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Linearizations, normalizations and isochrones of planar differential systems[1]

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Abstract. *In the first section we collect some unpublished results presented in [17], related to linearizations and normalizations of planar centers. In the second section we consider both the problem of finding isochrones of isochronous systems (centers or not) and its inverse, i.e. given a family of curves filling an open set, how to construct a system having such curves as isochrones. In particular, we show that for every family of curves $y = mx + d(x)$, $m \in \mathbb{R}$, there exists a Liénard system having such curves as isochrones.*

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1. Introduction

Let Ω be an open connected subset of the real plane. Let us consider a differential system

$$z' = V(z), \qquad z \equiv (x, y) \in \Omega, \tag{1}$$

$V(z) = (v_1(z), v_2(z)) \in C^\infty(\Omega, \mathbb{R}^2)$. We denote by $\phi_V(t, z)$, the local flow defined by (1). A connected subset $\mathcal{P} \subset \Omega$ covered with concentric non-trivial cycles is said to be a *period annulus*. If O is an isolated critical point of (1), we say that O is a *center* if it has a punctured neighbourhood which is a period annulus of Ω. The largest neighbourhood N_O of O such that $N_O \setminus \{O\}$ is a period annulus of Ω is said to be the its *central region*. On every period annulus one can define the *period function* $\tau(z)$, defined as the minimum positive period of the cycle starting at z. It can be proved that τ has the same regularity as the

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system. A period annulus is said to be *isochronous* if τ is constant. The study of τ, and in particular isochronicity, is related to boundary value problems and stability theory. In [1] several methods and results related to isochronicity theory were reviewed. One of the oldest ones is the linearization one, dating back to Poincaré. It consist in looking for a transformation that takes (1) into a linear system. Since every linear center is isochronous, if such a map exists, (1) has an isochronous center. Poincaré proved that if (1) is analytical and O is a non-degenerate critical point, then it admits a local linearization at O if and only if O is isochronous. Such a result is purely existential, giving no hints about how such a linearization could be obtained, in order to prove O actually to be isochronous. Linearizations of special classes of isochronous centers were found later by applying different techniques, as in [13].

A different method to prove isochronicity was introduced in [16, 21], based on the use of Lie brackets. Let us consider a second differential system

$$z' = W(z), \qquad z \equiv (x, y) \in \Omega, \tag{2}$$

$W(z) = (w_1(z), w_2(z)) \in C^\infty(\Omega, \mathbb{R}^2)$, $\phi_W(s, z)$ the local flow defined by (2). We say that (1) and (2) *commute*, or that V and W are *commutators*, if their Lie brackets $[V, W]$ vanish identically on Ω. A center is isochronous if and only if V it has a non-trivial (transversal at non-critical points) commutator W [16]. In several cases looking for a commutator turns out to be easier than looking for a linearization [1]. Also, as shown in [8], isochronicity is equivalent to the existence of a vector field W *normalized* by V, i.e. of a vector field W and a function μ such that $[V, W] = \mu W$. Every commutator is a normalizer, but the converse is not true, since the normalizing condition is expressed by one equality, the commutation condition by two.

Poincaré linearization theorem implies that an isochronous analytical center has a non-trivial commutator, since every linear center commutes with a transversal (at non-critical points) linear system. Conversely, if an analytical center has a non-trivial commutator, then it is isochronous, hence by Poincaré theorem it has an analytical linearization. The extension of such a relationship to non-analytical systems was studied in [22]. Procedures to get the linearization, starting form a given commutator, were studied in [5, 6, 9, 15], for several classes of analytical and non-analytical systems. In such papers it was always assumed the commutator W to have a non-degenerate critical point at O, usually having a linear part of star-node type. In the first section of this paper we present an approach, first presented in the unpublished preprint [17], where such an assumption is not required. The absence of a non-degeneracy assumption does not allow us to prove the existence of a linearizing diffeomorphism. In fact, we only prove the existence of a bijective linearizing map which fails to be a diffeomorphism at the critical point, where we lose the differentiability of the inverse map. In this section we also consider the existence of *normalizations*, i.e. maps that take (1) into a system of the form

$$\dot{u} = v\,\varphi(u^2 + v^2), \qquad \dot{v} = -u\,\varphi(u^2 + v^2).$$

Such a question was considered in [12].

In the second section we are concerned with the existence of *isochrones*, or *isochronous sections*, i.e. curves met by the local flow of (1) at equal time intervals. If O is an isochronous center, then every curve meeting its cycles exactly at a single point, even if not transversal, is an isochrone. The existence of isochrones becomes less obvious when dealing with cycles, isolated (limit cycles) or not, or with rotation points, or boundaries of attraction regions [19]. The existence of isochrones in a neighbourhood U_γ of a cycle γ, in relation to the existence of commutators or normalizers, was considered in [19, 20].

Following [2], we say that a point $z^* \in U_\gamma$ has *asymptotic phase* with respect to γ if there exists a point $z_* \in \gamma$ such that $\lim_{t\to+\infty} |\phi_V(t, z_*) - \phi_V(t, z^*)| = 0$, or $\lim_{t\to-\infty} |\phi_V(t, z_*) - \phi_V(t, z^*)| = 0$. In such a case, z^* is said to be *in phase* with z_*. In [2] a cycle is said to be isochronous if it has a neighbourhood U_γ such that every point of U_γ is in phase with some point of γ. A cycle is isochronous if and only if it has an isochrone, since the set of points in phase with a given $z_* \in \gamma$ is an isochrone, and vice-versa. Every hyperbolic limit cycle is isochronous, in such a sense [11]. Even non-hyperbolic limit cycles can be isochronous, under some additional conditions on the first return time map [2, 4]. The asymptotic phase approach cannot be extended to some other situations, as attraction boundaries, since if the boundary of the attraction region of an isochronous system is unbounded, then for every z in the boundary, $\phi_V(t, z)$ does not exist for all $t \in \mathbb{R}$.

If a system has an isochrone, then it has infinitely many ones, obtained from the given one by means of the local flow ϕ_V. If a cycle $\phi_V(t, z)$ is isochronous, such curves cover a neighbourhood of $\phi_V(t, z)$. If a critical point O is isochronous, then the system's isochrones cover a punctured neighbourhood of O. If a boundary is isochronous, then the system's isochrones cover a one-sided neighbourhood of such a boundary.

Given a family of curves covering an open set, one can consider an inverse problem, consisting in finding a differential system having such curves as isochrones. In the second section we describe an elementary approach to such a problem, with special regard to Liénard systems.

2. Linearizations and normalizations

Let Ω be an open connected subset of the real plane. We assume systems (1) and (2) to have the same, isolated critical points. We denote by $\phi_V(t, z)$, $\phi_W(s, z)$ the local flows of (1) and (2). If $I \in C^\infty(\Omega, \mathbb{R})$, we denote by $\partial_V I$, $\partial_W I$, the derivatives of I along the solutions of (1), (2), respectively. Similarly for $\partial_W I$ and for the derivative of a vector field along the solutions of (1) or (2). We write $[V, W] = \partial_V W - \partial_W V$, $A = V \wedge W = v_1 w_2 - v_2 w_1$. We say that W is a *non-trivial normalizer* of V if $A \neq 0$ at regular points and $V \wedge [V, W] = 0$. In this case, we define the function μ as follows,

$$\mu = \frac{V \wedge [V, W]}{|V|^2}.$$

If W is a normalizer of V, then the time-map $\phi_W(s, z)$ takes locally arcs of V-orbits into arcs of V-orbits. When both vector fields are non-trivial normalizers of each other we say that they are non-trivial *commutators*. By the transversality of V and W, this occurs when $[V, W] = 0$. In such a case, if $\phi_V(t, \phi_W(s, z))$ and $\phi_W(s, \phi_V(t, z))$ are defined for all $(s, t) \in J_s \times J_t$, J_s, J_t intervals containing 0, then one has the following commutativity property

$$\phi_V(t, \phi_W(s, z)) = \phi_W(s, \phi_V(t, z)).$$

We say that a function $I \in C^\infty(\Omega, \mathbb{R})$ is an *first integral* of (1), or V, if I is non-constant on any open subset of Ω, and $\partial_V I = 0$ in Ω. We say that a function $F \in C^\infty(\Omega, \mathbb{R})$ is an *integrating factor* of (1) if the divergence of the field FV vanishes in Ω. In such a case the differential form $\omega = -Fv_2 dx + Fv_1 dy$ is closed, and a potential exists on every simply connected subset of Ω. If FV does not vanish identically on any open subset of Ω, then such a potential is a first integral of (1). We say that a function $G \in C^\infty(\Omega, \mathbb{R})$, $G(z) \neq 0$ for all $z \in \Omega$, is an *inverse integrating factor* of (1) if $\frac{1}{G}$ is an integrating factor of (1).

If W is a normalizer of V, then $A = V \wedge W$ is an inverse integrating factor of V [7]. Similarly, if V is a normalizer of W, then $A = V \wedge W$ is an inverse integrating factor of W, so that, if V and W commute, then $A = V \wedge W$ is an inverse integrating factor both of V and W. Let us denote by $\mathcal{T}$ the set of points where V and W are transversal:

$$\mathcal{T} = \{z \in U : A(z) \neq 0\}.$$

For every $z \in \mathcal{T}$, we set $B(z) = \dfrac{1}{A(z)}$.

If W is a non-trivial normalizer of V, then for every point $z \in \mathcal{T}$ there exists a disk U_w^z and a function $S^z \in C^\infty(U_w^z, \mathbb{R})$, determined up to an additive constant κ_w^z, such that $\nabla S^z = B(-v_2, v_1)$. As a consequence, $\partial_V S^z = 0$. Similarly, if V is a non-trivial normalizer of W, then for every point $z \in \mathcal{T}$ there exists a disk U_w^z and a function $T^z \in C^\infty(U_w^z, \mathbb{R})$, determined up to an additive constant κ_w^z, such that $\nabla T^z = B(w_2, -w_1)$ and $\partial_W T^z = 0$.

If V and W commute, something more can be said, as in next lemma. We say that a map *rectifies* a vector field V if it takes (1) into a non-zero constant one. We say that a map *linearizes* a vector field V if it takes (1) into a linear one. We say that a map *normalizes* a vector field V if it takes (1) into a system of the following form

$$\dot{u} = v\,\varphi(u^2+v^2), \qquad \dot{v} = -u\,\varphi(u^2+v^2).$$

The orbits of such a system are contained in circles centered at O. If $\varphi(u^2+v^2) \neq 0$ on a given circle, then its minimal period is $\dfrac{1}{\varphi(u^2+v^2)}$. As a consequence, if such a system is defined in a neighbourhood of O, its period function is bounded only if $\varphi(u^2+v^2)$ does not approach 0. In the following we shall take into account also bijective C^∞ maps which fail to be diffeomorphisms just at a point.

For every point $z \in \Omega \cap \mathcal{T}$, let us set $U^z = U_v^z \cap U_w^z$. Then, for every point $z \in \Omega \cap \mathcal{T}$, we can define the map $\Gamma^z = (S^z, T^z) \in C^\infty(U^z, \mathbb{R}^2)$.

LEMMA 2.1. *Let V and W be non-trivial commutators. Then, for every choice of κ_v^z, κ_w^z, Γ^z is a local diffeomorphism that rectifies locally both (1) and (2). Moreover, for every $\zeta \in U^z$, $\zeta = \phi_V(t_\zeta, \phi_W(s_\zeta, z)) = \phi_W(s_\zeta, \phi_V(t_\zeta, z))$, one has:*

$$\phi_V(t,\zeta) = (\Gamma^z)^{-1}(t+t_\zeta, s_\zeta), \qquad \phi_W(s,\zeta) = (\Gamma^z)^{-1}(t_\zeta, s+s_\zeta). \tag{3}$$

Proof. The regularity of Γ^z comes from those of S^z, T^z. The map Γ^z has jacobian matrix:

$$J_{\Gamma^z} = \begin{pmatrix} -Bv_2 & Bv_1 \\ Bw_2 & -Bw_1 \end{pmatrix}$$

whose determinant is $-B$, that does not vanish on $\mathcal{T}$. Hence Γ^z is locally invertible on all of $\mathcal{T}$, that is at every regular point. As for the transformed systems, we know from what above that $\partial_V S^z = 0$, $\partial_W T^z = 0$. Moreover,

$$\begin{cases} \partial_V T^z & = Bw_2v_1 - Bw_1v_2 = BA = 1 \\ \partial_W S^z & = -Bv_2w_1 + Bv_1w_2 = BA = 1. \end{cases}$$

This shows that Γ rectifies locally both systems.

We prove only the first equality in (3), the second one can be proved similarly. We have: $\Gamma^z(\phi_V(t,\zeta)) = \Gamma^z(\phi_V(t, \phi_V(t_\zeta, \phi_W(s_\zeta, z)))) = \Gamma^z(\phi_V(t + t_\zeta, \phi_W(s_\zeta, z)))) = (t+t_\zeta, s_\zeta)$. By the local invertibility of Γ^z we get $\phi_V(t,\zeta) = \Gamma^{z\,-1}(t+t_\zeta, s_\zeta)$. □

LEMMA 2.2. *Let $\mathcal{P}$ is an open isochronous period annulus of (1). Then, for every vector field W such that $[V,W]=0$ on $\mathcal{P}$, there exists a map $\Lambda_W \in C^\infty(\mathcal{P}, \mathbb{R}^2)$ that linearizes both (1) and (2).*

Proof. Possibly multiplying V by $\dfrac{\tau(z)}{2\pi}$, we may assume the cycles of V to have minimal period 2π. Let us consider $z_0 \in \mathcal{P}$. The W-orbit $\phi_W(s, z_0)$ meets all the V-cycles in $\mathcal{P}$ exactly once. Let T^{z_0}, S^{z_0} be the maps of Lemma 2.1, defined in a suitable neighbourhood U_{z_0} of z_0. Let us choose the integration constants so that $T(z_0) = 0$, $S(z_0) = 0$. By Lemma 2.1, S^{z_0} and T^{z_0} coincide, respectively, with s and t of $\phi_W(s, z_0)$, $\phi_V(t, z_0)$. Hence S^{z_0} can be extended in a unique way to all of $\mathcal{P}$, by using the commutativity of the local flows ϕ_V and ϕ_W. Let us denote again by S^{z_0} and T^{z_0} the extended maps. The function T^{z_0} is not continuous at some point of every cycle, since $\phi_V(2\pi, z_0) = z_0$. Anyway, the functions $\cos T^{z_0}$, $\sin T^{z_0}$ are well-defined on all of $\mathcal{P}$. Their regularity comes from Lemma 2.1, since at every point they coincide, up to an additive constant, with some $\cos T^{z}$, $\sin T^{z}$.

Let us define Λ_W as follows,

$$\Lambda_W(z) = \left(e^{S^{z_0}(z)} \cos\left(T^{z_0}(z)\right), e^{S^{z_0}(z)} \sin\left(T^{z_0}(z)\right)\right) = (u, v).$$

Then Λ_W takes V-cycles into circles, and is one-to-one on cycles. This implies that Λ_W is one-to-one on all of $\mathcal{P}$.

Λ_W linearizes both (1) and (2). In fact, writing S and T for $S^{z_0}(z)$ and $T^{z_0}(z)$, one has

$$\begin{cases} \partial_V u & = e^S\ \partial_V S \cos T - e^S \sin T\ \partial_V T = -e^S \sin T = -v \\ \partial_V v & = e^S\ \partial_V S \sin T + e^S \cos T\ \partial_V T = e^S \cos T = u, \end{cases}$$

$$\begin{cases} \partial_W u & = e^S\ \partial_W S \cos T - e^S \sin T\ \partial_W T = e^S \cos T = u \\ \partial_W v & = e^S\ \partial_W S \sin T + e^S \cos T\ \partial_W T = e^S \sin T = v. \end{cases}$$

□

In next theorem we prove that starting from a commutator of (1) one can find a linearization, even without the non-degeneracy assumption on the commutator.

Theorem 2.3. *Let O be an isochronous center of (1), with central region N_O. Then, for every vector field W such that $[V, W] = 0$ on $N_O \setminus \{O\}$, there exists a map $\Lambda^0_W \in C^\infty(N_O, R)$ that linearizes (1).*

Proof. Let z_0 be a point of $\mathcal{P} = N_O \setminus O$, and Λ be defined as in Lemma 2.2. Possibily multiplying the vector field W by -1, in order to make its orbits tend to O as $s \to -\infty$, we may assume O to be asymptotically stable for (2). Let us define the map Λ^*_W as follows,

$$\Lambda^*_W(z) = \begin{cases} O & \text{if } \ z = O, \\ \Lambda_W(z) & \text{if } \ z \neq O. \end{cases}$$

Then $\Lambda_W^* \in C^0(N_O, \mathbb{R}) \cap C^\infty(\mathcal{P}, \mathbb{R})$. Working as in [14], thm 1.3, one can prove the existence of a first integral $I \in C^\infty(N_O, \mathbb{R})$, such that $\Lambda_W^0 = I\Lambda_W^* \in C^\infty(N_O, \mathbb{R})$. By Lemma 2.2, the map $w = \Lambda_W(z)$ transforms (1) into the linear system

$$\dot{u} = -v, \qquad \dot{v} = u.$$

Then, setting $\varepsilon = \Lambda_W^0(z) = I(z)\Lambda_W^*(z) = Iw$, one has

$$\dot{\varepsilon} = \dot{(Iw)} = \dot{I}w + I\dot{w} = IMw = M(Iw) = M\varepsilon,$$

hence Λ_W^0 linearizes (1). $\square$

The above theorem allows to prove the existence of a normalization for every system with a center at O.

Corollary 2.4. *Let O be a center of (1), with central region N_O. Then there exists a map $\Lambda_0 \in C^\infty(N_O, R)$ that normalizes (1).*

Proof. Let us consider the system

$$\dot{z} = \frac{\tau(z)}{2\pi} V(z). \tag{4}$$

Such a system is of class C^∞ in $\mathcal{P} = N_O \setminus \{O\}$, since $\tau \in C^\infty(\mathcal{P}, \mathbb{R})$. $\mathcal{P}$ is an isochronous annulus, with minimal period 2π. By Theorem 2.3, there exists a map $\Lambda_0 \in C^\infty(N_O, R)$ that linearizes (4), taking it into the system

$$\dot{u} = -v, \qquad \dot{v} = u.$$

As a consequence, system (1) is taken into the system

$$\dot{u} = -\frac{2\pi}{\tau(\Lambda_0(z))}\, v, \qquad \dot{v} = \frac{2\pi}{\tau(\Lambda_0(z))}\, u. \tag{5}$$

The function $\tau(z)$ is a first integral of (4), hence $\tau(\Lambda_0(z))$ is a first integral of (5). The orbits of (5) are circles centered at the origin, hence there exists a function $\beta \in C^\infty((0, +\infty), \mathbb{R})$ such that $\tau(\Lambda_0(z)) = \beta(u^2 + v^2)$. Then, setting

$$\varphi(u^2 + v^2) = -\frac{2\pi}{\beta(u^2 + v^2)}$$

satisfies the definition of normalized system. $\square$

We consider now the special case of hamiltonian systems

$$\dot{x} = H_y \qquad \dot{y} = -H_x, \tag{6}$$

where $H \in C^\infty(\Omega, {\rm I\!R})$. A map is said to be a *canonical transformation* if it transforms every hamiltonian system into a hamiltonian system. A diffeomorphism is a canonical transformation if and only if its jacobian determinant is a non-zero constant. The approach of Theorem 2.3 does not allow to get a canonical linearization on all of N_O, since the smoothing procedure affects the value of the jacobian determinant. On the other hand, one can characterize hamiltonian systems with commutators in terms of jacobian maps, i.e. maps wih constant non-vanishing jacobian determinant [17].

COROLLARY 2.5. *Let $H \in C^\infty(\Omega, R)$. Let z be a regular point of the hamiltonian system (6). Then (6) has a nontrivial commutator in a neighbourhood U^z of z if and only if there exist $P, Q \in C^\infty(U^z, {\rm I\!R})$ such that:*

i) the map $\Lambda(z) = (P(z), Q(z))$ has jacobian determinant $\equiv 1$ in U^z;

ii) $H = \frac{P^2+Q^2}{2}$.

If (6) has an isochronous period annulus $\mathcal{P}$, then Λ can be extended to all of $\mathcal{P}$, and is a canonical linearization of (6) on $\mathcal{P}$. If (6) has a non-isochronous period annulus $\mathcal{P}$, then such a Λ is a canonical normalization of (6) on $\mathcal{P}$.

Proof. Assume that $H = \frac{P^2+Q^2}{2}$, with $P_x Q_y - P_y Q_x \equiv 1$. Then the hamiltonian system (6) has the form

$$\begin{cases} \dot{x} &= PP_y + QQ_y \\ \dot{y} &= -PP_x - QQ_x. \end{cases} \tag{7}$$

and commutes with the system:

$$\begin{cases} \dot{x} &= -PQ_y + QP_y \\ \dot{y} &= PQ_x - QP_x. \end{cases} \tag{8}$$

Conversely, assume (6) to commute with (2). Let z be a non-critical point of (6). Then the function $A = H_y w_2 + H_x w_1$ is an inverse integrating factor for both (6) and (2). Hence there exist a neighbourhood U^z of z, and functions S and T, local first integrals of (6) and (2). In particular:

$$\nabla H = A \nabla S.$$

This implies that $A_x S_y + A S_{yx} = H_{yx} = H_{xy} = A_y S_x + A S_{xy}$, so that $A_y S_x - A_x S_y = 0$. Hence the level sets of A and S coincide, so that A is a first integral of (6), too. Since the gradient of S does not vanish, there exist two scalar functions h, a such that $H = h(S)$, $A = a(S)$. We have:

$$h'(S) \nabla S = \nabla H = a(S) \nabla S.$$

that gives $h' = a$. Now let us consider the map

$$\Lambda(\zeta) = (P(\zeta), Q(\zeta)) = (\sqrt{2h(S(\zeta))}\cos T(\zeta), \sqrt{2h(S(\zeta))}\sin T(\zeta)).$$

The jacobian determinant of Λ is identically 1:

$$\det \Lambda(\zeta) = \begin{vmatrix} \frac{h'(S)S_x}{\sqrt{2h(S)}}\cos T - \sqrt{2h(S)}T_x \sin T & \frac{h'(S)S_y}{\sqrt{2h(S)}}\cos T - \sqrt{2h(S)}T_y \sin T \\ \frac{h'(S)S_x}{\sqrt{2h(S)}}\sin T + \sqrt{2h(S)}T_x \cos T & \frac{h'(S)S_y}{\sqrt{2h(S)}}\sin T + \sqrt{2h(S)}T_y \cos T \end{vmatrix}$$

$$= h'(S)\left[S_xT_y - S_yT_x\right] = h'(S)\left[\frac{H_x}{A}\frac{w_1}{A} + \frac{H_y}{A}\frac{w_2}{A}\right]$$

$$= h'(S)\frac{H_xw_1 + H_yw_2}{a(S)^2} = h'(S)\frac{a(S)}{a(S)^2} = 1.$$

Moreover $P^2 + Q^2 = 2h(S) = 2H$, as required.

Now, let $\mathcal{P}$ be an isochronous period annulus. Without loss of generality, we may assume the period to be 2π. Working as in Lemma 2.2, one proves that Λ can be extended to all of $\mathcal{P}$, and that it linearizes (2).

If $\mathcal{P}$ is a non-isochronous period annulus, then working as in Corollary 2.4 one obtains a new system

$$\dot{x} = \frac{\tau H_y}{2\pi}, \qquad \dot{x} = -\frac{\tau H_x}{2\pi}, \tag{9}$$

which is itself a hamiltonian system, since $\dfrac{\tau(z)}{2\pi}$ is a first integral of (6). $\mathcal{P}$ is an isochronous period annulus of (9), hence there exists a canonical map Λ that linearizes (9) on $\mathcal{P}$. As in Corollary 2.4, such a linearization is a canonical normalization of (6) on $\mathcal{P}$. $\square$

A different, and more satisfactory approach to canonical linearizations for hamiltonian systems can be found in [12].

3. Isochrones

When dealing with centers the natural definition of isochronicity is given by requiring T to be constant. This is no longer possible when dealing with systems having non-periodic oscillations, as systems with foci. In such a case one can extend the isochronicity definition by considering *isochrones*, or *isochronous sections*, i.e. curves δ such that $\phi_V(T, \delta) \subset \delta$ for a fixed T, not necessarily positive. This in turn implies $\phi_V(nT, \delta) \subset \delta$, for every positive integer n. Usually such isochrones are taken transversal to V, but this is not necessary, in order to identify the existence of isochronous oscillations. Isochrones can exist in a neighbourhood of a rotation point, or a cycle, or a boundary (of a central or attraction region). In a neighbourhood of a semi-stable cycle one can consider $\phi_V(T,\delta) \subset \delta$ for $T > 0$ on one side of the cycle, $\phi_V(-T,\delta) \subset \delta$ on the opposite side. If a system (1) admits a linearization Λ, then the half-lines l_θ originating at O are isochrones of the linear system, hence the curves $\Lambda^{-1}(l_\theta)$ are isochrones of (1). The linearization method can be adapted to deal with non-periodic solutions, as in the case of foci [5]. On the other hand, it cannot be applied to the study of a limit cycle's isochrones, since linear systems do not have limit cycles. The same happens for attraction boundaries, since if a linear system has an asymptotically stable point, then it is globally attractive. A different approach can be based on normalizers, since if V is a normalizer of W, then the orbits of W are isochrones of V [8]. Looking for a normalizer is an effective way both to prove a system's isochronicity, and for attacking the inverse problem, i.e. to construct an isochronous system with a given family of curves as isochrones. In fact, one can consider two problems naturally related to isochrones:

- given a system with isochronous oscillations, find a family of isochrones covering a (punctured) neighbourhood, or a one-sided neighbourhood, of a point, or cycle, or boundary;
- given a family of curves covering an open set, find a system admitting such curves as isochrones.

A related question is that of constructing an isochronous system with some prescribed dynamic properties, as centers, foci, or limit cycles. All such problems are strictly related. We first show a simple procedure to construct new isochronous systems starting from a given one.

Lemma 3.1. *If V normalizes W on an open set U, then for every function $J \in C^\infty(U, \mathbb{R})$, and for every first integral of (2) $I^W \in C^\infty(U, \mathbb{R})$, the vector field $I^W V + JW$ normalizes W.*

Proof. Assume $[V, W] = \mu W$ on U. Then one has

$$[I^W V + JW, W] = \big(I^W \mu - \partial_W J\big) W.$$

□

If (2) is isochronous, passing from V to $I^W V + JW$ we can modify V's dynamics getting a new isochronous system with different properties. For instance we can pass form a center to a system with a focus and one or more limit cycles. In order to construct smooth vector fields, one has to consider only constant first integrals I^W. In fact, a non-constant first integral of (2) is not continuous at the critical point, since it assumes different values on different orbits. This is not an issue if one looks for an isochronous perturbation in a neighbourhood of a cycle, neglecting the effects of such a perturbation at the critical point located inside the cycle.

One can construct several examples, starting form any couple of commuting vector fields [1]. In order to get the desired dynamics, one has to choose the proper function J, which determines the attractive or repulsive effect of JW. Starting with a jacobian map $\Lambda(x, y) = (P(x, y), Q(x, y))$, we consider the hamiltonian systems (7) and (8) of the previous section. Then we perturb (7) choosing J as a function of H, so that the limit cycles of the new system, corresponding to the zeroes of J, are cycles of (7). For example, if H assumes the value 1 in the period annulus, we can take $J(x, y) = H(x, y)^2 - 1$, obtaining the system

$$\left\{ \begin{array}{ll} \dot{x} & = PP_y + QQ_y + (H^2 - 1)(-PQ_y + QP_y) \\ \dot{y} & = -PP_x - QQ_x + (H^2 - 1)(PQ_x - QP_x), \end{array} \right. \tag{10}$$

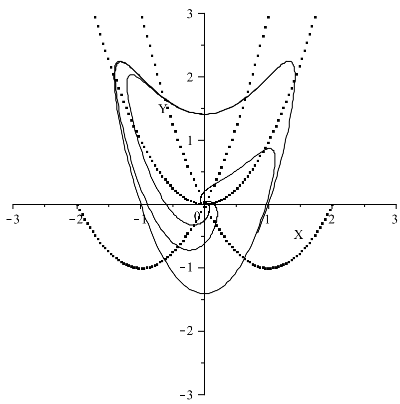
with a limit cycle coinciding with the level set $H = 1$.

If the jacobian map is $\Lambda(x, y) = (x, y - x^2)$, then system (10) has the form

$$\left\{ \begin{array}{ll} \dot{x} & = x + y - x^2 - \frac{xy^4}{4} - \frac{x^3y^2}{2} - \frac{x^5}{4} + x^3y^3 + x^5y - \frac{3y^2x^5}{2} - \frac{x^7}{2} + x^7y - \frac{x^9}{4} \\ \dot{y} & = -x + y + x^2 + 2xy - 2x^3 - \frac{x^2y^3}{2} - \frac{y^5}{4} + \frac{3x^2y^4}{4} - \frac{x^4y}{4} + \frac{x^4y^2}{2} - \frac{x^4y^3}{2} + \\ & \quad + \frac{x^4y^2}{2} - \frac{x^4y^3}{2} - \frac{x^6}{4} + \frac{x^6y}{2} - \frac{x^6y^2}{2} - \frac{x^8y}{2}x^8 + \frac{3x^8y}{4} - \frac{x^{10}}{4}. \end{array} \right. \tag{11}$$

Its isochrones are the curves $ax + b(y - x^2) = 0$, for $a, b \in \mathbb{R}$. In next figure we have plotted in continuous line some orbits of (11), and in dotted line the isochrones contained in the curves $y = -2x + x^2$, $y = x^2$, $y = 2x + x^2$. The system has a limit cycle contained in the level set $x^2 + (y - x^2)^2 = 1$.

Figure 1: The system (11)



By Poincaré's theorem, system (11) is linearizable at O, but its linearization is no longer Λ, which linearizes (7), but transforms (11) into the system

$$\begin{cases} \dot u &= v + u\Big(1 - H^2(\Lambda^{-1}(u,v))\Big) \\ \dot v &= -u + v\Big(1 - H^2(\Lambda^{-1}(u,v))\Big), \end{cases}$$

A normalizer can be also produced by means of a different procedure. In next statement we characterize normalizers in terms of first integrals. We do not know whether such a statement already appeared elsewhere.

THEOREM 3.2. *Let K be a first integral of (2) on an open set A. Assume W and ∇K not to vanish on A. Then V is a non-trivial normalizer of W if and only if for all $z^* \in A$ there exists a neighbourhood U^* and a function $\nu^* : U^* \to \mathbb{R}$, $\nu^* \neq 0$ such that*

$$\partial_V K = \nu^*(K).$$

Proof. Let V be a non-trivial normalizer of W. Let us choose arbitrarily a W-orbit γ^* and a point $z^* \in \gamma^*$. Every point z in a neighbourhood U^* of z^* can be written as $z = \phi_W(s, \phi_V(t, z^*))$. V is a normalizer, hence the parameter t depends only on the orbit to which z belongs. Hence the function that associates to a point $z \in U^*$ the value $t(z)$ of the parameter such that $z = \phi_V(t(z), \phi_W(s, z^*))$ is a first integral of (2). By construction, one has

$$\partial_V t(z) = 1.$$

The above formula also implies that ∇t does not vanish on A. Hence there exists a scalar function χ such that $K(z) = \chi(t(z))$, with $\chi'(t) \neq 0$ because both ∇t and ∇K do not vanish. Then

$$\partial_V K(z) = \chi'(t(z))\partial_V t(z) = \chi'(t(z)) = \chi'(\chi^{-1}(K(z))).$$

Then it is sufficient to set $\nu^*(K) = \chi'(\chi^{-1}(K))$.

Conversely, let us assume that there exists a scalar function ν^* such that $\partial_V K = \nu^*(K)$. Since ∇K does not vanish on A, locally K does not has the same value on different orbits, so that every arc of orbit in U^* can be identified as $K^{-1}(l) \cap U^*$, for some $l \in \mathbb{R}$. This establishes a one-to-one correspondence between the W-orbits of in U^* and the values of K. The relationship $\partial_V K = \nu^*(K)$ implies that $K(\phi_V(t,z))$ depends only on the initial value of K (in particular, it does not depend on the initial point z), hence the local flow $\phi_V(t,\cdot)$ takes arcs of orbits of (2) into arcs of orbits of (2), that is, V is a normalizer of W. $\square$

Theorem 3.2 allows to construct systems with prescribed isochrones without referring to any smooth linearization. In fact, the system we consider now do not necessarily admit linearizations, since they are not regular enough.

Corollary 3.3. *Assume that for every non-critical point z of (2) there exist a neighbourhood $U_z \subset \Omega$ and functions $K \in C^\infty(U_z, \mathbb{R})$, $\xi \in C^0(U_z, \mathbb{R})$, $\nu \in C^0(\mathbb{R}, \mathbb{R})$, such that in U_z one has $|\nabla K| \neq 0$ and*

$$W = \left(\frac{K_x}{|\nabla K|^2}\nu(K) + \xi K_y, \frac{K_y}{|\nabla K|^2}\nu(K) - \xi K_x \right). \tag{12}$$

Then (2) is an isochronous system, whose isochrones are locally defined by the level curves of K.

Proof. On every U_z, one has $\dot{K} = \nu(K)$, hence by Lemma 3.2, system (12) normalizes the hamiltonian system having K as hamiltonian function. Hence its isochrones are the orbits of such a hamiltonian system, i.e. K's level sets. □

Corollary 3.3 provides a tool for constructing systems with pre-assigned isochrones. In this case the system's attractors depend on the function ξ. We give some examples generating rational vector fields. Let us consider a one-to-one-map $\Lambda \in C^\infty(\Omega, \mathbb{R}^2)$, such that $\Lambda(0,0) = (0,0)$. Setting $\Lambda(x,y) = (P(x,y), Q(x,y))$, we may consider polar coordinates (ρ, θ) in the (P,Q)-plane. Let us consider a strictly increasing function η, and K locally defined as follows,

$$K(x,y) = \eta(\theta(P(x,y), Q(x,y))).$$

Such a function is defined only locally, since $\theta(P(x,y), Q(x,y))$ is not a single-valued function, but the corresponding system (12), for an arbitrary choice of ν and ξ, is well defined on all of $\Omega \setminus O$. It can be extended to all of Ω by adding the origin as a stationary point. The new vector field can be discontinuous at O, but the dynamics at regular points do not change. Adapting the usual terminology, we say that O is a center if it surrounded by non-trivial cycles, or a focus if every orbit in a neighbourhood of O spirals towards O or away from O. If it has a section, then it is isochronous. The isochrones are locally contained in K's level curves, which coincide with those of $\theta(P(x,y), Q(x,y))$, i.e. half-lines starting at the origin in the (P,Q)-plane, as for system (10):

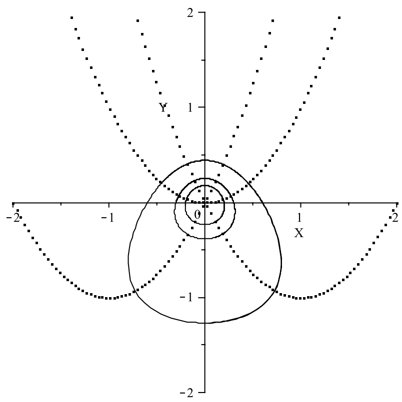
$$aP(x,y) + bQ(x,y) = 0, \qquad a, b \in \mathbb{R}.$$

If $\Lambda(x,y) = (x, y - x^2)$, $\eta(t) = t$, $\nu(t) = 1$, $\xi(x,y) = 0$, then O is a center of (12), since its orbits are symmetric with respect to the y-axis:

$$\dot{x} = -\frac{(y+x^2)(x^4 - 2yx^2 + y^2 + x^2)}{x^2 + y^2 + 2yx^2 + x^4}, \qquad \dot{y} = \frac{x(x^4 - 2yx^2 + y^2 + x^2)}{x^2 + y^2 + 2yx^2 + x^4}$$

Its isochrones are the parabolas $ax + b(y - x^2) = 0$. In Figure 2 we show three cycles and six isochrones contained in $y = -2x + x^2$, $y = x^2$, $y = 2x + x^2$. If

Figure 2: $\Lambda(x,y) = (x, y - x^2)$, $\eta(t) = t$, $\nu(t) = 1$, $\xi(x,y) = 0$.



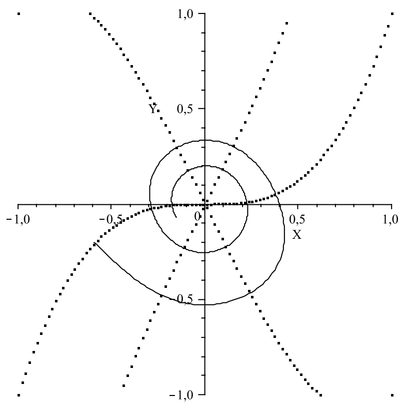
$\Lambda(x,y) = (x, y - x^3)$, $\eta(t) = t$, $\nu(t) = 1$, $\xi(x,y) = \frac{x^2+y^2-1}{500}$, then O is a focus of (12):

$$\dot{x} = -\frac{(y+x^2)(x^4 - 2yx^2 + y^2 + x^2)}{x^2 + y^2 + 2yx^2 + x^4} + \frac{x(x^2+y^2-1)}{500(x^2+y^2-2x^3y+x^6)},$$

$$\dot{y} = \frac{x(x^4 - 2yx^2 + y^2 + x^2)}{x^2 + y^2 + 2yx^2 + x^4} + \frac{(2x^3+y)(x^2+y^2-1)}{500(x^2+y^2-2x^3y+x^6)},$$

Its isochrones are the cubics $ax + b(y - x^3) = 0$. In Figure 3 we show a spiralling orbit and the isochrones contained in $y = -2x + x^3$, $y = x^3$, $y = 2x + x^3$. The

Figure 3: $\Lambda(x,y) = (x, y - x^3)$, $\eta(t) = t$, $\nu(t) = 1$, $\xi(x,y) = \frac{x^2+y^2-1}{500}$.



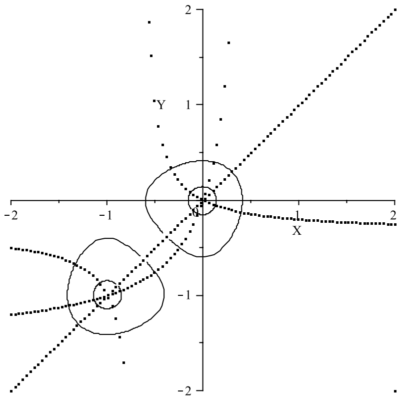
last two examples are constructed starting with globally invertible maps. This is not the case with next one, where we use the map $\Lambda(x,y) = (x+xy, y+xy)$, $\eta(t) = t$, $\nu(t) = 1$, $\xi(x,y) = 0$. Λ is only locally invertible at O, where we find a family of local isochrones defined by $a(x+xy) + b(y+xy) = 0$, $a, b \in \mathbb{R}$. Moreover, there exist other isochrones defined by the same equation, passing through the point $(-1,-1)$, where the system has another center.

$$\dot{x} = -\frac{y(1+y)(x^2+y^2+2xy^2+2x^2y+2x^2y^2)}{x^2+y^2+2x^3+2y^3+x^4+y^4},$$

$$\dot{y} = \frac{x(1+x)(x^2+y^2+2xy^2+2x^2y+2x^2y^2)}{x^2+y^2+2x^3+2y^3+x^4+y^4},$$

In Figure 4 we show both centers and the isochrones contained in the curves $x - y = 0$, $(x+xy) + 2(y+xy) = 0$, $-3(x+xy) + (y+xy) = 0$.

Figure 4: $\Lambda(x,y) = (x+xy, y+xy)$, $\eta(t) = t$, $\nu(t) = 1$, $\xi(x,y) = 0$.



The above procedure may not be the most efficient way to find a system with a given family of isochrones, in particular if one is looking for systems of a special form. In [18] some sufficient conditions for isochronicity of Liénard systems were given. In particular, it was proved that if

$$\sigma(x) = 2x^2 f(x) \int_0^x sf(s)ds - 4\left(\int_0^x sf(s)ds\right)^2 + x^3 g_n(x) - x^4 g_n'(x) \quad (13)$$

vanishes identically, then all the oscillations around the origin of the Liénard system

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x), \quad (14)$$

where $F'(x) = f(x)$, are isochronous. The paper [18] was concerned with centers, but its conclusions are valid for more general systems, since they are based on the properties a differential system equivalent to (14),

$$\dot{x} = y - xb(x), \qquad \dot{y} = -c(x) - yb(x), \tag{15}$$

under some additional conditions. The equivalence conditions of (14) and (15) are the following ones,

$$b(x) = \frac{\int_0^x sf(s)ds}{x^2} = \frac{I(x)}{x^2}, \qquad c(x) = g(x) - xb(x)^2.$$

Without loss of generality we may assume $g(x) = x + h.\ o.\ t.$. In this case the isochronicity condition (13) is equivalent to $c(x) = x$, so that (15) has the following form

$$\dot{x} = y - xb(x), \qquad \dot{y} = -x - yb(x). \tag{16}$$

Such a system has constant angular speed. If $b(x)$ is an odd function, then O is a center, hence an isochronous one. If $b(x)$ is not odd, the system can have a focus at O, with attraction (repulsion) region possibly bounded by a limit cycle or an unbounded orbit. Also, it is possible that several concentric limit cycles surround O. In all such cases, the half-lines starting at the origin are isochrones of (16). These allows to find isochrones for system (14), when (13) holds, since the transformation $(x, y) \mapsto (x, y + F(x) - xb(x))$ takes (15) into (14). Such a transformation is canonical, and its inverse is a canonical normalization of (14). In next theorem, we consider the converse statement. For a special class of curves filling an open region, we find a Liénard system having such curves as isochrones.

THEOREM 3.4. *For every function $d \in C^\infty(\mathcal{I}, \mathbb{R})$, $\mathcal{I}$ open interval containing* 0*, the Liénard system*

$$\dot{x} = y - (xd(x))', \qquad \dot{y} = -x(1 + d'(x)^2), \tag{17}$$

has the curves

$$y = mx + d(x), \qquad m \in \mathbb{R},$$

as isochrones.

Proof. The isochrones $ax + by = 0$ of (16) are taken into the curves $ax + b(y - F(x) + xb(x)) = 0$, so that the graphs of the functions

$$y = mx + F(x) - xb(x)$$

are isochrones of (14), under the condition (13). Imposing the equality $F(x) - xb(x) = d(x)$ leads to

$$d(x) = F(x) - xb(x) = F(x) - \frac{\int_0^x sf(s)ds}{x}.$$

Multiplying the first and last terms by x and differentiating, one has

$$F(x) = (xd(x))' = d(x) + xd'(x).$$

Substituting this expression into $d(x) = F(x) - xb(x)$ one obtains $b(x) = d'(x)$. In order to find an isochronous system having the curves $y = mx + d(x)$ as isochrones, we have to find $g(x)$ such that (13) holds. From [18] one has the isochronicity condition that relates $g(x)$ to $f(x)$. If $g'(0) = 1$, one has

$$g(x) = x + \frac{1}{x^3}\left(\int_0^x sf(s)ds\right)^2 = x + \frac{I(x)^2}{x^3}.$$

Since, from what above, $I(x) = x(F(x) - d(x))$, one has

$$\frac{I(x)^2}{x^3} = \frac{x^2(F(x) - d(x))^2}{x^3} = \frac{(xd'(x))^2}{x} = xd'(x)^2,$$

that gives

$$g(x) = x + xd'(x)^2.$$

□

System (17) is equivalent to the Liénard equation

$$\ddot{x} + (xd(x))''\dot{x} + x(1 + d'(x)^2) = 0.$$

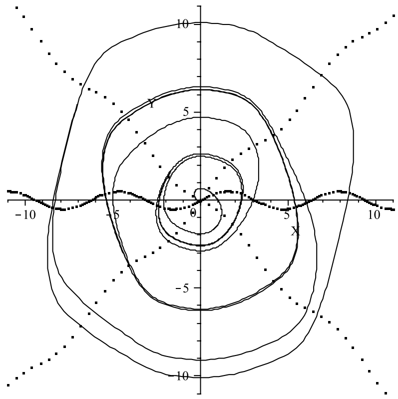
The function $d(x)$ determines the above system's dynamics. If $d(x)$ is even, then $F(x) = (xd(x))'$ is even, hence the origin is a center. If $d(x)$ is not even, then the origin is a focus.

In Figure 1 we have chosen $d(x) = \dfrac{\sin x}{2}$, and plotted the orbits of (14) as continuous lines. The dotted lines are the isochrones contained in $y = -x + \dfrac{\sin x}{2}$, $y = \dfrac{\sin x}{2}$, $y = x + \dfrac{\sin x}{2}$. The figure shows three limit cycles and six isochrones. Presumably the system has infinitely many limit cycles all meeting such isochrones.

After finding the explicit form of system (17), one can check that it normalizes a transversal system. By Lemma 3.2, it is sufficient to find two functions K and ν such that $\dot{K} = \nu(K)$. This implies that the hamiltonian system having K as hamiltonian is normalized by (17). Since the isochrones can be seen as the level sets of the function $H(x, y) = \dfrac{y - d(x)}{x}$, for $x \neq 0$, one can take $K(x, y) = \arctan\left(\dfrac{y - d(x)}{x}\right)$. The derivative of $H(x, y)$ along the solutions of (17) is

$$\dot{H} = -\frac{(y - d(x))^2 + x^2}{x^2} = -H^2 - 1,$$

Figure 5: $d(x) = \frac{\sin x}{2}$.



hence one has

$$\dot{K} = -1.$$

The hamiltonian system having K as hamiltonian function is

$$\dot{x} = \frac{x}{x^2 + (y - d(x))^2}, \qquad \dot{y} = \frac{y - d(x) + xd'(x)}{x^2 + (y - d(x))^2}.$$

Its orbits are the system's isochrones.

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