  $u$  of (1) such that  $u \in H_0^2(0, 1)$ .

When the  $x$ -variable belongs to a compact interval, problems of the form (1) have been very widely studied. A more limited number of contributions is available in the literature when the  $x$ -variable belongs to a (semi)-open interval, as it is the case in the present paper, or to an unbounded interval [7, 8].

We treat (1) in the framework of bifurcation theory. For this reason, we first discuss in Section 2 the eigenvalue problem

$$-u'' + q(x)u = \lambda u, \quad x \in (0, 1], \quad \lambda \in \mathbb{R}. \quad (4)$$

For such singular problems, the well-known embedding of (4) (by an elementary application of the integration by parts rule, together with the boundary condition  $u(0) = 0 = u(1)$ ) in the setting of eigenvalue problems for compact self-adjoint operators cannot be performed. Thus, the questions of the existence of eigenvalues and of the nodal properties of the associated eigenfunctions have various delicate features. For a comprehensive account on the spectral properties of the Schrödinger operator we refer to the books [12] and [10]; for more specific results on singular problems in  $(0, 1)$  we refer, among many others, to [5, 14].

However, the linear spectral theory for singular problems is well-established and can be found, among others, in the classical book by Coddington and Levinson [4] and in the (relatively) more recent text by Weidmann [17]. The former monograph focuses on a generalization of the so-called “expansion theorem” valid for functions in  $L^2([0, 1])$  and, by doing this, a sort of “generalized shooting method” is performed. On the other hand, in [17] the singular problem is tackled from an abstract point of view; more precisely, it is considered the general question of the existence of a self-adjoint realization of the formal differential expression  $\tau u = -u'' + q(x)u$  and the important Weyl alternative theorem [17, Theorem 5.6] is used. It is interesting to observe that the approach in [4] (based on more elementary ODE techniques) and the abstract one in [17] lead in different ways to the important concepts of “limit point case” and “limit circle case”. The knowledge of one (or the other) case is ensured by suitable assumptions on  $q$  and leads to information on the boundary conditions to be added to (4) in order to have a self-adjoint realization of  $\tau$ .

In the setting of the present paper, the operator  $\tau$  is regular at  $x = 1$ ; this implies that it is in the limit circle case. Moreover, under assumption (2), from [17, Theorem 6.4] it follows that  $\tau$  is in the limit circle case also in  $x = 0$ . Thus, the differential operator  $A : u \mapsto \tau u$  with

$$D(A) = \{u \in L^2(0, 1) : u, u' \in AC(0, 1), \tau u \in L^2(0, 1), \\ \lim_{x \rightarrow 0^+} (xu'(x) - u(x)) = 0 = u(1)\}$$

is a self-adjoint realization of  $\tau$  ([17, p. 287-288]). We prove in Proposition 2.2 that in fact  $D(A) = H_0^2(0, 1)$ ; to do this, we need some knowledge of the behaviour of the solutions of (4) near zero. These estimates are developed in Proposition 2.1 by means of the classical Levinson theorem [6, Theorem 1.8.1]. Finally, at the end of Section 2 we focus on the nodal properties of a solution to (4); more precisely, in Proposition 2.4 we prove that (4) is non-oscillatory and conclude in Proposition 2.5 that the spectrum of  $A$  is purely discrete and that, for every  $n \in \mathbb{N}$ , the eigenfunction associated to the eigenvalue  $\lambda_n$  has  $(n - 1)$  simple zeros in  $(0, 1)$ .

Section 3 contains a global bifurcation result (Theorem 3.2) which follows in a rather straightforward manner as an application of the celebrated Rabinowitz theorem in [11].

In order to exclude alternative (2) in Theorem 3.2, we use a technique that we already applied for Hamiltonian systems in  $\mathbb{R}^{2N}$  in [2] and for planar Dirac-type systems in [3]. More precisely, we introduce a continuous integer-valued functional defined on the set of solutions to (1). Due to the singularity at  $x = 0$ , some care is necessary in order to prove its continuity; this is the content of Proposition 3.4. We can then state and prove our main result (Theorem 3.5).

In what follows, for a given function  $p$  we write  $p(x) \sim \frac{m}{x^a}, x \rightarrow 0^+$ , when

$$\lim_{x \rightarrow 0^+} \frac{p(x)}{m/x^a} = 1 \tag{5}$$

for some  $m, a \in \mathbb{R}^+$ .

Finally, we write

$$H_0^2(0, 1) = \{u \in H^2(0, 1) : u(0) = 0 = u(1)\},$$

equipped with the norm defined by

$$\|u\|^2 = \|u\|_{L^2(0,1)}^2 + \|u''\|_{L^2(0,1)}^2, \quad \forall u \in H_0^2(0, 1).$$

## 2. The linear equation

In this section we study a linear second order equation of the form

$$-u'' + q(x)u = \lambda u, \quad x \in (0, 1], \quad \lambda \in \mathbb{R}. \tag{6}$$

We will assume that  $q \in C((0, 1])$  and that

$$q(x) \sim \frac{l}{x^\alpha}, \quad x \rightarrow 0^+, \tag{7}$$

for some  $l > 0$  and  $\alpha \in (0, 5/4)$ . Without loss of generality we may suppose that

$$q(x) > 0, \quad \forall x \in (0, 1]. \quad (8)$$

For every  $u : (0, 1] \rightarrow \mathbb{R}$  we denote by  $\tau u$  the formal expression

$$\tau u = -u'' + q(x)u;$$

First of all, we study the asymptotic behaviour of solutions of (6) when  $x \rightarrow 0^+$ ; to this aim, let us introduce the change of variables  $t = -\log x$  and let

$$w(t) = u(e^{-t}), \quad \forall t > 0.$$

From the relations

$$\begin{aligned} w'(t) &= -e^{-t}u'(e^{-t}) \\ w''(t) &= e^{-t}u'(e^{-t}) + e^{-2t}u''(e^{-t}), \end{aligned} \quad (9)$$

we deduce that  $u$  is a solution of (6) on  $(0, 1)$  if and only if  $w$  is a solution of

$$-w'' - w' + e^{-2t}q(e^{-t})w = \lambda e^{-2t}w \quad (10)$$

on  $(0, +\infty)$ . Equation (10) can be written in the form

$$Y' = (C + R(t))Y, \quad (11)$$

where  $Y = (w, z)^T$  and

$$C = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad R(t) = \begin{pmatrix} 0 & 0 \\ e^{-2t}q(e^{-t}) - \lambda e^{-2t} & 0 \end{pmatrix}, \quad \forall t > 0. \quad (12)$$

Now, let us observe that  $C$  has eigenvalues  $\lambda_1 = 0$ ,  $\lambda_2 = -1$  and corresponding eigenvectors  $u_1 = (1, 0)$ ,  $u_2 = (1, -1)$  and that  $R \in L^1(0, +\infty)$ ; therefore, an application of [6, Theorem 1.8.1] implies that (11) has two linearly independent solutions  $Y_1, Y_2$  such that

$$\begin{aligned} Y_1(t) &= u_1 + o(1), \quad t \rightarrow +\infty, \\ Y_2(t) &= (u_2 + o(1))e^{-t}, \quad t \rightarrow +\infty. \end{aligned} \quad (13)$$

As a consequence, we obtain the following result:

PROPOSITION 2.1. *For every  $\lambda \in \mathbb{R}$  the equation (6) has two linearly independent solutions  $u_{1,\lambda}$ ,  $u_{2,\lambda}$  such that*

$$u_{1,\lambda}(x) = 1 + o(1), \quad u'_{1,\lambda}(x) = o\left(\frac{1}{x}\right) \quad x \rightarrow 0^+, \quad (14)$$

$$u_{2,\lambda}(x) = x + o(x), \quad u'_{2,\lambda}(x) = 1 + o(1), \quad x \rightarrow 0^+,$$

and  $u_{2,\lambda} \in H^2(0, 1)$ .

For every  $f \in L^2(0, 1)$  the solutions of  $\tau u = f$  are given by

$$u(x) = c_1 u_{1,0}(x) + c_2 u_{2,0}(x) + u_f(x), \quad \forall x \in (0, 1), \quad c_1, c_2 \in \mathbb{R}, \quad (15)$$

where

$$u_f(x) = \int_0^x G(x, t) f(t) dt, \quad \forall x \in (0, 1), \quad (16)$$

$$G(x, t) = u_{1,0}(t)u_{2,0}(x) - u_{2,0}(t)u_{1,0}(x), \quad \forall x \in (0, 1), \quad t \in (0, 1)$$

fulfill  $G \in L^\infty((0, 1)^2)$ ,  $u_f(0) = 0 = u'_f(0)$  and  $u_f \in H^2(0, 1)$ .

*Proof.* The estimates in (14) follow from (9) and (13), while (16) is the usual variation of constants formula. Moreover, from (14) we obtain that  $u_{2,\lambda}, u'_{2,\lambda} \in L^2(0, 1)$ . On the other hand we have

$$q(x)u_{2,\lambda}(x) \sim x^{1-\alpha}, \quad x \rightarrow 0^+, \quad (17)$$

which implies that  $qu_{2,\lambda} \in L^2(0, 1)$ , since  $\alpha < 5/4$  (cf. Remark 2.3 for comments on this restriction); using the fact that  $\tau u_{2,\lambda} = \lambda u_{2,\lambda}$ , we deduce that

$$u''_{2,\lambda} = \lambda u_{2,\lambda} - qu_{2,\lambda} \in L^2(0, 1).$$

From now on, we will indicate  $u_i = u_{i,0}$ ,  $i = 1, 2$ . The fact that the function  $G$  defined in (16) belongs to the space  $L^\infty((0, 1)^2)$  is a consequence of the asymptotic estimates (14). Moreover, from (16) we also deduce that  $u_f(0) = 0$  and that

$$u'_f(x) = \int_0^x (u_1(t)u'_2(x) - u_2(t)u'_1(x))f(t) dt, \quad \forall x \in (0, 1), \quad (18)$$

which implies  $u'_f(0) = 0$ .

Finally, the condition  $u_f(0) = 0 = u'_f(0)$  guarantees that  $u_f, u'_f \in L^2(0, 1)$ ; as far as the second derivative of  $u_f$  is concerned, let us observe that we have

$$\tau u_f = f$$

and so

$$u''_f = f - qu_f. \quad (19)$$

Using the fact that  $u_f(0) = 0 = u'_f(0)$  and (7), it follows that  $qu_f \in L^2(0, 1)$ ; hence  $u_f \in H^2(0, 1)$ .  $\square$

In what follows, we study the spectral properties of suitable self-adjoint realizations of  $\tau$ ; to this aim, let us first observe that the differential operator  $\tau$  is regular at  $x = 1$ . As a consequence, it is in the limit circle case at  $x = 1$ ; moreover, from (7), according to [17, Theorem 6.4],  $\tau$  is in the limit circle case also in  $x = 0$ .

The differential operator  $A$  defined by

$$D(A) = \{u \in L^2(0, 1) : u, u' \in AC(0, 1), \tau u \in L^2(0, 1), \\ \lim_{x \rightarrow 0^+} (xu'(x) - u(x)) = 0 = u(1)\}$$

$$Au = \tau u, \quad \forall u \in D(A),$$

is then a self-adjoint realization of  $\tau$  ([17, p. 287-288]). We can show the validity of the following Proposition:

PROPOSITION 2.2. *The relation*

$$D(A) = H_0^2(0, 1)$$

*holds true. Moreover,  $A$  has a bounded inverse  $A^{-1} : L^2(0, 1) \rightarrow H_0^2(0, 1)$ .*

*Proof.* 1. Let us start proving that  $H_0^2(0, 1) \subset D(A)$ . It is well known that  $H_0^2(0, 1) \subset C^1(0, 1)$ ; hence, for every  $u \in H_0^2(0, 1)$  we have  $u, u' \in AC(0, 1)$ . Moreover, using the fact that  $u(0) = 0$  we deduce that

$$u(x) = u'(0)x + o(x), \quad x \rightarrow 0^+$$

and

$$q(x)u(x) = u'(0)x^{1-\alpha} + o(x^{1-\alpha}), \quad x \rightarrow 0^+;$$

the condition  $\alpha < 5/4$  guarantees again that  $qu \in L^2(0, 1)$  and therefore  $\tau u = -u'' + qu \in L^2(0, 1)$ . Finally, the regularity of  $u$  and  $u'$  imply that

$$\lim_{x \rightarrow 0^+} (xu'(x) - u(x)) = 0$$

and so also the boundary condition in the definition of  $D(A)$  is satisfied.

Now, let us prove that  $D(A) \subset H_0^2(0, 1)$ ; for every  $u \in D(A)$  let  $f = \tau u \in L^2(0, 1)$ . From (15) we deduce that  $u$  can be written as

$$u = c_1 u_1 + c_2 u_2 + u_f, \tag{20}$$

for some  $c_1, c_2 \in \mathbb{R}$ ; it is easy to see that the function  $u_1$  does not satisfy the boundary condition given in  $x = 0$  in the definition of  $D(A)$ , while  $u_2$  and  $u_f$  do. Hence  $u \in D(A)$  if and only if  $c_1 = 0$ ; the last statement of Proposition 2.1 implies then that  $u \in H^2(0, 1)$ . As in the first part of the proof, the regularity

of  $u$  allows to conclude that the boundary condition in  $x = 0$  given in  $D(A)$  reduces to  $u(0) = 0$ .

2. Let us study the invertibility of  $A$ ; the existence of a bounded inverse of  $A$  is equivalent to the fact that  $0 \in \rho_A$ , being  $\rho_A$  the resolvent of  $A$ . Since  $A$  is self-adjoint on  $H_0^2(0, 1)$ , this follows from the surjectivity of  $A$  (cf. [16, Theorem 5.24]); hence, it is sufficient to prove that  $A$  is surjective.

To this aim, let us first observe that condition (8) guarantees that 0 cannot be an eigenvalue of  $A$ . Now, let us fix  $f \in L^2(0, 1)$  and let us prove that there exists  $u \in H_0^2(0, 1)$  such that  $Au = f$ , i.e.  $\tau u = f$ ; by applying Proposition 2.1 we deduce again that (20) holds true and the same argument of the first part of the proof implies that  $c_1 = 0$ .

Hence we obtain  $u = c_2 u_2 + u_f$ ; from Proposition 2.1 we deduce that this function belongs to  $H^2(0, 1)$  and satisfies the boundary condition  $u(0) = 0$ . In order to prove that the missing condition  $u(1) = 0$  is fulfilled for every  $f \in L^2(0, 1)$ , let us observe that  $u_2(1) \neq 0$ , otherwise  $u_2$  would be an eigenfunction of  $A$  associated to the zero eigenvalue. Therefore,  $u(1) = 0$  is satisfied if

$$c_2 = -\frac{u_f(1)}{u_2(1)},$$

for every  $f \in L^2(0, 1)$ . □

REMARK 2.3. *As for the restriction  $\alpha < 5/4$ , we observe that for the proofs of Proposition 2.1 and Proposition 2.2 it is sufficient to require the milder condition  $\alpha < 3/2$ . The fact that  $\alpha < 5/4$  is used (cf. [17, p. 287-288]) in order to obtain that  $D(A)$  is the one described above. Finally, we observe that in the particular case when  $\alpha < 1$  the problem is regular (cf., among others, [9]).*

The spectral properties of  $A$  are related to the oscillatory behaviour of solutions of (6). We first recall the following definition:

DEFINITION 2.4. *The differential equation (6) is oscillatory if every solution  $u$  has infinitely many zeros in  $(0, 1)$ . It is non-oscillatory when it is not oscillatory.*

We observe that the regularity assumptions on  $q$  imply that solutions of (6) have a finite number of zeros in any interval of the form  $[a, 1)$ , for every  $0 < a < 1$ . Moreover, from (7) we infer that for every  $\lambda \in \mathbb{R}$  there exists  $c(\lambda) \in (0, 1]$  such that

$$\lambda - q(x) < 0, \quad \forall x \in (0, c(\lambda)).$$

An application of the Sturm comparison theorem proves that every solution of (6) has at most one zero in  $(0, c(\lambda))$ ; as a consequence, we obtain the following result:

PROPOSITION 2.5. *For every  $\lambda \in \mathbb{R}$  the differential equation (6) is non-oscillatory.*

Once Proposition 2.5 is obtained, we can provide in a straightforward way some useful information on the spectral properties of  $A$ ; more precisely, denoting by  $\sigma_{ess}$  the essential spectrum of a given operator, we have:

PROPOSITION 2.6. *([17, Theorem 14.3, Theorem 14.6 and Theorem 14.9], [12, Theorem XIII.1]) The differential operator  $A$  is bounded-below and satisfies*

$$\sigma_{ess}(A) = \emptyset.$$

Moreover, there exists a sequence  $\{\lambda_n\}_{n \in \mathbb{N}}$  of simple eigenvalues of  $A$  such that

$$\lim_{n \rightarrow +\infty} \lambda_n = +\infty$$

and for every  $n \in \mathbb{N}$  the eigenfunction  $u_n$  of  $A$  associated to the eigenvalue  $\lambda_n$  has  $(n - 1)$  simple zeros in  $(0, 1)$ .

REMARK 2.7. *According to [17], operators of the form  $\tau$  (defined on functions whose domain is  $(0, +\infty)$ ) arise when the time independent Schrödinger equation with spherically symmetric potential*

$$-\Delta u(x) + V(|x|)u(x) = \lambda u(x), \quad u \in L^2(\mathbb{R}^m) \quad (21)$$

is reduced to an infinite system of eigenvalue problems associated to the ordinary differential operators in  $L^2(0, +\infty)$

$$\tau_i = -\frac{d^2}{dr^2} + \frac{1}{r^2} \left[ i(i + m - 2) + \frac{1}{4}(m - 1)(m - 3) \right] + V(r)$$

( $i \in \mathbb{N}$ ). In Appendix 17.F of [17] it is treated the case of a potential  $V$  satisfying assumptions (which enable to consider Coulomb potentials) that lead to (7). More precisely, it is shown that for  $m = 3, i = 0$  the operator is in the limit circle case at zero and self-adjoint extensions of  $\tau_0$  are described.

### 3. The main result

In this section we are interested in proving a global bifurcation result for a nonlinear eigenvalue problem of the form

$$-u'' + q(x)u = \lambda u + g(x, u)u, \quad \lambda \in \mathbb{R}, \quad x \in (0, 1], \quad (22)$$

where  $q \in C((0, 1])$  satisfies (7) and  $g \in C([0, 1] \times \mathbb{R})$  is such that

$$\lim_{u \rightarrow 0} g(x, u) = 0, \quad \text{uniformly in } x \in [0, 1]. \quad (23)$$



We will look for solutions  $u$  of (22) such that  $u \in H_0^2(0, 1)$ . To this aim, let  $\Sigma$  denote the set of nontrivial solutions of (22) in  $H_0^2(0, 1) \times \mathbb{R}$  and let  $\Sigma' = \Sigma \cup \{(0, \lambda) \in H_0^2(0, 1) \times \mathbb{R} : \lambda \text{ is an eigenvalue of } A\}$ , where  $A$  is as in Section 2.

Let  $M$  denote the Nemitskii operator associated to  $g$ , given by

$$M(u)(x) = g(x, u(x))u(x), \quad \forall x \in [0, 1],$$

for every  $u \in H_0^2(0, 1)$ . We can show the validity of the following:

PROPOSITION 3.1. *Assume  $g \in C([0, 1] \times \mathbb{R})$  and (23). Then  $M : H_0^2(0, 1) \rightarrow L^2(0, 1)$  is a continuous map and satisfies*

$$M(u) = o(\|u\|), \quad u \rightarrow 0. \tag{24}$$

*Proof.* 1. We first show that  $Mu \in L^2(0, 1)$  when  $u \in H_0^2(0, 1)$ . When this condition holds,  $u \in L^\infty(0, 1)$  and the continuity of  $g$  implies that there exists  $C_u > 0$  such that

$$|g(x, u(x))u(x)| \leq C_u, \quad \forall x \in [0, 1].$$

As a consequence we obtain  $Mu \in L^\infty(0, 1) \subset L^2(0, 1)$ .

2. Let us prove that  $M$  is continuous. Let us fix  $u_0 \in X$  and let  $u_n \in X$  such that  $u_n \rightarrow u_0$  when  $n \rightarrow +\infty$ ; the continuous embedding

$$H_0^2(0, 1) \subset L^\infty(0, 1)$$

and the uniform continuity of  $g$  on compact subsets of  $[0, 1] \times \mathbb{R}$  ensure that

$$g(x, u_n(x)) \rightarrow g(x, u_0(x)) \quad \text{in } L^\infty(0, 1). \tag{25}$$

This is sufficient to conclude that  $Mu_n \rightarrow Mu_0$  in  $L^\infty(0, 1)$  and hence  $Mu_n \rightarrow Mu_0$  in  $L^2(0, 1)$ .

3. Finally, let us prove (24): using again the fact that  $H_0^2(0, 1) \subset L^\infty(0, 1)$ , we have

$$\|Mu\|_{L^2(0,1)} \leq \|g(x, u(x))\|_{L^\infty(0,1)} \|u\|_{L^2(0,1)} \leq \|g(x, u(x))\|_{L^\infty(0,1)} \|u\|,$$

for all  $u \in H_0^2(0, 1)$ ; hence, we deduce that

$$\frac{\|Mu\|_{L^2(0,1)}}{\|u\|} \leq \|g(x, u(x))\|_{L^\infty(0,1)}, \quad \forall u \in H_0^2(0, 1), \quad u \neq 0.$$

Therefore the result follows from (23) and (25). □

Now, let us observe that the search of solutions  $u \in H_0^2(0, 1)$  of (22) is equivalent to the search of solutions of the abstract equation

$$Au = \lambda u + M(u), \quad (u, \lambda) \in H_0^2(0, 1) \times \mathbb{R}; \tag{26}$$

on the other hand, (26) can be written in the form

$$w = \lambda R w + M(Rw), \quad (w, \lambda) \in L^2(0, 1) \times \mathbb{R}, \tag{27}$$

where  $R : L^2(0, 1) \rightarrow H_0^2(0, 1)$  is the inverse of  $A$  (cf. Proposition 2.2).

Now, from [17, Theorem 7.10] we deduce that  $R$  is compact; this fact and the continuity of  $M$  guarantee that the operator  $MR : L^2(0, 1) \rightarrow H_0^2(0, 1)$  is compact. Moreover, the condition

$$M(Rw) = o(\|w\|_{L^2(0,1)}), \quad w \rightarrow 0, \tag{28}$$

is a consequence of (24). From an application of the global bifurcation result of Rabinowitz (cfr. [11]) to (27) we then obtain the following result:

**THEOREM 3.2.** *Assume (7) and (23). Then, for every eigenvalue  $\lambda_n$  of  $A$  there exists a continuum  $C_n$  of nontrivial solutions of (22) in  $H_0^2(0, 1) \times \mathbb{R}$  bifurcating from  $(0, \lambda_n)$  and such that one of the following conditions holds true:*

- (1)  $C_n$  is unbounded in  $H_0^2(0, 1) \times \mathbb{R}$ ;
- (2)  $C_n$  contains  $(0, \lambda_{n'}) \in \Sigma'$ , with  $n' \neq n$ .

Now, let us observe that a more precise description of the bifurcating branch, eventually leading to exclude condition (2), can be obtained when there exists a continuous functional  $j : \Sigma' \rightarrow \mathbb{N}$  (cf. [2, Pr. 2.1]). In order to define such a functional, we will use the fact that nontrivial solutions of (22) have a finite number of zeros in  $(0, 1)$ ; this will be a consequence of our next result.

For every  $\lambda \in \mathbb{R}$  and for every nontrivial solution  $u \in H_0^2(0, 1)$  of (22) let us define  $q_{u,\lambda} : (0, 1] \rightarrow \mathbb{R}$  by  $q_{u,\lambda}(x) = q(x) - \lambda - g(x, u(x))$ , for every  $x \in (0, 1]$ . The following Lemma holds true:

**LEMMA 3.3.** *For every  $\lambda \in \mathbb{R}$  and for every nontrivial solution  $u \in H_0^2(0, 1)$  of (22) there exists a neighborhood  $U \subset H_0^2(0, 1) \times \mathbb{R}$  of  $(u, \lambda)$  and  $x_{u,\lambda} \in (0, 1)$  such that*

$$q_{v,\mu}(x) > 0, \quad \forall (v, \mu) \in U, \quad x \in (0, x_{u,\lambda}]. \tag{29}$$

*Proof.* Let  $(u, \lambda) \in H_0^2(0, 1) \times \mathbb{R}$ ,  $u \neq 0$ , be fixed and let  $U$  be the neighborhood of radius 1 of  $(u, \lambda)$  in  $H_0^2(0, 1) \times \mathbb{R}$ ; from the continuous embedding  $L^\infty(0, 1) \subset H_0^2(0, 1)$  we deduce that if  $(w, \mu) \in \Sigma \cap U_1$  then

$$\|w\|_{L^\infty(0,1)} \leq 1 + \|u\|_{L^\infty(0,1)}, \quad |\mu| \leq 1 + |\lambda|$$

and

$$q(x) - \mu - g(x, w(x)) \geq q(x) - |\lambda| - 1 - \max_{\substack{x \in [0,1], \\ |s| \leq 1 + \|u\|_{L^\infty(0,1)}}} |g(x, s)|, \quad \forall x \in (0, 1).$$

From (7) we then deduce that there exists  $x_{(u,\lambda)} \in (0, 1)$ , depending only on  $(u, \lambda)$ , such that

$$q(x) - \mu - g(x, w(x)) > 0, \quad \forall x \in (0, x_{(u,\lambda)}].$$

□

Now, let us observe that for every  $\lambda \in \mathbb{R}$  and for every nontrivial solution  $u \in H_0^2(0, 1)$  of (22) the function  $u$  is a nontrivial solution of the linear equation

$$-w'' + (q(x) - g(x, u(x)) - \lambda)w = 0. \tag{30}$$

From Lemma 3.3, with an argument similar to the one which led to Proposition 2.5, we deduce that all the nontrivial solutions of (30) (in particular  $u$ ) have a finite number of zeros in  $(0, 1)$ . We denote by  $n(u)$  this number.

We are then allowed to define the functional  $j$  by setting

$$j(u, \lambda) = \begin{cases} n(u) & \text{if } u \not\equiv 0 \\ n - 1 & \text{if } u \equiv 0 \text{ and } \lambda = \lambda_n, \end{cases} \tag{31}$$

for every  $(u, \lambda) \in \Sigma'$ . Let us observe that the definition  $j(0, \lambda_n) = n - 1$  is suggested by Proposition 2.6.

PROPOSITION 3.4. *The function  $j : \Sigma' \rightarrow \mathbb{N}$  is continuous.*

*Proof.* 1. As for the continuity of  $j$  in every point of the form  $(0, \lambda_n)$ ,  $n \in \mathbb{N}$ , we refer to [15, Lemma 2.5].

2. Let us now fix  $(u_0, \lambda_0) \in \Sigma$  and let  $(u, \lambda) \in U$ , with  $U$  as in Lemma 3.3; this Lemma guarantees that both  $u$  and  $u_0$  have no zeros in  $(0, x_{u_0, \lambda_0})$ .

On the other hand, in the interval  $[x_{u_0, \lambda_0}, 1]$  a standard continuous dependence argument (cf. also [11]) ensures that  $u$  and  $u_0$  have the same numbers of zeros if  $(u, \lambda)$  is in a sufficiently small neighborhood of  $(u_0, \lambda_0)$ . As a consequence, we obtain that there exists a neighborhood  $U_0$  of  $(u_0, \lambda_0)$  such that

$$j(u, \lambda) = j(u_0, \lambda_0), \quad \forall (u, \lambda) \in U_0.$$

□

As a consequence, from Theorem 3.2 and Proposition 3.4 we deduce the final result:

**THEOREM 3.5.** *Assume (7) and (23). Then, for every eigenvalue  $\lambda_n$  of  $A$  there exists a continuum  $C_n$  of nontrivial solutions of (22) in  $H_0^2(0, 1) \times \mathbb{R}$  bifurcating from  $(0, \lambda_n)$  and such that condition (1) of Theorem 3.2 holds true and*

$$j(u, \lambda) = n - 1, \quad \forall (u, \lambda) \in C_n. \quad (32)$$

**REMARK 3.6.** *Theorem 3.2 can be proved as an application of Stuart's result [15, Theorem 1.2] as well. However, since in the situation considered in this paper the singularity at zero does not affect the compactness of the operator  $R$  defined after (27), we chose to apply Rabinowitz theorem [11]. We finally mention the interesting paper [1], where global branches of solutions, with prescribed nodal properties, are obtained for a second order degenerate problem in  $(0, 1)$ .*

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Authors' addresses:

Anna Capietto  
Dipartimento di Matematica  
Università di Torino  
Via Carlo Alberto 10, 10123 Torino, Italy  
E-mail: [anna.capietto@unito.it](mailto:anna.capietto@unito.it)

Walter Dambrosio  
Dipartimento di Matematica  
Università di Torino  
Via Carlo Alberto 10, 10123 Torino, Italy  
E-mail: [walter.dambrosio@unito.it](mailto:walter.dambrosio@unito.it)

Duccio Papini  
Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche,  
Università di Siena  
Via Roma 56, 53100 Siena, Italy  
E-mail: [papini@dii.unisi.it](mailto:papini@dii.unisi.it)

Received May 28, 2012  
Revised September 3, 2012