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Splitting the Fučík Spectrum and the Number of Solutions to a Quasilinear ODE[1]

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Abstract. *For ϕ an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and $f \in C(\mathbb{R})$, we consider the problem*

$$(\phi(u'))' + f(u) = 0, \quad t \in (0, L), \quad u(0) = 0 = u(L).$$

The aim is to study multiplicity of solutions by means of some generalized Pseudo Fučík spectrum (at infinity, or at zero). New insights that lead to a very precise counting of solutions are obtained by splitting these spectra into two parts, called Positive Pseudo Fučík Spectrum (PPFS) and Negative Pseudo Fučík Spectrum (NPFS) (at infinity, or at zero, respectively), in this form we can discuss separately the two cases $u'(0) > 0$ and $u'(0) < 0$.

Keywords: Fučík Spectrum, Quasilinear, p-Laplacian, Multiplicity.
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1. Introduction

In this paper we study the number of the solutions to the two-point (Dirichlet) boundary value problem

$$(P) \quad \begin{cases} (\phi(u'))' + f(u) = 0, \quad t \in (0, L), \\ u(0) = 0 = u(L), \end{cases}$$

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where ϕ is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and $f \in C(\mathbb{R})$.

In case that the differential operator in (P) is linear, i.e., when $\phi(s) = s$, or more generally when the differential operator is the one-dimensional p-Laplacian, i.e., when $\phi = \phi_p$ with

$$\phi_p(s) := |s|^{p-2}s, \quad s \neq 0, \quad \phi_p(0) = 0, \quad p > 1,$$

it is known that multiplicity results are obtained by assuming a suitable interaction of the nonlinearity with the Fučík spectrum of the corresponding operator. For instance, when the differential operator is linear, one is usually led to consider the limits of $\frac{f(s)}{s}$ as $s \to 0^{\pm}$ and as $s \to \pm\infty$. If these limits are all finite, say $a^{\pm}$ and $A^{\pm}$, respectively, then the number of the solutions for the boundary value problem depends on the existence of a suitable gap between the pairs (a^+, a^-) and (A^+, A^-). We recall that, in this situation, the Fučík spectrum is the set of all the pairs (μ, ν) such that the problem

$$\begin{cases} u'' + \mu u^+ - \nu u^- = 0, & t \in (0, L), \\ u(0) = 0 = u(L), \end{cases}$$

has nontrivial solutions. As it is well known this set is the union of the critical sets $\mathcal{C}_{i,j} = \{(\mu, \nu) \in (\mathbb{R}_0^+)^2 : \frac{i}{\sqrt{\mu}} + \frac{j}{\sqrt{\nu}} = \frac{L}{\pi}\}$ for i, j nonnegative integers with $|i - j| \leq 1$ (see [15]).

Here and henceforth the following notation is used: $\mathbb{R}_0^+ :=]0, +\infty[$ and $\mathbb{R}^+ := [0, +\infty[$, also $\mathbb{R}$ and $\mathbb{N}$ will denote the sets of real numbers and the set positive integers, respectively.

Similar results (see [2], [11]) have been developed for the p-Laplacian. For this case the Fučík spectrum is given by the union of the critical sets $\mathcal{C}_{i,j} = \{(\mu, \nu) \in (\mathbb{R}_0^+)^2 : \frac{i}{\mu^{1/p}} + \frac{j}{\nu^{1/p}} = \frac{L}{\pi_p}\}$ for i, j nonnegative integers with $|i - j| \leq 1$, where

$$\pi_p := 2(p-1)^{\frac{1}{p}} \int_0^1 \frac{ds}{(1 - s^p)^{\frac{1}{p}}} = 2(p-1)^{\frac{1}{p}} \frac{\pi}{p \sin(\pi/p)} \,,$$

(see [10]). As for the linear differential operator, this spectrum is the set of all the pairs (μ, ν) such that the problem

$$(F_{\mu,\nu}) \qquad \begin{cases} (\phi_p(u'))' + \mu\phi_p(u^+) - \nu\phi_p(u^-) = 0, & t \in (0, L), \\ u(0) = 0 = u(L), \end{cases}$$

has nontrivial solutions, see [13]. Note that, for any $p > 1$, if $u(\cdot)$ is a nontrivial solution of the above problem, then so is $\lambda u(\cdot)$ for any positive λ. The critical sets $\mathcal{C}_{i,j}$ intersect the diagonal of the (μ, ν)-plane exactly at the sequence of the eigenvalues $\Lambda_j = (j\pi_p/L)^p$ of the differential operator $u \mapsto -(\phi_p(u'))'$ with the associated Dirichlet boundary conditions.

In the rest of this paper the following main assumptions concerning the functions ϕ and f will be considered:

(ϕ_1) *$\phi : \mathbb{R} \to \mathbb{R}$ is an increasing (not necessarily odd) bijection with $\phi(0) = 0$. We also define $\Phi(s) = \int_0^s \phi(\xi)\, dx$;*

(f_1) *$f : \mathbb{R} \to \mathbb{R}$ is continuous with $f(0) = 0$, $f(s)s > 0$ for $s > 0$ and $F(s) \to +\infty$ as $s \to \pm\infty$, where $F(s) = \int_0^s f(\xi)\, d\xi$;*

(ϕ_0) $\limsup_{s\to 0^\pm} \dfrac{\phi(\sigma s)}{\phi(s)} < +\infty$ *and* $\liminf_{s\to 0^\pm} \dfrac{\phi(\sigma s)}{\phi(s)} > 1$; *for each $\sigma > 1$,*

(ϕ_∞) $\limsup_{s\to \pm\infty} \dfrac{\phi(\sigma s)}{\phi(s)} < +\infty$ *and* $\liminf_{s\to \pm\infty} \dfrac{\phi(\sigma s)}{\phi(s)} > 1$, *for each $\sigma > 1$.*

The conditions in (ϕ_∞) were previously introduced in [16], where they were called respectively the upper and lower σ-conditions at infinity.

In connection to (P) a natural generalization of problem $(F_{\mu,\nu})$ is given by the problem

$$(P_{\mu,\nu}) \qquad \begin{cases} (\phi(u'))' + \mu\phi(u^+) - \nu\phi(u^-) = 0 & t \in (0, L), \\ u(0) = 0 = u(L). \end{cases}$$

We note that in doing this we loose homogeneity, a property which is naturally present in the definition of the Fučík spectrum for the linear or p-Laplacian cases. Nevertheless with the idea in mind that what matters is to compare some asymptotic properties of the nonlinearity f with respect to ϕ, we proposed in [18] (for the case of ϕ odd) a definition for a critical set, by using a time-mapping approach, which we called the Pseudo Fučík Spectrum (PFS) for $(P_{\mu,\nu})$ and denoted by $\mathcal{S}(\subset (\mathbb{R}_0^+)^2)$.

Next we briefly review the construction of this Spectrum. Let us consider the equation

$$(\phi(u'))' + g(u) = 0, \tag{1}$$

and recall that under suitable growth assumptions on $g \in C(\mathbb{R}, \mathbb{R})$, we can define the time-mapping

$$\mathrm{T}_g(R) := 2 \left| \int_0^R \frac{ds}{\mathcal{L}_r^{-1}(G(R) - G(s))} \right| ,$$

where $G(s) = \int_0^s g(\xi)\, d\xi$ and $\mathcal{L}(s) = s\phi(s) - \int_0^s \phi(\xi)\, d\xi$. The functions $\mathcal{L}_r^{-1}$ and $\mathcal{L}_l^{-1}$, denote, respectively, the right and left inverses of $\mathcal{L}$. The number $\mathrm{T}_g(R)$

gives the distance between two consecutive zeros of a solution of (1) which attains a maximum $R > 0$ (respectively a minimum $R < 0$).

In [17], [18], for ϕ odd, we considered the problem

$$(P_\Lambda) \qquad \begin{cases} (\phi(u'))' + \Lambda\phi(u) = 0 & t \in (0, L), \\ u(0) = 0 = u(L). \end{cases}$$

Assuming that for all positive Λ the limit

$$\mathrm{T}(\Lambda) := \lim_{R \to \pm\infty} \mathrm{T}_{\Lambda\phi}(R), \tag{2}$$

exist and is a strictly decreasing functions of Λ, we considered those Λ such that $n\mathrm{T}(\Lambda) = L$ for some integer n, and called them pseudo eigenvalues for (P_Λ). With this at hand we defined the (PFS) for $(P_{\mu,\nu})$ as the set

$$\mathcal{S} = \{(\mu, \nu) \in (\mathbb{R}_0^+)^2 \mid i\mathrm{T}(\mu) + j\mathrm{T}(\nu) = L\},$$

for i, j nonnegative integers with $|i - j| \leq 1$, (a somewhat more explicit description of the (PFS) will be recalled in section 2).

We also mention that we proved in [18] that for any compact set $\mathcal{K} \subset \mathbb{R}^2 \setminus \mathcal{S}$ the set of all the possible solutions for $(P_{\mu,\nu})$, with $(\mu, \nu) \in \mathcal{K}$ is a priori bounded. Furthermore, as it is easy to see, the (PFS) $\mathcal{S}$ coincides with the standard Fučík Spectrum when $\phi = \phi_p$.

We note at this point that it is more appropriate to call the set $\mathcal{S}$ a Pseudo Fučík spectrum *at infinity.* Indeed, the set $\mathcal{S}$ does not take into account any information about solutions with small norm. In Section 2 we will consider a corresponding Pseudo Fučík spectrum *at zero.* In this context we recall [20] where pseudo eigenvalues at zero were defined.

This paper is organized as follows. In Section 2 we present our main results for multiplicity of solutions for problem (P), assuming some suitable behavior at zero and at infinity of the nonlinearities involved. Indeed setting

$$\lim_{s \to 0^\pm} \frac{f(s)}{\phi(s)} = a^\pm \qquad \lim_{s \to \pm\infty} \frac{f(s)}{\phi(s)} = A^\pm,$$

our results are based on some key lemmas relating the position of the limit pairs (A^+, A^-), (a^+, a^-), in the "classical" Fučík spectrum with respect to their position in a "universal" Fučík spectrum. This comparison is possible even if the points coincide (as considered in an example at the end of this section).

In order to treat in an independent manner solutions starting with positive slope, with those starting with a negative slope, we split the Fučík spectrum (at infinity, or at zero) into two parts, that we shall call Positive Pseudo Fučík Spectrum (PPFS) and Negative Pseudo Fučík Spectrum (NPFS) (at infinity, or at zero, respectively).

In Section 3 we give a result for the strict monotonicity of time-maps, which we use to obtain the exact number of solutions for some cases, but which may also be of some independent interest. Section 4 is devoted to some examples which illustrate our results. We end the paper in Section 5 by proving some technical lemmas, of comparison type, which are needed to obtain our results.

We finish this section with an illustrative example of some of the concepts we have introduced. Let ϕ be the map defined as the odd extension to $\mathbb{R}$ of

$$s \mapsto \begin{cases} \phi_q(s), & 0 \le s \le 1, \\ \phi_p(s), & s \ge 1, \end{cases}$$

where $p, q > 1$ and $p \neq q$. Then the (PFS) at infinity is exactly the one corresponding to the p-laplacian, while the (PFS) at zero is the one corresponding to the q-laplacian. Both spectra look alike but for some choices of p and q it becomes clear that both spectra which quite different in scale. Thus in Figure 1, for $q = 1.3$, $p = 6.5$, and $L = \pi$, we have plotted the Fučík spectrum at zero, using different colours for the curves of type $j = i-1$, $i = j-1$ and $j = i$. Note that the square $[0, 40]^2$ of the positive quadrant contains portions of eight critical sets of the type $\mathcal{C}_{i,i}$ (the darkest ones) as well as the same number of asymmetric curves of the class $\mathcal{C}_{i,i-1}$ and of the class $\mathcal{C}_{i-1,i}$, respectively.

In Figure 2, for the same values of p, q, and L, we have plotted the Fučík spectrum at infinity. Note that now only three curves ($\mathcal{C}_{1,0}$, $\mathcal{C}_{0,1}$ and $\mathcal{C}_{1,1}$) appear in the square $[0, 40]^2$ of the positive quadrant. In Figure 3, we have put together the two previous cases in order to stress the difference in scale.

Finally let us consider the interesting situation where

$$\lim_{s\to 0^+,\,+\infty} \frac{f(s)}{\phi(s)} = A^+, \qquad \lim_{s\to 0^-,\,-\infty} \frac{f(s)}{\phi(s)} = A^-,$$

i.e., a situation where the limits at 0^+ and $+\infty$ and also the limits at 0^- and $-\infty$ coincide. To make things precise, let us assume that $A^+ = 37$ and $A^- = 19$. Then the point (37,19) belongs to different non-critical regions (at zero and infinity) as it can be observed in Figures 1 and 2, respectively. Of course, this situation cannot occur for the linear or p-laplacian operator. Actually for this case there are at least 12 solutions starting with positive slope and at least 11 solutions starting with negative slope as we will see from Theorems 2.1 and 2.2 below.

2. Main results

We consider the two-point boundary value problem (P) which by convenience we recall next,

$$(P) \qquad \begin{cases} (\phi(u'))' + f(u) = 0, \\ u(0) = 0 = u(L). \end{cases}$$

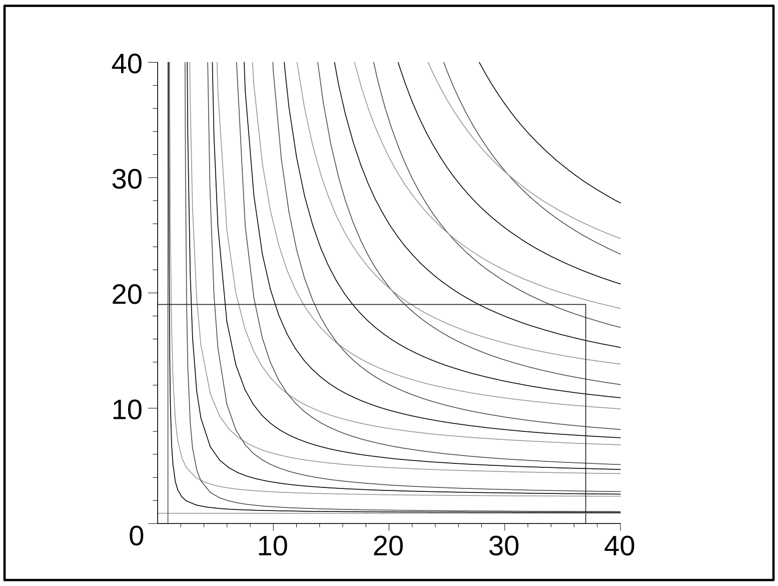

Figure 1: Fučík Spectrum for $q = 1.3$ and $L = \pi$ (PFS at zero).

We begin our analysis by assuming (ϕ_1) and (f_1) only. Under these assumptions we have that for each κ there is a unique solution $u = u(\cdot, \kappa)$ to the initial value problem

$$(\phi(u'))' + f(u) = 0, \quad u(0) = 0, \ u'(0) = \kappa. \tag{3}$$

This, indeed follows easily by writing equation (3) as an equivalent planar system and by using a result from [25]. Moreover, for any $\kappa \neq 0$, $u(\cdot, \kappa)$ is a nontrivial periodic solution. We will denote by $T(\kappa)$ its minimal period and by $\tau(\kappa)$ its first zero after $t = 0$.

Let us set $\Phi^*(s) = \int_0^s \phi^{-1}(\xi)\, d\xi$. Then $\mathcal{L}(s) = (\Phi^* \circ \phi)(s)$, where we recall $\mathcal{L}(s) = s\phi(s) - \Phi(s)$. As in [17], it is known that the following energy relation holds

$$\mathcal{L}(u'(t)) + F(u(t)) = \mathcal{L}(\kappa), \tag{4}$$

and it follows that $u'(\tau(\kappa)) = (\mathcal{L}_l)^{-1}(\mathcal{L}(\kappa))$ when $\kappa > 0$ and $u'(\tau(\kappa)) = (\mathcal{L}_r)^{-1}(\mathcal{L}(\kappa))$ if $\kappa < 0$, so that

$$T(\kappa) = \tau(\kappa) + \tau((\mathcal{L}_l)^{-1}(\mathcal{L}(\kappa))), \ \text{ for } \kappa > 0$$

and

$$T(\kappa) = \tau(\kappa) + \tau((\mathcal{L}_r)^{-1}(\mathcal{L}(\kappa))), \ \text{ for } \kappa < 0.$$

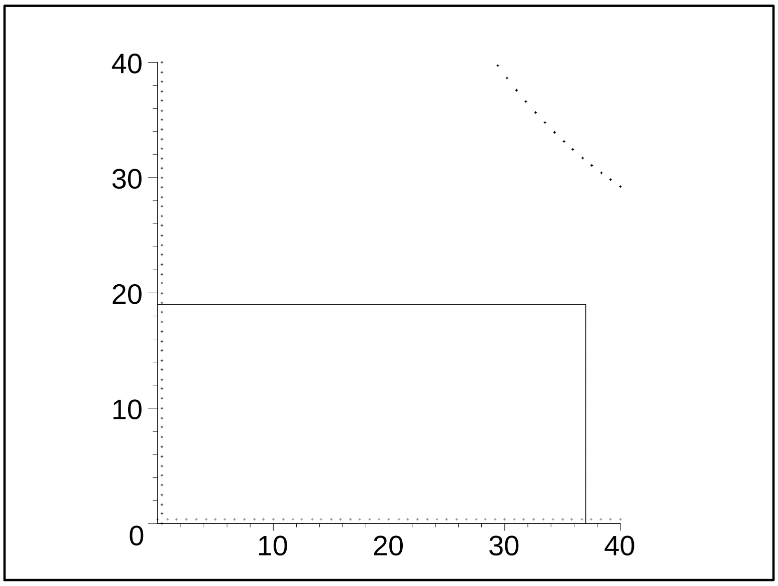

Figure 2: Fučík Spectrum for $p = 6.5$ and $L = \pi$ (PFS at infinity).

Conversely, if we fix an energy level $h > 0$, we have two solutions of (3) with $u(0) = 0$ and

$$\mathcal{L}(u'(t)) + F(u(t)) = h.$$

One of the two solutions corresponds to an initial positive slope $u'(0) = (\mathcal{L}_r)^{-1}(h)$ and the other to an initial negative slope $u'(0) = (\mathcal{L}_l)^{-1}(h)$. Clearly, any of the two solutions is a time-shift of the other. We denote by $L \cdot \mathcal{T}(h)$ (L is the length of the interval) the period of any of such solutions and also by $L \cdot x(h)$ and $L \cdot y(h)$ the distance of two consecutive zeros in an interval where the solution is respectively positive and negative.

The relationship among all the above definitions is the following:

$$x(h) + y(h) = \mathcal{T}(h) = \frac{T((\mathcal{L}_r)^{-1}(h))}{L} = \frac{T((\mathcal{L}_l)^{-1}(h))}{L},$$

and

$$x(h) = \frac{\tau((\mathcal{L}_r)^{-1}(h))}{L}, \quad y(h) = \frac{\tau((\mathcal{L}_l)^{-1}(h))}{L}.$$

By the fundamental theory of ODEs it follows that the maps $\kappa \mapsto T(\kappa)$ and $\kappa \mapsto \tau(\kappa)$, as well as $h \mapsto x(h)$ and $h \mapsto y(h)$ are continuous. Indeed, let us consider, for example, the map τ and let $\kappa_0 > 0$ be given. For any $\varepsilon > 0$, we can take $t_1, t_2 \in]0, T(\kappa_0)[$, with $\tau(\kappa_0) - \varepsilon < t_1 < \tau(\kappa_0) < t_2 < \tau(\kappa_0) + \varepsilon$, so that

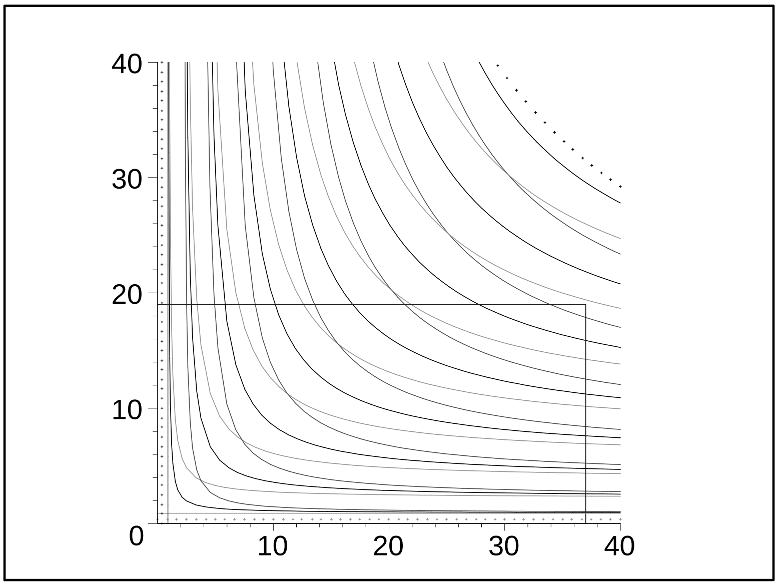

Figure 3: Overlapping the two previous Fučík Spectra.

$u(t_1, \kappa_0) > 0 > u(t_2, \kappa_0)$. Now, from the continuous dependence of the solutions on the initial data (which comes from the uniqueness of the solutions for the Cauchy problem) we have that there is $\delta > 0$ ($\delta < \kappa_0$), such that for each $\kappa \in]\kappa_0 - \delta, \kappa_0 + \delta[$ it holds that $u(t, \kappa) > 0$ for all $t \in [0, t_1]$ and $u(t_2, \kappa) < 0$. Hence, for the first zero of $u(\cdot, \kappa)$ in $]0, t_2]$, which is actually $\tau(\kappa)$, we have $\tau(\kappa) \in]t_1, t_2[\subset]\tau(\kappa_0) - \varepsilon, \tau(\kappa_0) + \varepsilon[$. The proof of the continuity for $\kappa_0 < 0$ follows the same argument and therefore we have also the continuity of $\kappa \mapsto T(\kappa)$.

In the next argument we will consider in detail the case $\kappa > 0$. Our first aim is to define a kind of "universal" Fučík spectrum in terms of the time-mappings $x(h)$ and $y(h)$. The use of universal critical sets to study boundary value problems for the equation $u'' + g(u) = 0$, was initiated in [6] and has been already developed in [4], [8] for the linear differential operator and in [9] for a Neumann problem containing the differential equation in (3). We also quote previous results which have were obtained in [3], [7], [19] and [22] for the superlinear case, and in [4], [9] for asymmetric and one-sided superlinear problems.

In this paper, as a difference with previous ones, we will "decompose" the pseudo spectrum (this will be also done for the classical Fučík spectrum) into two critical sets, corresponding respectively to solutions starting with positive and negative slope. This decomposition combined with shooting techniques will permit us to obtain precise information about the number of solutions.

Lemma 2.1. *Let ϕ and f satisfy (ϕ_1) and (f_1), respectively. Then, there exists $\kappa > 0$ such that problem (P) has a solution with $u'(0) = \kappa$ if and only if there are $n \in \mathbb{N}$ and $j \in \{0, 1\}$ such that*

$$nx(h) + (n + j - 1)y(h) = 1, \quad h = \mathcal{L}(\kappa). \tag{5}$$

Moreover, in this case, $u(\cdot)$ has exactly $2n + j - 2$ zeros in $]0, L[$ and there are n intervals in which $u > 0$ and $n + j - 1$ intervals where $u < 0$.

The proof of this lemma is straightforward and is left to the reader. Based on this result, we define the "critical lines"

$$H_i^+ = \{(x, y) \in (\mathbb{R}_0^+)^2 : \exists\, (n \in \mathbb{N}, j = 0, 1) : 2n+j-1 = i,\ nx+(n+j-1)y = 1\}.$$

Thus, H_1^+ is the half-line $x = 1$, with $y > 0$; H_2^+ is the open segment $x + y = 1$, with $x, y > 0$; H_3^+ is the open segment $2x + y = 1$, with $x, y > 0$, and so forth. The superscript "+" is to remember that these critical lines are tied up with solutions starting with positive slopes.

We will denote the set $H^+ = \cup_{i=1}^{\infty} H_i^+$ as the *Universal Positive Pseudo Fučík Spectrum* for problem $(P_{\mu,\nu})$.

Observe that if u is a solution of (3) with $u'(0) = \kappa > 0$ such that $(x(\mathcal{L}(\kappa)), y(\mathcal{L}(\kappa))) \in H_i^+$ then u is a solution of (P) having exactly $i - 1$ zeros in $]0, L[$.

Now we can divide the open first quadrant (x, y) into a countable number of open regions W_i^+ which form the complement in $(\mathbb{R}_0^+)^2$ of the critical set H^+. We label such regions as follows, see Figure 4:

- $W_1^+ = \{(x, y) : x > 1, y > 0\}$ is the part of the open first quadrant at the right hand side of H_1^+;
- $W_2^+ = \{(x, y) : 0 < x < 1, x+y > 1\}$ is the part of the open first quadrant between H_1^+ and H_2^+;
- ...
- W_i^+ is the part of the open first quadrant between H_{i-1}^+ and H_i^+, more precisely, $W_i^+ = \{(x, y) : x > 0, y > 0, kx + (k - 1)y < 1 < k(x + y)\}$ for $i = 2k$ and $W_i^+ = \{(x, y) : x > 0, y > 0, kx + ky < 1 < (k + 1)x + ky\}$ for $i = 2k + 1$.

Lemma 2.2. *Let ϕ and f satisfy (ϕ_1) and (f_1), respectively. Suppose that there are numbers $h_1, h_2 > 0$ and integers $k, \ell \geq 1$ with $k \neq \ell$ such that*

$$(x(h_1), y(h_1)) \in W_k^+, \quad (x(h_2), y(h_2)) \in W_\ell^+.$$

Then, problem (P) has at least $|k - \ell|$ solutions with $u'(0) > 0$.

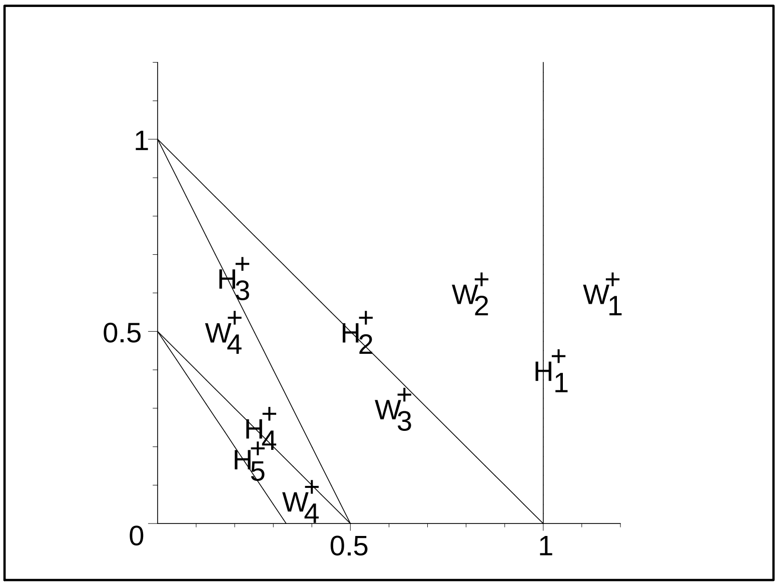

Figure 4: Universal Positive Pseudo Fučík Spectrum.

Proof. Just to fix a case, let us assume that $h_1 < h_2$ and $k < \ell$. By the assumptions, we have that the point $(x(h_1), y(h_1))$ is above H_k^+ and therefore it is above any of the H_i^+ for each $i \geq k$. On the other hand, $(x(h_2), y(h_2))$ is below $H_{\ell-1}^+$ and hence it is also below any of the H_i^+ for each $i \leq \ell - 1$.

Now, we fix an integer $i \in [k, \ell - 1]$ and observe that $(x(h_1), y(h_1))$ and $(x(h_2), y(h_2))$ belong to different open components of $(\mathbb{R}_0^+)^2 \setminus H_i^+$. Since the connected set $\{(x(h), y(h))\colon\ h_1 \leq h \leq h_2\}$ has points in both components of $(\mathbb{R}_0^+)^2 \setminus H_i^+$, it must intersect H_i^+ at least once. By Lemma 2.1, this means that there is solution of (P) with positive slope at $t = 0$ and exactly $i-1$ zeros in $]0, L[$. So, all together, there are at least $\ell - k$ solutions of (P) starting with a positive slope. ☐

Remark 2.1. *Actually, the solutions given by the lemma are such that*

$$\alpha = \min\{(\mathcal{L}_r)^{-1}(h_1), (\mathcal{L}_r)^{-1}(h_2)\} < u'(0) < \max\{(\mathcal{L}_r)^{-1}(h_1), (\mathcal{L}_r)^{-1}(h_2)\} = \beta.$$

If $\tau(\cdot)$ is strictly monotone in $]\alpha, \beta[$ and in $](\mathcal{L}_l)^{-1}((\mathcal{L}(\beta)), (\mathcal{L}_l)^{-1}((\mathcal{L}(\alpha))[$, then the number of the solutions is exactly $|k - \ell|$, for $u'(0) = \kappa$ ranging in $]\alpha, \beta[$ Indeed, if $\tau(\cdot)$ is strictly increasing (decreasing), also the maps $x(\cdot)$ and $y(\cdot)$ are strictly increasing (decreasing) with respect to h.

Our argument continues by introducing some further maps and properties that we need for the definition of the pseudo Fučík spectrum.

We deal, in first place, with the pseudo Fučík spectrum at infinity. Following [18], we assume conditions (ϕ_1) and (ϕ_∞) to hold and, moreover, that the limits

(T_∞) :

$$\mathrm{T}^{\pm}_{1,\infty}(\Lambda) = \lim_{R\to\pm\infty}\left|\int_0^R \frac{ds}{\mathcal{L}_r^{-1}(\Lambda\Phi(R)-\Lambda\Phi(s))}\right|$$

and

$$\mathrm{T}^{\pm}_{2,\infty}(\Lambda) = \lim_{R\to\pm\infty}\left|\int_0^R \frac{ds}{\mathcal{L}_l^{-1}(\Lambda\Phi(R)-\Lambda\Phi(s))}\right|$$

exist. Furthermore, we define

$$\mathrm{T}^{\pm}_{\infty}(\Lambda) = \mathrm{T}^{\pm}_{1,\infty}(\Lambda) + \mathrm{T}^{\pm}_{2,\infty}(\Lambda).$$

For each of the $\mathrm{T}^{\pm}_{\infty}$ *, we assume that either* $\mathrm{T}^{\pm}_{\infty}(\Lambda) = +\infty$ *for all* Λ*, or it is continuous and strictly decreasing with respect to* $\Lambda \in \mathbb{R}^+_0$ *. Moreover for this case, we assume that*

$$\lim_{\Lambda\to 0^+} \mathrm{T}^{\pm}_{\infty}(\Lambda) = +\infty \quad \textit{and} \quad \lim_{\Lambda\to+\infty} \mathrm{T}^{\pm}_{\infty}(\Lambda) = 0.$$

Conditions under which these hypotheses are fulfilled are given in [17], [18] and [20] for ϕ odd and T finite.

Next we define the *Positive Pseudo-Fučík Spectrum at infinity.* For $\mu, \nu > 0$ let us consider the problem $(P_{\mu,\nu})$ of the Introduction. Let us observe first that in [18], for the case of ϕ an odd function, the four numbers $\mathrm{T}^{\pm}_{i,\infty}(\Lambda)$ are all the same, finite and strictly positive, for any given Λ. Denoting this common value by $\mathrm{T}_\infty(\Lambda)$ (it corresponds to $\frac{\mathrm{T}(\Lambda)}{2}$ as defined in (2)), we can describe the (PFS) $\mathcal{S}$ (see the Introduction) as the union of the sets $\mathcal{C}_{i,i-1}$, $\mathcal{C}_{i-1,i}$, and $\mathcal{C}_{i,i}$, for $i \in \mathbb{N}$, contained in the positive (μ, ν)-quadrant, where

$$\begin{aligned}
\mathcal{C}_{i,i-1} &= \{(\mu,\nu) \mid i\mathrm{T}_\infty(\mu) + (i-1)\mathrm{T}_\infty(\nu) = L/2\}\\
C_{i-1,i} &= \{(\mu,\nu) \mid (i-1)\mathrm{T}_\infty(\mu) + i\mathrm{T}_\infty(\nu) = L/2\}\\
C_{i,i} &= \{(\mu,\nu) \mid i\mathrm{T}_\infty(\mu) + i\mathrm{T}_\infty(\nu) = L/2\}.
\end{aligned}$$

We want next to extend this definition to that of the Positive Pseudo-Fučík Spectrum (PPSF) at infinity, denoted by $\mathcal{S}^+(\infty)$, and where ϕ is not necessarily odd. We set

$$\mathcal{S}^+(\infty) = \cup_{i=1}^{\infty}\mathcal{C}^+_i(\infty),$$

where we consider only the case of solutions with positive (and large) slopes at $t = 0$. Here,

$$\begin{aligned}
\mathcal{C}^+_1(\infty) &= \{(\mu,\nu) \in (\mathbb{R}^+_0)^2 :\ \mathrm{T}^+_\infty(\mu) = L\},\\
\mathcal{C}^+_2(\infty) &= \{(\mu,\nu) \in (\mathbb{R}^+_0)^2 :\ \mathrm{T}^+_\infty(\mu) + \mathrm{T}^-_\infty(\nu) = L\},
\end{aligned}$$

and, in general for $i = j + k$, with $k = j - 1$ when i is odd, or $k = j$ when i is even, we have

$$\mathcal{C}_i^+(\infty) = \{(\mu, \nu) \in (\mathbb{R}_0^+)^2 : \ j\mathrm{T}_\infty^+(\mu) + k\mathrm{T}_\infty^-(\nu) = L\}.$$

We observe that the sets $\mathcal{C}_i^+(\infty)$ are all non-empty (and actually continuous curves) only in the case that both T_∞^+ and T_∞^- are finite (and hence continuous and strictly decreasing, as assumed above). In this situation the sets $\mathcal{C}_i^+(\infty)$ look similar with the corresponding $\mathcal{C}_{j,k}$ for the case ϕ odd, for $i = j + k$, with $k = j - 1$ when i is odd, or $k = j$ when i is even, see [18].

On the other hand, if T_∞^+ is finite but $\mathrm{T}_\infty^-(\Lambda) = +\infty$ for all Λ, then only $\mathcal{C}_1^+(\infty)$ is non-empty and if $\mathrm{T}_\infty^+(\Lambda) = +\infty$ for all Λ, then all the $\mathcal{C}_i^+(\infty)$'s are empty.

A qualitative description of the (PPFS) at infinity is given in the following table.

Curve	Vert. Asympt.	Horiz. Asympt.	Intersect. with Diagonal
$C_1^+(\infty)$	(Λ_1, ν)	$\emptyset$	(Λ_1, Λ_1)
$C_{2p}^+(\infty)$	(Λ_p, ν)	(μ, Λ_p)	$(\Lambda_{2p}, \Lambda_{2p})$
$C_{2p-1}^+(\infty)$	(Λ_p, ν)	(μ, Λ_{p-1})	$(\Lambda_{2p-1}, \Lambda_{2p-1})$

Arguing like in [18], we can prove the following result which corresponds to [18, Th.3.1].

LEMMA 2.3. *Let f satisfy (f_1) and ϕ satisfy (ϕ_1), (ϕ_∞) and (T_∞). Then, for any compact set $\mathcal{R}$ contained in $(\mathbb{R}^+)^2 \setminus \cup_{i=1}^\infty \mathcal{C}_i^+$ there is $\kappa_\mathcal{R}$ such that for each $(\mu, \nu) \in \mathcal{R}$ any solution of $(P_{\mu,\nu})$ with $0 < u'(0)$ satisfies $u'(0) \leq \kappa_\mathcal{R}$. The same result is true is we perturb the equation in $(P_{\mu,\nu})$ by adding at the right hand side a term $q(t, u)$ such that $q(t, s)/\phi(s) \to 0$ as $s \to \pm\infty$, uniformly with respect to $t \in [0, L]$.*

This result was proved in [18] for a situation corresponding to the case when T_∞^+ and T_∞^- are both finite. If T_∞^+ is finite and $\mathrm{T}_\infty^- = +\infty$, the proof is exactly the same, while if $\mathrm{T}_\infty^+ = +\infty$ we simply have a priori bounds for all the solutions starting with positive slope.

We now divide the open first quadrant (μ, ν) into a countable number of open regions $Z_i^+(\infty)$ which are the complement in $(\mathbb{R}_0^+)^2$ of the (PPFS) at infinity and label them as follows:

- $Z_1^+(\infty)$ is the part of the open first quadrant at the left hand side of $\mathcal{C}_1^+(\infty)$;
- $Z_2^+(\infty)$ is the part of the open first quadrant between $\mathcal{C}_1^+(\infty)$ and $\mathcal{C}_2^+(\infty)$;

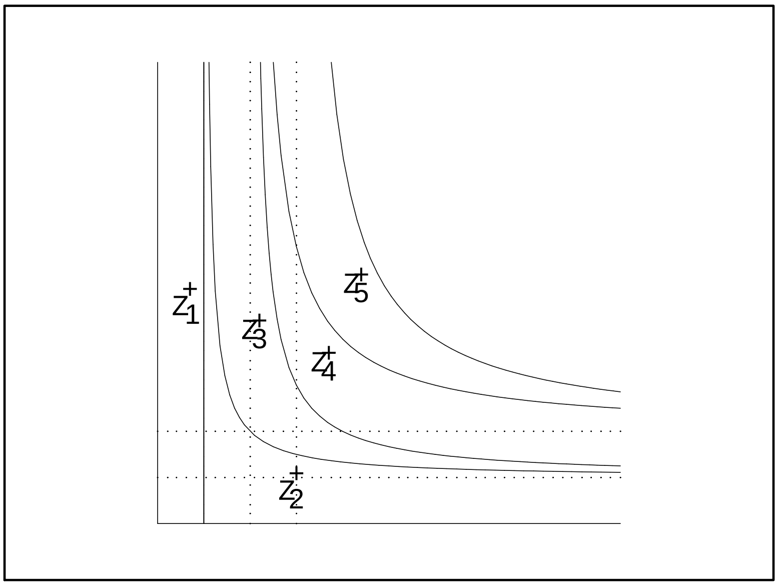

Figure 5: Positive Pseudo Fučík Spectrum at infinity.

- ...
- $Z_i^+(\infty)$ is the part of the open first quadrant between $\mathcal{C}_{i-1}^+(\infty)$ and $\mathcal{C}_i^+(\infty)$,

see Figure 5. In order to avoid a separated discussion for the cases when $\mathrm{T}_\infty^\pm$ are not finite, we point out that we have only two regions $Z_1^+(\infty)$ at the left of $\mathcal{C}_1^+(\infty)$ and $Z_2^+(\infty)$ at the right of $\mathcal{C}_1^+(\infty)$ when $\mathrm{T}_\infty^+ < +\infty$ and $\mathrm{T}_\infty^- = +\infty$. If $\mathrm{T}_\infty^+ = +\infty$ there is only one region $Z_1^+(\infty) = (\mathbb{R}_0^+)^2$.

In the special case $\phi = \phi_p$, this set is clearly a part of the standard Fučík spectrum for the one-dimensional p-Laplacian with Dirichlet boundary conditions on $]0, L[$.

Lemma 2.4. *Let f satisfy (f_1) and let ϕ satisfy (ϕ_1), (ϕ_∞) and (T_∞). Assume that*

$$\lim_{s\to+\infty} \frac{f(s)}{\phi(s)} = A^+, \quad \lim_{s\to-\infty} \frac{f(s)}{\phi(s)} = A^-,$$

with $(A^+, A^-) \in Z_i^+(\infty)$ for some index i. Then there is $\kappa^* > 0$ such that for each $\kappa \geq \kappa^*$ it follows that $(x(\mathcal{L}(\kappa)), y(\mathcal{L}(\kappa))) \in W_i^+$.

Proof. For simplicity, we only give the proof in the case that T_∞^+ and T_∞^- are both finite. Let $\kappa > 0$ and consider the solution $u(\cdot, \kappa)$ of (3) with $u(0) = 0$ and $u'(0) = \kappa$.

First of all, we claim that the distance of two consecutive zeros of u in an interval where $u > 0$ is given by $\mathrm{T}^+_\infty(A^+) + \varepsilon_1(\kappa)$ and the distance of two consecutive zeros of u for an interval where $u < 0$ is given by $\mathrm{T}^-_\infty(A^-) + \varepsilon_2(\kappa)$, where $\varepsilon_1(\kappa) \to 0$ and $\varepsilon_2(\kappa) \to 0$ as $\kappa \to +\infty$. Hence, we have that $x(\mathcal{L}(\kappa)) - L^{-1} \cdot \mathrm{T}^+_\infty(A^+) \to 0$ and $y(\mathcal{L}(\kappa)) - L^{-1} \cdot \mathrm{T}^-_\infty(A^-) \to 0$ as $\kappa \to +\infty$.

To see this, we observe that for each $\varepsilon > 0$ there is $M_\varepsilon > 0$ such that

$$(A^+ - \varepsilon)\phi(s) \le f(s) \le (A^+ + \varepsilon)\phi(s), \quad \forall\, s \ge M_\varepsilon$$

and

$$(A^+ - \varepsilon)|\phi(s)| \le |f(s)| \le (A^+ + \varepsilon)|\phi(s)|, \quad \forall\, s \le -M_\varepsilon\,.$$

Hence, by Corollary A.1 in the appendix, and since by the energy relation (4), $\max u$, $|\min u| \to +\infty$ as $\kappa \to +\infty$, we find that

$$\mathrm{T}^+_\infty(A^+ - \varepsilon) \le \liminf_{\kappa\to+\infty} \tau(\kappa) \le \limsup_{\kappa\to+\infty} \tau(\kappa) \le \mathrm{T}^+_\infty(A^+ + \varepsilon).$$

(Here the assumption (T_∞) which guarantees the existence of the limits $\mathrm{T}^+_\infty(A^+ \pm \varepsilon)$ has been used.) Then, from these inequalities and the continuity of the function T^+_∞, we immediately obtain that

$$\lim_{\kappa\to+\infty} \tau(\kappa) = \lim_{h\to+\infty} L \cdot x(h) = \mathrm{T}^+_\infty(A^+).$$

In a completely similar manner, one can see that

$$\lim_{\kappa\to+\infty} \tau(\mathcal{L}^{-1}(\kappa)) = \lim_{h\to+\infty} L \cdot y(h) = \mathrm{T}^-_\infty(A^-),$$

concluding the proof of our claim.

Suppose next that

$$(A^+, A^-) \in Z^+_i(\infty) \quad \text{for some } \ i \in \mathbb{N}.$$

Without loss of generality we assume also that $i > 1$ and that $i = 2k$ is an even number (the case $i = 1$ is simpler as it requires only a one-sided estimate). This means that (A^+, A^-) is above the curve

$$\mathcal{C}^+_{i-1}(\infty) = \{(\mu,\nu) \in (\mathbb{R}^+_0)^2 :\ k\mathrm{T}^+_\infty(\mu) + (k-1)\mathrm{T}^-_\infty(\nu) = L\}$$

and below the curve

$$\mathcal{C}^+_i(\infty) = \{(\mu,\nu) \in (\mathbb{R}^+_0)^2 :\ k\mathrm{T}^+_\infty(\mu) + k\mathrm{T}^-_\infty(\nu) = L\}$$

and hence,

$$L < k\mathrm{T}^+_\infty(A^+) + k\mathrm{T}^-_\infty(A^-), \quad k\mathrm{T}^+_\infty(A^+) + (k-1)\mathrm{T}^-_\infty(A^-) < L.$$

Now, using the estimates in [18] for the time-mappings recalled at the beginning of this proof, we find that

$$kx(\mathcal{L}(\kappa)) + ky(\mathcal{L}(\kappa)) \to k\frac{\mathrm{T}^{+}_{\infty}(A^{+})}{L} + k\frac{\mathrm{T}^{-}_{\infty}(A^{-})}{L} > 1$$

and

$$kx(\mathcal{L}(\kappa)) + (k-1)y(\mathcal{L}(\kappa)) \to k\frac{\mathrm{T}^{+}_{\infty}(A^{+})}{L} + (k-1)\frac{\mathrm{T}^{-}_{\infty}(A^{-})}{L} < 1$$

as $\kappa \to +\infty$. This, in turns, implies that there is $\kappa^* > 0$ such that for each $\kappa > \kappa^*$ the pair $(x(\mathcal{L}(\kappa)), y(\mathcal{L}(\kappa)))$ belongs to a compact subset of W_i^+.

The case i is an odd number is clearly treated in the same way and therefore it is omitted. □

By repeating the same reasoning at zero, and using the estimates in [20], we can define *the Positive Pseudo-Fučík Spectrum at zero* $\mathcal{S}^+(0)$ as the union of a countable number of critical curves $\mathcal{C}_i^+(0)$ obtained in a similar manner as for the critical sets of the (PPFS) at infinity, but this time, with the asymptotic estimates made at zero.

More precisely let us assume the conditions (ϕ_1) and (ϕ_0), and that the limits

(T_0) :

$$\mathrm{T}^{\pm}_{1,0}(\Lambda) = \lim_{R\to 0^{\pm}} \left| \int_0^R \frac{ds}{\mathcal{L}_r^{-1}(\Lambda\Phi(R) - \Lambda\Phi(s))} \right|$$

and

$$\mathrm{T}^{\pm}_{2,0}(\Lambda) = \lim_{R\to 0^{\pm}} \left| \int_0^R \frac{ds}{\mathcal{L}_l^{-1}(\Lambda\Phi(R) - \Lambda\Phi(s))} \right|$$

exist. Furthermore, define

$$\mathrm{T}^{\pm}_{0}(\Lambda) = \mathrm{T}^{\pm}_{1,0}(\Lambda) + \mathrm{T}^{\pm}_{2,0}(\Lambda),$$

and assume that for each of the $\mathrm{T}^{\pm}_{0}$ *, either* $\mathrm{T}^{\pm}_{0}(\Lambda) = +\infty$ *for all* Λ*, or it is continuous and strictly decreasing with respect to* $\Lambda \in \mathbb{R}^+_0$ *. Moreover we assume that,*

$$\lim_{\Lambda\to 0^+} \mathrm{T}^{\pm}_{0}(\Lambda) = +\infty \quad \textit{and} \quad \lim_{\Lambda\to +\infty} \mathrm{T}^{\pm}_{0}(\Lambda) = 0.$$

Similarly to Lemma 2.3, we can prove the following lemma.

LEMMA 2.5. *Let f satisfy (f_1) and let ϕ satisfy (ϕ_1), (ϕ_0) and (T_0). Then, for any compact set $\mathcal{K}$ contained in $(\mathbb{R}_0^+)^2 \setminus \cup_{i=1}^{\infty} \mathcal{C}_i^+(0)$ there is $\kappa_{\mathcal{R}}$ such that for each $(\mu, \nu) \in \mathcal{R}$ any solution of $(P_{\mu,\nu})$ with $0 < u'(0)$ satisfies $u'(0) \geq \kappa_{\mathcal{R}}$. The same result is true is we perturb the equation in $(P_{\mu,\nu})$ by adding a term $q(t, u)$ at the right hand side such that $q(t, s)/\phi(s) \to 0$ as $s \to 0$, uniformly with respect to $t \in [0, L]$.*

As before, we can define the corresponding regions in the complementary parts of the (PPFS) at zero. Accordingly, we denote by $Z_i^+(0)$ the components in the complement of the (PPFS) at zero.

We have the following result which is analogous to Lemma 2.4 and whose proof follows the same lines of that of Lemma 2.4, by using the comparison result of Lemma A.1.

LEMMA 2.6. *Let f satisfy (f_1) and let ϕ satisfy (ϕ_1), (ϕ_0) and (T_0). Assume that*

$$\lim_{s\to 0^+} \frac{f(s)}{\phi(s)} = a^+, \quad \lim_{s\to 0^-} \frac{f(s)}{\phi(s)} = a^-,$$

with $(a^+, a^-) \in Z_i^+(0)$ for some index i. Then there is $\kappa_* > 0$ such that for each $0 < \kappa \leq \kappa_*$, it follows that $(x(\mathcal{L}(\kappa)), y(\mathcal{L}(\kappa))) \in W_i^+$.

We are now in a position to state our first main result.

THEOREM 2.1. *Let f satisfy (f_1) and let ϕ satisfy (ϕ_1), (ϕ_0), (ϕ_∞) and (T_0), (T_∞). Suppose that there are positive numbers a^+, a^-, A^+, A^- such that*

$$\lim_{s\to 0^+} \frac{f(s)}{\phi(s)} = a^+, \quad \lim_{s\to 0^-} \frac{f(s)}{\phi(s)} = a^-$$

and

$$\lim_{s\to +\infty} \frac{f(s)}{\phi(s)} = A^+, \quad \lim_{s\to -\infty} \frac{f(s)}{\phi(s)} = A^-.$$

Assume also that there are $k, \ell \in \mathbb{N}$ with $k \neq \ell$ with $(a^+, a^-) \in Z_k^+(0)$ and $(A^+, A^-) \in Z_\ell^+(\infty)$. Then, problem (P) has at least $|k - \ell|$ solutions with $u'(0) > 0$.

Proof. The proof is a direct consequence of the Lemmas 2.4, 2.6, and 2.2, and by using the Positive Pseudo Fučík Spectrum (PPFS) at zero and at infinity. □

At this point we can establish corresponding results for solutions of (3) with $\kappa < 0$, by using a similar reasoning. To this end, with $x(h)$ and $y(h)$ as defined above, we need the following lemma which will take the place of Lemma 2.1.

Lemma 2.7. *Let ϕ and f satisfy (ϕ_1) and (f_1), respectively. Then, there exists $\kappa < 0$ such that problem (P) has a solution with $u'(0) = \kappa$ if and only if there are $n \in \mathbb{N}$ and $j \in \{0, 1\}$ such that*

$$(n + j - 1)x(h) + ny(h) = 1, \quad h = \mathcal{L}(\kappa). \tag{6}$$

Moreover, in this case, $u(\cdot)$ has exactly $2n + j - 2$ zeros in $]0, 1[$ and also there are n intervals in which $u < 0$ and $n + j - 1$ intervals where $u > 0$.

From this result we can define, as before, the critical lines:

$$H_i^- = \{(x, y) \in (\mathbb{R}_0^+)^2 : \exists\, (n \in \mathbb{N}, j = 0, 1) : 2n+j-1 = i,\ (n+j-1)x+ny = 1\}.$$

Thus H_1^- is the half-line $y = 1$, with $x > 0$; H_2^- is the open segment $x + y = 1$, with $x, y > 0$; H_3^- is the open segment $x + 2y = 1$, with $x, y > 0$, and so forth. The superscript "-" is to remember that these critical lines are related with solutions starting with negative slopes.

We will denote the set $H^- = \cup_{i=1}^{\infty} H_i^-$ as the *Universal Negative Pseudo Fučík Spectrum*.

As before, we can divide the open first quadrant (x, y) into a countable number of open regions W_i^- which are the complement in $(\mathbb{R}_0^+)^2$ of the critical set H^-. We label such zones as follows, see Figure 6:

- $W_1^- = \{(x, y) : x > 0, y > 1\}$ is the part of the open first quadrant above H_1^-;
- $W_2^- = \{(x, y) : x+y > 1, 0 < y < 1\}$ is the part of the open first quadrant between H_1^- and H_2^-;
- ...
- W_i^- is the part of the open first quadrant between H_{i-1}^- and H_i^- and, more precisely, we have $W_i^- = \{(x, y) : x > 0, y > 0, (k - 1)x + ky < 1 < k(x + y)\}$ for $i = 2k$ and $W_i^- = \{(x, y) : x > 0, y > 0, kx + ky < 1 < kx + (k + 1)y\}$ for $i = 2k + 1$.

Remark 2.2. *The set $H = H^+ \cup H^-$ has the same shape like the set $\mathcal{F}$ drawn in [4, p.874]. It represents a "universal" model of Fučík spectrum for the two-point boundary value problem in terms of time-maps. What we have done here is to distinguish the parts H^+ and H^- in H in order to treat separately the solutions with positive slope and those with negative slope. The same procedure is feasible for the standard Fučík spectrum $\mathcal{C}$ which splits as a "positive" part (concerning solutions with positive slope at $t = 0$) and a "negative" part (for the solutions with negative slope at $t = 0$).*

The following lemma is the equivalent to Lemma 2.2, it is proved similarly.

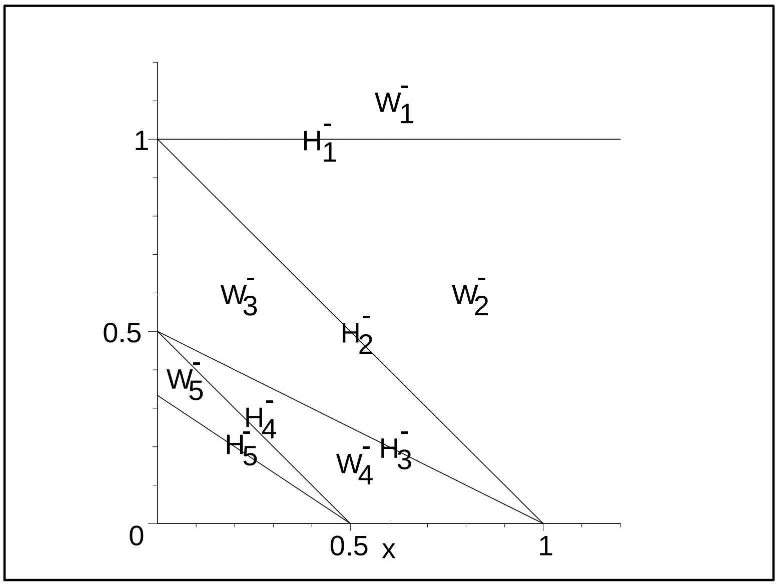

Figure 6: Universal Negative Pseudo Fučík Spectrum.

LEMMA 2.8. *Let ϕ and f satisfy (ϕ_1) and (f_1), respectively. Suppose that there are numbers $h_1, h_2 > 0$ and integers $k, \ell \geq 1$ with $k \neq \ell$ such that*

$$(x(h_1), y(h_1)) \in W_k^- , \quad (x(h_2), y(h_2)) \in W_\ell^- .$$

Then, problem (P) has at least $|k-\ell|$ solutions with $u'(0) < 0$.

REMARK 2.3. *In a similar way as in Remark 2.1, we find that these solutions are such that*

$$\alpha = \min\{(\mathcal{L}_l)^{-1}(h_1), (\mathcal{L}_l)^{-1}(h_2)\} < u'(0) < \max\{(\mathcal{L}_l)^{-1}(h_1), (\mathcal{L}_l)^{-1}(h_2)\} = \beta.$$

The number of solutions is exactly $|k-\ell|$, for $u'(0) = \kappa$ ranging in $]\alpha, \beta[$ if $\tau(\cdot)$ is strictly monotone in $]\alpha, \beta[$ and in $](\mathcal{L}_r)^{-1}((\mathcal{L}(\beta)), (\mathcal{L}_r)^{-1}((\mathcal{L}(\alpha))[$.

Next we define `the Negative Pseudo-Fučík Spectrum at infinity`. We set

$$\mathcal{S}^-(\infty) = \cup_{i=1}^{\infty} \mathcal{C}_i^-(\infty),$$

where we only consider solutions with negative (and large in absolute value) slopes at $t = 0$. Here,

$$\mathcal{C}_1^-(\infty) = \{(\mu, \nu) \in (\mathbb{R}_0^+)^2 : \ \mathrm{T}_\infty^-(\mu) = L\},$$
$$\mathcal{C}_2^-(\infty) = \{(\mu, \nu) \in (\mathbb{R}_0^+)^2 : \ \mathrm{T}_\infty^+(\mu) + \mathrm{T}_\infty^-(\nu) = L\},$$

and, in general, for $i = k + l$ with $k = l - 1$ (when i is odd) or $k = l$ (when i is even), we have

$$\mathcal{C}_i^-(\infty) = \{(\mu, \nu) \in (\mathbb{R}_0^+)^2 : \; k\mathrm{T}_\infty^+(\mu) + l\mathrm{T}_\infty^-(\nu) = L\}.$$

By this definition, $\mathcal{C}_i^-(\infty) = \mathcal{C}_i^+(\infty)$, when i is even.

As before the sets $\mathcal{C}_i^-(\infty)$ are all non-empty (and actually continuous curves) only in the case that both T_∞^+ and T_∞^- are finite (and hence continuous and decreasing by assumption). On the other hand, if T_∞^- is finite but $\mathrm{T}_\infty^+(\Lambda) = +\infty$ for all Λ, then only $\mathcal{C}_1^-(\infty)$ is non-empty and if $\mathrm{T}_\infty^-(\Lambda) = +\infty$ for all Λ, then all the $\mathcal{C}_i^-(\infty)$'s are empty.

A qualitative description of the (NPFS) at infinity is given in the following table.

Curve	Vert. Asympt.	Horiz. Asympt.	Intersect. with Diagonal
$C_1^-(\infty)$	(μ, Λ_1)	$\emptyset$	(Λ_1, Λ_1)
$C_{2p}^-(\infty)$	(Λ_p, ν)	(μ, Λ_p)	$(\Lambda_{2p}, \Lambda_{2p})$
$C_{2p-1}^-(\infty)$	(Λ_{p-1}, ν)	(μ, Λ_p)	$(\Lambda_{2p-1}, \Lambda_{2p-1})$

Corresponding to Lemma 2.3, we now have.

LEMMA 2.9. *Let f satisfy (f_1) and let ϕ satisfy (ϕ_1), (ϕ_∞) and (T_∞). Then, for any compact set $\mathcal{R}$ contained in $(\mathbb{R}^+)^2 \setminus \cup_{i=1}^\infty \mathcal{C}_i^-$ there is $\kappa_\mathcal{R}$ such that for each $(\mu, \nu) \in \mathcal{R}$ any solution of $(P_{\mu,\nu})$ with $u'(0) < 0$ satisfies $|u'(0)| \le \kappa_\mathcal{R}$. The same result is true is we perturb the equation in $(P_{\mu,\nu})$ by adding at the right hand side a term $q(t, u)$ such that $q(t, s)/\phi(s) \to 0$ as $s \to \pm\infty$, uniformly with respect to $t \in [0, L]$.*

We split next the open first quadrant (μ, ν) into a countable number of open regions $Z_i^-(\infty)$, which are the complement in $(\mathbb{R}_0^+)^2$ of the (NPFS) at infinity, see Figure 7, and labeled as follows:

- $Z_1^-(\infty)$ is the part of the open first quadrant below $\mathcal{C}_1^-(\infty)$;
- $Z_2^-(\infty)$ is the part of the open first quadrant between $\mathcal{C}_1^-(\infty)$ and $\mathcal{C}_2^-(\infty)$;
- ...
- $Z_i^-(\infty)$ is the part of the open first quadrant between $\mathcal{C}_{i-1}^-(\infty)$ and $\mathcal{C}_i^-(\infty)$.

Also as before and in order to avoid to discuss separately the cases when the $\mathrm{T}_\infty^\pm$ are not finite, we point out that there are only two regions $Z_1^-(\infty)$ below $\mathcal{C}_1^-(\infty)$ and $Z_2^-(\infty)$ above $\mathcal{C}_1^-(\infty)$ when $\mathrm{T}_\infty^- < +\infty$ and $\mathrm{T}_\infty^+ = +\infty$. If $\mathrm{T}_\infty^- = +\infty$ there is only one region $Z_1^-(\infty) = (\mathbb{R}_0^+)^2$.

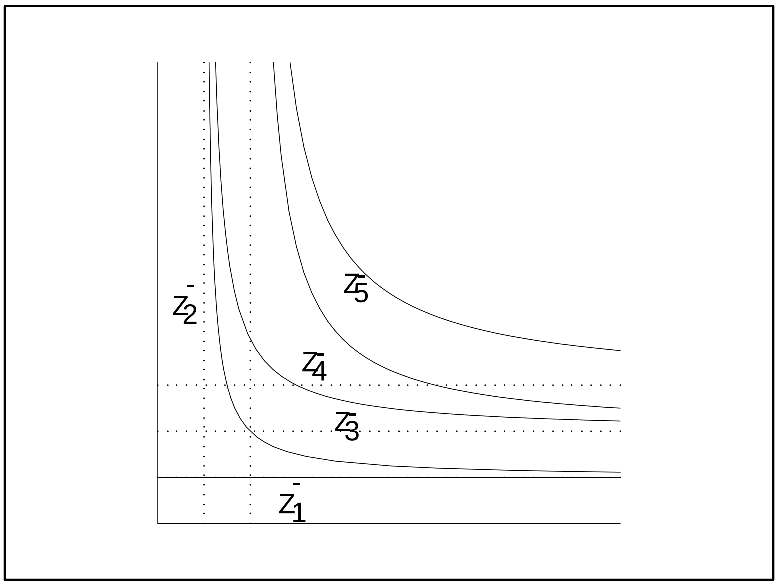

Figure 7: Negative Pseudo Fučík Spectrum at infinity.

The following lemma is the counterpart of Lemma 2.4 and it is proved in the same form.

LEMMA 2.10. *Let f satisfy (f_1) and let ϕ satisfy (ϕ_1), (ϕ_∞) and (T_∞). Assume that*

$$\lim_{s\to+\infty}\frac{f(s)}{\phi(s)} = B^+, \quad \lim_{s\to-\infty}\frac{f(s)}{\phi(s)} = B^-,$$

with $(B^+, B^-) \in Z_i^-(\infty)$ for some index i. Then there is $\kappa^* > 0$ such that for each $\kappa \le -\kappa^*$ it follows that $(x(\mathcal{L}(\kappa)), y(\mathcal{L}(\kappa))) \in W_i^-$.

REMARK 2.4. *The critical set $\mathcal{S}$ defined in [18], that is the (PFS) at infinity, is exactly $\mathcal{S}^+(\infty) \cup \mathcal{S}^-(\infty)$.*

By repeating the argument, and using estimates like in [5] we can define `the Negative Pseudo-Fučík Spectrum at zero` $\mathcal{C}^-(0)$ as the union of a countable number of critical curves $\mathcal{C}_i^-(0)$ defined in a similar manner as the critical sets of the (NPFS) at infinity, but, this time, with the estimates made at zero. Also we can define the corresponding zones $Z_i^-(0)$ as the complementary parts of the (NPFS) at zero.

Similarly to Lemmas 2.9 and 2.10, we now have.

LEMMA 2.11. *Let f satisfy (f_1) and let ϕ satisfy (ϕ_1), (ϕ_0) and (T_0). Then, for any compact set $\mathcal{K}$ contained in $(\mathbb{R}^+)^2 \setminus \cup_{i=1}^{\infty}\mathcal{C}_i^-(0)$ there is $\kappa_{\mathcal{R}}$ such that for* *each* $(\mu,\nu)\in\mathcal{R}$ *any solution of* $(P_{\mu,\nu})$ *with* $u'(0)<0$ *satisfies* $|u'(0)|\geq\kappa_{\mathcal{R}}$. *The same result is true is we perturb the equation in* $(P_{\mu,\nu})$ *by adding at the right hand side a term* $q(t,u)$ *such that* $q(t,s)/\phi(s)\to 0$ *as* $s\to 0$, *uniformly with respect to* $t\in[0,L]$.

LEMMA 2.12. *Let* f *satisfy* (f_1) *and let* ϕ *satisfy* (ϕ_1), (ϕ_0) *and* (T_0). *Assume that*

$$\lim_{s\to 0^+}\frac{f(s)}{\phi(s)}=b^+,\quad \lim_{s\to 0^-}\frac{f(s)}{\phi(s)}=b^-,$$

with $(b^+,b^-)\in Z_i^-(0)$ *for some index* i. *Then there is* $\kappa_*>0$ *such that for each* $-\kappa_*\leq\kappa<0$ *it follows that* $(x(\mathcal{L}(\kappa)),y(\mathcal{L}(\kappa)))\in W_i^-$.

We thus have reached a point where we can establish and prove our second main theorem.

THEOREM 2.2. *Let* f *satisfy* (f_1) *and let* ϕ *satisfy* (ϕ_1), (ϕ_0), (ϕ_∞) *and* (T_0), (T_∞). *Suppose that there are positive numbers* b^+,b^-,B^+,B^- *such that*

$$\lim_{s\to 0^+}\frac{f(s)}{\phi(s)}=b^+,\quad \lim_{s\to 0^-}\frac{f(s)}{\phi(s)}=b^-$$

and

$$\lim_{s\to +\infty}\frac{f(s)}{\phi(s)}=B^+,\quad \lim_{s\to -\infty}\frac{f(s)}{\phi(s)}=B^-.$$

Assume also that there are $k,\ell\in\mathbb{N}$ *with* $k\neq\ell$ *with* $(b^+,b^-)\in Z_k^-(0)$ *and* $(B^+,B^-)\in Z_\ell^-(\infty)$. *Then, problem* (P) *has at least* $|k-\ell|$ *solutions with* $u'(0)<0$.

Proof. Direct consequence of Lemmas 2.11, 2.12, and 2.8. □

Note that given a pair $(\mu,\nu)\in(\mathbb{R}_0^+)^2$, not belonging to the critical set at zero $\mathcal{S}^+(0)$, we can determine the region $Z_i^+(0)$ to which it belongs, by the following criterion. Let us set

$$\rho(0)=\frac{L}{T_0^+(\mu)+T_0^-(\nu)}$$

and observe that there is only one integer, say j, belonging to the open interval

$$\left]\rho(0)-\frac{T_0^+(\mu)}{T_0^+(\mu)+T_0^-(\nu)},\rho(0)+\frac{T_0^-(\nu)}{T_0^+(\mu)+T_0^-(\nu)}\right[.$$

Then, we have that (μ,ν) belongs to $Z_{2j}^+(0)$ or $Z_{2j+1}^+(0)$, according to whether $j>\rho(0)$ or $j<\rho(0)$.

Avoiding to consider the semi-trivial cases in which one or both of the two maps $T_0^{\pm}$ are infinite, we observe that $\rho(0)$ as a function of μ or ν is continuous, strictly increasing and such that $\rho(0) \to 0$ as $\mu \to 0^+$ or $\nu \to 0^+$. Moreover $\rho(0) \to \frac{L}{T_0^-(\nu)}$ as $\mu \to +\infty$ and $\rho(0) \to \frac{L}{T_0^+(\mu)}$ as $\mu \to +\infty$. On the other hand, for $\alpha(0) = \frac{T_0^+(\mu)}{T_0^+(\mu)+T_0^-(\nu)}$, we have that $0 < \alpha(0) < 1$, with $\alpha(0)$ decreasing in μ and increasing in ν and such that $\alpha(0) \to 1$ as $\mu \to 0^+$, or $\mu \to +\infty$ and $\alpha(0) \to 0$ as $\mu \to +\infty$, or $\nu \to 0^+$.

Clearly, the same procedure can be followed in order to determine the region $Z_i^+(\infty)$ to which the pair (μ, ν) belongs.

Similarly, given a pair $(\mu, \nu) \in (\mathbb{R}_0^+)^2$ not belonging to the critical set at zero $\mathcal{S}^-(0)$, we can determine the region $Z_i^-(0)$ it belongs, by the following criterion. Let us set

$$\rho(0) = \frac{L}{T_0^+(\mu) + T_0^-(\nu)}\,,$$

and as before observe that there is only one integer, say j, belonging to the open interval

$$\left]\rho(0) - \frac{T_0^-(\nu)}{T_0^+(\mu) + T_0^-(\nu)}\,, \rho(0) + \frac{T_0^+(\mu)}{T_0^+(\mu) + T_0^-(\nu)}\right[\,.$$

Then, we have that (μ, ν) belongs to $Z_{2j}^-(0)$ or $Z_{2j+1}^-(0)$, according to whether $j > \rho^+(0)$ or $j < \rho^+(0)$.

The same procedure can be followed in order to determine the region $Z_i^-(\infty)$ the pair (μ, ν) belongs to.

Remark 2.5. *In the proof of the main theorems, it is not important that the limiting pairs (a^+, a^-) and (A^+, A^-) in Theorem 2.1, or (b^+, b^-), and (B^+, B^-) in Theorem 2.2 be in "nonresonance" zones out of the (PFS) sets. What really matters is that there is a suitable gap between the corresponding positions of these limits at zero and at infinity with respect to the universal Fučík spectrum. This situation reminds the occurrence of a twist condition between zero and infinity which permits the application of the Poincaré-Birkhoff fixed point theorem in the periodic case [11].*

In view of the above remark, the following rule can be given.

The set $\mathcal{U}^+ := \{H_i^+, W_i^+ : i \in \mathbb{N}\}$ represents a partition of $(\mathbb{R}_0^+)^2$. Given two points $P, Q \in (\mathbb{R}_0^+)^2$, we have that the number of transverse intersections between the *open* segment $]P, Q[$ and the critical lines $\cup_{i=1}^{\infty} H_i^+$ depends only on which of the classes of the above partition P and Q belong to (notice that, in this way, if P and Q are both on the same critical line H_i^+, then the intersection counts like zero).

The sets $\mathcal{M}^+(0) := \{\mathcal{C}_i^+(0), Z_i^+(0) \,:\, i \in \mathbb{N}\}$ and $\mathcal{M}^+(\infty) := \{\mathcal{C}_i^+(\infty), Z_i^+(\infty) \,:\, i \in \mathbb{N}\}$ also determine a partition of $(\mathbb{R}_0^+)^2$. We can define now the maps

$$\mathcal{M}^+(0) \to \mathcal{U}^+, \quad \mathcal{M}^+(\infty) \to \mathcal{U}^+$$

by

$$\mathcal{C}_i^+(0),\ \mathcal{C}_i^+(\infty) \mapsto H_i^+ \quad \text{and} \quad Z_i^+(0),\ Z_i^+(\infty) \mapsto W_i^+ .$$

Note that these maps make a correspondence between curves to lines and open sets to open sets.

Take a point $(a,b) \in (\cup_{i=1}^{\infty}\mathcal{C}_i^+(0)) \cup (\cup_{i=1}^{\infty} Z_i^+(0))$. We can associate to (a,b) the set in $\mathcal{M}^+(0)$ to which it belongs and hence, to this one, the set in $\mathcal{U}^+$ which is associated to it via the above map. Call this set $[(a,b)]$. Similarly, given a point $(A,B) \in (\cup_{i=1}^{\infty}\mathcal{C}_i^+(\infty)) \cup (\cup_{i=1}^{\infty} Z_i^+(\infty))$ we can map it to a set $[(A,B)] \in \mathcal{U}^+$, where $[(A,B)]$ is the set corresponding to that one in $\mathcal{M}^+(\infty)$ to which (A,B) belongs. Since, as observed before, the number of transverse intersections of the open segment $]P,Q[$ with the set $\cup_{i=1}^{\infty} H_i^+$ is the same for each $P \in [(a,b)]$ and $Q \in [(A,B)]$, we have that this number is well determined by the initial choice of the pairs (a,b) and (A,B). We denote this number by $i^+[(a,b),(A,B)]$ and call it the *positive intersection index* for the pairs (a,b) and (A,B).

We remark that this definition requires that the pairs (a,b) and (A,B) are "thought" in relation with the (PPFS) at zero and at infinity, respectively.

In a similar manner, one can define the partitions $\mathcal{U}^-$, $\mathcal{M}^-(0)$, $\mathcal{M}^-(\infty)$ of $(\mathbb{R}_0^+)^2$ and the maps

$$\mathcal{C}_i^-(0),\ \mathcal{C}_i^-(\infty) \mapsto H_i^- \quad \text{and} \quad Z_i^-(0),\ Z_i^-(\infty) \mapsto W_i^- ,$$

in order to define an index $i^-[(c,d),(C,D)]$ as the *negative intersection index* for the pairs (c,d) and (C,D), where (c,d) is related to the (NPFS) at zero and (CD) to the (NPFS) at infinity.

We point out that this procedure works also in the "degenerate" case in which some of the $\mathrm{T}_0^{\pm}(\Lambda)$ or $\mathrm{T}_{\infty}^{\pm}(\Lambda)$ considered in (T_0) and in (T_∞) is constantly equal to infinity. In this situation, some of the maps $\mathcal{M}^{\pm}(0), \mathcal{M}^{\pm}(0) \to \mathcal{U}^{\pm}$ will be not surjective, but the definition of the intersection indexes is well posed too.

Then, we have:

Theorem 2.3. *Let f satisfy (f_1) and let ϕ satisfy (ϕ_1), (ϕ_0), (ϕ_∞) and (T_0), (T_∞). Suppose that there are positive numbers d^+, d^-, D^+, D^- such that*

$$\lim_{s\to 0^+} \frac{f(s)}{\phi(s)} = d^+, \quad \lim_{s\to 0^-} \frac{f(s)}{\phi(s)} = d^-$$

and

$$\lim_{s\to +\infty} \frac{f(s)}{\phi(s)} = D^+, \quad \lim_{s\to -\infty} \frac{f(s)}{\phi(s)} = D^-.$$

Then, problem (P) has at least so many solutions with $u'(0) > 0$ like the positive intersection index $i^+[(d^+, d^-), (D^+, D^-)]$ and at least so many solutions with $u'(0) < 0$ like the negative intersection index $i^-[(d^+, d^-), (D^+, D^-)]$.

3. A result for strictly monotone time-mappings

We present in this section some suitable conditions under which we have a strictly monotone time-map. Hence, according to the remarks after Lemmas 2.2 and 2.8, they can be applied to obtain an exact number of solutions, .

We shall confine ourselves only to the consideration of $\tau(\kappa)$ for $\kappa > 0$. The situation in which $\kappa < 0$ is completely symmetric and therefore can be discussed using the same arguments.

For simplicity, we suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism which is of class C^1 in $\mathbb{R}_0^+$ and such that

$$(\phi_2) \qquad \lim_{s \to 0^+} s^2 \phi'(s) = 0$$

and we also assume that $f : \mathbb{R} \to \mathbb{R}$ satisfy (f_1). Let $u(\cdot)$ be a solution of (3) for some $\kappa > 0$.

From the energy relation (4) we can compute the distance $\tau(\kappa)$ of two consecutive zeros of u in an interval where $u > 0$ by the formula

$$\tau(\kappa) = 2 \int_0^R \frac{ds}{\mathcal{L}_r^{-1}(F(R) - F(s))}, \quad \text{with } F(R) = \mathcal{L}(\kappa),\ R > 0,$$

where the number "2" comes from the fact that, by the symmetry of ϕ, we have that the time from $t = 0$ to the point of maximum of u is the same like the time from the point of maximum of u and $t = \tau(\kappa)$.

For convenience in the next proof, we also introduce the function

$$\Psi(s) := \phi'(\mathcal{L}_r^{-1}(s))(\mathcal{L}_r^{-1}(s))^2,$$

which, by the assumptions on ϕ, is defined and continuous for every $s > 0$ and, by (ϕ_2) can be extended by continuity to the origin, putting $\Psi(0) = 0$.

Then, the following result can be proved:

Lemma 3.1. *Assume (ϕ_2) and suppose that the function Ψ is convex (respectively, concave) in $\mathbb{R}^+$. Then, the map $\kappa \mapsto \tau(\kappa)$ is strictly decreasing (respectively, strictly increasing) if the map $s \mapsto sf(s) - \Psi(F(s))$ is strictly increasing (respectively, strictly decreasing).*

Proof. First of all, via the change of variables $s = Rt$, we write the integral for $\tau(\kappa)$ as

$$\tau(\kappa) = 2 \int_0^1 \frac{R\, dt}{\mathcal{L}_r^{-1}(F(R) - F(Rt))},$$

so that, in order to prove that $\tau(\cdot)$ is strictly decreasing (respectively, strictly increasing) for κ in a certain interval, say $\kappa \in]\alpha, \beta[$, with $0 \leq \alpha < \beta \leq +\infty$, we may prove that, for each $t \in]0,1[$ the map $R \mapsto (\mathcal{L}_r^{-1}(F(R) - F(Rt)))/R$ is strictly increasing (or strictly decreasing) on $]\alpha_1, \beta_1[$, where we have set $\alpha_1 = F_r^{-1}(\mathcal{L}(\alpha))$ and $\beta_1 = F_r^{-1}(\mathcal{L}(\beta))$. To do this, we compute the derivative with respect to R for any fixed $t \in]0,1[$. Then, at the numerator we find

$$\frac{Rf(R) - Rtf(Rt)}{\mathcal{L}'(\mathcal{L}_r^{-1}(F(R) - F(Rt)))} - \mathcal{L}_r^{-1}(F(R) - F(Rt)).$$

Recalling that $\mathcal{L}'(\xi) = \xi\phi'(\xi)$ and the definition of Ψ, we conclude that τ is strictly decreasing (strictly increasing) in $]\alpha, \beta[$, if

$$Rf(R) - Rtf(Rt) - \Psi(F(R) - F(Rt)) > 0 \tag{7}$$

holds every $R \in]\alpha_1, \beta_1[$ and all $t \in]0,1[$. Now, according to the assumption, we have that Ψ is a convex function and recall also that $\Psi(0) = 0$. From this, it follows that $\Psi(a+b) \geq \Psi(a) + \Psi(b)$ for all $a, b \geq 0$ and therefore, $\Psi(x-y) \leq \Psi(x) - \Psi(y)$ for all $0 < y < x$. Hence, we have that $\Psi(F(R) - F(Rt)) \leq \Psi(F(R)) - \Psi(F(Rt))$ holds for every $R \in]\alpha_1, \beta_1[$ and all $t \in]0,1[$ and therefore, in order to prove (7) it will be sufficient to prove that

$$Rf(R) - Rtf(Rt) - [\Psi(F(R)) - \Psi(F(Rt))] > 0$$

holds for every $R \in]\alpha_1, \beta_1[$ and all $t \in]0,1[$.

Now, to have this last inequality satisfied it will be enough to have that the function $R \mapsto Rf(R) - \Psi(F(R))$ is strictly increasing on the interval $]0, \beta_1[$.

In case that the function Ψ is concave, we use the inequality $\Psi(x-y) \geq \Psi(x) - \Psi(y)$ for all $0 < y < x$ and obtain that the time-mapping is strictly increasing on $]\alpha, \beta[$ provided that the map $R \mapsto Rf(R) - \Psi(F(R))$ is strictly decreasing in the interval $]0, \beta_1[$. □

Note that if ϕ is of class C^2 in $\mathbb{R}_0^+$, then the map Ψ is convex (respectively, concave) provided that $s\phi''(s)/\phi'(s)$ is increasing (respectively, decreasing) on $\mathbb{R}_0^+$. In the special case of $\phi = \phi_p$, for some $p > 1$, we have that $s\phi''(s)/\phi'(s)$ is a positive constant.

An elementary application of Lemma 3.1 is the following:

Corollary 3.1. *For $\phi = \phi_p$, with $p > 1$, the following holds: The map $\kappa \mapsto \tau(\kappa)$ is strictly decreasing (respectively, strictly increasing) if the map $s \mapsto \frac{f(s)}{s^{p-1}}$ is strictly increasing (respectively, strictly decreasing).*

Proof. In this case, by a direct computation, we have that $\Psi(s) = ps$, so that all the assumptions on Ψ are satisfied and the auxiliary function $s \mapsto sf(s) - \Psi(F(s))$ takes the form of $sf(s) - pF(s)$, which, in turn, is strictly increasing (respectively, strictly decreasing) if the map $s \mapsto \frac{f(s)}{s^{p-1}}$ is strictly increasing (respectively, strictly decreasing), too. □

Corollary 3.1 extends a classical result obtained by Opial [24] for the case $p = 2$. Other estimates for the time-mappings associated to the ϕ_p or the ϕ-operators can be found in [17], [18], [20] and [23].

A variant of Lemma 3.1 which gives as a consequence Corollary 3.1 as well, is the following.

LEMMA 3.2. *Assume* (ϕ_2) *and suppose that there is a constant* $\theta > 0$ *such that* $\Psi(s) \leq \theta s$ *(respectively* $\Psi(s) \geq \theta s$*) for all* $s \geq 0$*. Then, the map* $\kappa \mapsto \tau(\kappa)$ *is strictly decreasing (respectively, strictly increasing) if the map* $s \mapsto \frac{f(s)}{s^{\theta-1}}$ *is strictly increasing (respectively, strictly decreasing).*

This result still admits a little variant, in the sense that if we know that $\Psi(s) < \theta s$ for $s > 0$, then it will be sufficient to assume $s \mapsto \frac{f(s)}{s^{\theta-1}}$ increasing (weakly) in order to have τ strictly decreasing in κ and, conversely, if $\Psi(s) > \theta s$ for $s > 0$, then τ is strictly increasing when $s \mapsto \frac{f(s)}{s^{\theta-1}}$ is decreasing (weakly).

As a final remark for this section, we observe that all the results presented here can be extended, modulo suitable changes, if we assume that there are $-\infty \leq a < 0 < b \leq +\infty$ and $-\infty \leq c < 0 < d \leq +\infty$, such that $\phi :]a, b[\to]c, d[$ is a strictly increasing bijection with $\phi(0) = 0$ and also we suppose that ϕ is of class C^1 in $\mathbb{R}_0^+$ with $s^2\phi'(s) \to 0$, as $s \to 0^{\pm}$.

In this situation, however, the results of monotonicity for the time-mapping will have their range of validity only for a suitable neighborhood of the origin. More precisely, for $u'(0) = \kappa > 0$, we have to take $\kappa \in]0, \beta[$, with $\beta = \min\{b, \mathcal{L}_r^{-1}(\mathcal{L}(a))\}$ and the corresponding R's will vary in $]R_l^{-1}(\mathcal{L}(\beta)), F_r^{-1}(\mathcal{L}(\beta)[$. For $u'(0) = \kappa < 0$, we have to take $\kappa \in]\alpha, 0[$, with $\alpha = \max\{a, \mathcal{L}_l^{-1}(\mathcal{L}(b))\}$ and the corresponding R's will vary in $]F_l^{-1}(\mathcal{L}(\alpha), F_r^{-1}(\mathcal{L}(\alpha))[$.

4. Examples

In this section we illustrate some of our results through simple examples.

Example 1. Let $p,\ q > 1$ and define the homeomorphism ϕ by

$$\phi(s) = \begin{cases} \phi_p(s), & \text{for } s \geq 0, \\ \phi_q(s), & \text{for } s \leq 0. \end{cases} \tag{8}$$

Also, let us denote by B the well known beta function (see for example [1, p. 258])

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,$$

which we recall is convergent for $m,\ n > 0$. Then, for any positive h, we find that

$$\begin{aligned} L\cdot x(h) =& \left(\frac{p}{\mu}\right)^{\frac{1}{p}}\left[\left(\frac{1}{p'}\right)^{\frac{1}{p}}\int_0^1\frac{dz}{(1-z^p)^{\frac{1}{p}}}+\left(\frac{1}{q'}\right)^{\frac{1}{q}}\int_0^1\frac{dz}{(1-z^p)^{\frac{1}{q}}}\,(\mu h)^{\frac{1}{p}-\frac{1}{q}}\right]\\ =&\frac{\pi_p}{2\mu^{\frac{1}{p}}}+\frac{B(\frac{1}{p},\frac{1}{q'})}{p^{\frac{1}{p'}}(q')^{\frac{1}{q}}}\cdot\frac{h^{\frac{1}{p}-\frac{1}{q}}}{\mu^{\frac{1}{q}}}\end{aligned}$$

and

$$\begin{aligned} L\cdot y(h) =& \left(\frac{q}{\nu}\right)^{\frac{1}{q}}\left[\left(\frac{1}{q'}\right)^{\frac{1}{q}}\int_0^1\frac{dz}{(1-z^q)^{\frac{1}{q}}}+\left(\frac{1}{p'}\right)^{\frac{1}{p}}\int_0^1\frac{dz}{(1-z^q)^{\frac{1}{p}}}\,(\nu h)^{\frac{1}{q}-\frac{1}{p}}\right]\\ =&\frac{\pi_q}{2\nu^{\frac{1}{q}}}+\frac{B(\frac{1}{q},\frac{1}{p'})}{q^{\frac{1}{q'}}(p')^{\frac{1}{p}}}\cdot\frac{h^{\frac{1}{q}-\frac{1}{p}}}{\nu^{\frac{1}{p}}}\,,\end{aligned}$$

where $x(h)$ and $y(h)$ are referred to the equation in $(P_{\mu,\nu})$. We also have

$$\mathrm{T}^+_{1,\infty}(\Lambda)=\left(\frac{p}{\Lambda}\right)^{\frac{1}{p}}\left[\left(\frac{1}{p'}\right)^{\frac{1}{p}}\int_0^1\frac{dz}{(1-z^p)^{\frac{1}{p}}}\right]=\frac{\pi_p}{2\Lambda^{\frac{1}{p}}},$$

$$\mathrm{T}^+_{2,\infty}(\Lambda)=\begin{cases}0, & \text{if } q<p,\\ \dfrac{\pi_p}{2\Lambda^{\frac{1}{p}}}\,, & \text{if } q=p,\\ +\infty\,, & \text{if } q>p,\end{cases}$$

$$\mathrm{T}^-_{2,\infty}(\Lambda)=\left(\frac{q}{\Lambda}\right)^{\frac{1}{q}}\left[\left(\frac{1}{q'}\right)^{\frac{1}{q}}\int_0^1\frac{dz}{(1-z^q)^{\frac{1}{q}}}\right]=\frac{\pi_q}{2\Lambda^{\frac{1}{q}}},$$

$$\mathrm{T}^-_{1,\infty}(\Lambda)=\begin{cases}+\infty, & \text{if } q<p,\\ \dfrac{\pi_q}{2\Lambda^{\frac{1}{q}}}\,, & \text{if } q=p,\\ 0\,, & \text{if } q>p.\end{cases}$$

Finally,

$$\mathrm{T}^+_{\infty}(\Lambda)=\begin{cases}\dfrac{\pi_p}{2\Lambda^{\frac{1}{p}}}\,, & \text{if } q<p,\\[2ex] \dfrac{\pi_p}{\Lambda^{\frac{1}{p}}}\,, & \text{if } q=p,\\[2ex] +\infty\,, & \text{if } q>p\end{cases}\quad\text{and}\quad \mathrm{T}^-_{\infty}(\Lambda)=\begin{cases}+\infty\,, & \text{if } q<p,\\[2ex] \dfrac{\pi_p}{\Lambda^{\frac{1}{p}}}\,, & \text{if } q=p,\\[2ex] \dfrac{\pi_p}{2\Lambda^{\frac{1}{p}}}\,, & \text{if } q>p.\end{cases}$$

We note that when $p=q$, the (PPFS) at infinity is $\mathcal{S}^+(\infty)=\cup_{i=1}^{\infty}\mathcal{C}_i^+(\infty)$, where

$$\mathcal{C}^+_{2j-1}(\infty)=\left\{(\mu,\nu)\in(\mathbb{R}^+_0)^2:\ \frac{j}{\mu^{\frac{1}{p}}}+\frac{j-1}{\nu^{\frac{1}{p}}}=\frac{L}{\pi_p}\right\}$$

and

$$\mathcal{C}^+_{2j}(\infty) = \left\{(\mu,\nu) \in (\mathbb{R}^+_0)^2 : \ \frac{j}{\mu^{\frac{1}{p}}} + \frac{j}{\nu^{\frac{1}{p}}} = \frac{L}{\pi_p}\right\}$$

This set is clearly a part of the standard Fučík spectrum for the one-dimensional p-Laplacian with respect to the Dirichlet boundary conditions on $]0,L[$. On the other hand, when $p>q$, the (PPFS) at infinity is actually made only by the line

$$\mathcal{C}^+_1(\infty) = \left\{(\mu,\nu) \in (\mathbb{R}^+_0)^2 : \ \mu = \left(\frac{\pi_p}{2L}\right)^p\right\}$$

and when $p<q$, it is empty.

Similarly, we have that

$$\mathrm{T}^+_{1,0}(\Lambda) = \frac{\pi_p}{2\Lambda^{\frac{1}{p}}}, \quad \mathrm{T}^+_{2,0}(\Lambda) = \begin{cases} \infty\,, & \text{if } \ q<p, \\ \dfrac{\pi_p}{2\Lambda^{\frac{1}{p}}}\,, & \text{if } \ q=p, \\ 0\,, & \text{if } \ q>p, \end{cases}$$

$$\mathrm{T}^-_{2,0}(\Lambda) = \frac{\pi_q}{2\Lambda^{\frac{1}{q}}}, \quad \mathrm{T}^-_{1,0}(\Lambda) = \begin{cases} 0\,, & \text{if } \ q<p, \\ \dfrac{\pi_q}{2\Lambda^{\frac{1}{q}}}\,, & \text{if } \ q=p, \\ +\infty\,, & \text{if } \ q>p. \end{cases}$$

Hence,

$$\mathrm{T}^+_0(\Lambda) = \begin{cases} +\infty\,, & \text{if } \ q<p, \\ \dfrac{\pi_p}{\Lambda^{\frac{1}{p}}}\,, & \text{if } \ q=p, \\ \dfrac{\pi_p}{2\Lambda^{\frac{1}{p}}}\,, & \text{if } \ q>p \end{cases} \quad \text{and} \quad \mathrm{T}^-_0(\Lambda) = \begin{cases} \dfrac{\pi_q}{2\Lambda^{\frac{1}{q}}}\,, & \text{if } \ q<p, \\ \dfrac{\pi_p}{\Lambda^{\frac{1}{p}}}\,, & \text{if } \ q=p, \\ +\infty\,, & \text{if } \ q>p. \end{cases}$$

Clearly, when $p=q$, the (PPFS) at zero is the same as the one at infinity, while, when $p>q$ it is empty and, when $p<q$ it consists of the single line

$$\mathcal{C}^+_1(0) = \left\{(\mu,\nu) \in (\mathbb{R}^+_0)^2 : \ \mu = \left(\frac{\pi_q}{2L}\right)^q\right\}.$$

Again, we can explicitly describe the (NPFS) at infinity and at zero. In particular, when $p=q$, the (NPFS) at infinity is $\mathcal{S}^+(\infty) = \cup_{i=1}^\infty \mathcal{C}^-_i(\infty)$, where

$$\mathcal{C}^-_{2j-1}(\infty) = \left\{(\mu,\nu) \in (\mathbb{R}^+_0)^2 : \ \frac{j-1}{\mu^{\frac{1}{p}}} + \frac{j}{\nu^{\frac{1}{p}}} = \frac{L}{\pi_p}\right\}$$

and

$$\mathcal{C}_{2j}^{-}(\infty) = \left\{(\mu,\nu) \in (\mathbb{R}_0^+)^2 : \frac{j}{\mu^{\frac{1}{p}}} + \frac{j}{\nu^{\frac{1}{p}}} = \frac{L}{\pi_p}\right\}.$$

To show the role of the strong asymmetry when $p \neq q$ for the map ϕ defined in (8), we consider f satisfying (f_1) and assume that $\frac{f(s)}{\phi(s)} \to a^{\pm}$ as $s \to 0^{\pm}$ and $\frac{f(s)}{\phi(s)} \to A^{\pm}$ as $s \to \pm\infty$, for some positive constants $a^{\pm}$ and $A^{\pm}$. Then, if $p > q$, it follows that problem (P) has at least one solution with positive slope at $t = 0$ if $A^+ > \left(\frac{\pi_p}{2L}\right)^p$ and it has at least one solution with negative slope at $t = 0$ if $a^- > \left(\frac{\pi_q}{2L}\right)^q$.

We remark that, according to [20], the same spectra at infinity (or, respectively, at zero) is obtained, and therefore, the same application to problem (P) will occur for any function ϕ having the form of

$$\phi(s) = \begin{cases} \psi_1(s), & \text{for } s \geq 0, \\ \psi_2(s), & \text{for } s \leq 0, \end{cases}$$

where $\psi_1 : \mathbb{R}^+ \to \mathbb{R}^+$ and $\psi_2 : \mathbb{R}^- \to \mathbb{R}^-$ are increasing bijections with $\psi_i(0) = 0$ and

$$\lim_{s\to+\infty,\, 0^+} \frac{\psi_1(\sigma s)}{\psi_1(s)} = \sigma^{p-1}, \quad \text{for all}\, \sigma > 0,$$

and

$$\lim_{s\to-\infty,\, 0^-} \frac{\psi_2(\sigma s)}{\psi_2(s)} = \sigma^{q-1}, \quad \text{for all}\, \sigma > 0,$$

for some $p, q > 1$.

Example 2. Continuing along this direction, we could consider (like it was done in [20] for an odd ϕ), the case of a ϕ-function such that for some $p, q > 1$,

$$\lim_{s\to\pm\infty} \frac{\phi(\sigma s)}{\phi(s)} = \sigma^{p-1}, \quad \text{for all}\, \sigma > 0, \tag{9}$$

and

$$\lim_{s\to 0^{\pm}} \frac{\phi(\sigma s)}{\phi(s)} = \sigma^{q-1}, \quad \text{for all}\, \sigma > 0. \tag{10}$$

Hence, we have a (PPFS) and also a (NPFS) at infinity and one at zero, which are the same like those for the p-Laplacian and the q-Laplacian, respectively. Observe that here we do not assume the ϕ to be odd. For instance, a map of the form

$$\phi:\ s \mapsto \begin{cases} \phi_p(s), & s \leq -1, \\ \phi_q(s), & -1 \leq s \leq 0, \\ \log(1 + s^{q-1}), & 0 \leq s \leq 1, \\ \phi_p(s)\log(1+s), & s \geq 1, \end{cases}$$

is suitable for our applications.

As a consequence of Theorem 2.1, we have the following

COROLLARY 4.1. *Let ϕ satisfy (ϕ_1), (9), and (10). Suppose that there are positive numbers a^+, a^-, A^+, A^- such that*

$$\lim_{s\to 0^+}\frac{f(s)}{\phi(s)} = a^+, \quad \lim_{s\to 0^-}\frac{f(s)}{\phi(s)} = a^-$$

and

$$\lim_{s\to +\infty}\frac{f(s)}{\phi(s)} = A^+, \quad \lim_{s\to -\infty}\frac{f(s)}{\phi(s)} = A^-.$$

Assume also that there are $k, \ell \in \mathbb{N}$ with $k \neq \ell$ such that, for $k = 2j$

$$\frac{j}{(a^+)^{\frac{1}{q}}} + \frac{j-1}{(a^-)^{\frac{1}{q}}} < \frac{L}{\pi_q} < \frac{j}{(a^+)^{\frac{1}{q}}} + \frac{j}{(a^-)^{\frac{1}{q}}}$$

and for $k = 2j + 1$

$$\frac{j}{(a^+)^{\frac{1}{q}}} + \frac{j}{(a^-)^{\frac{1}{q}}} < \frac{L}{\pi_q} < \frac{j+1}{(a^+)^{\frac{1}{q}}} + \frac{j}{(a^-)^{\frac{1}{q}}}$$

while, for $\ell = 2i$,

$$\frac{i}{(A^+)^{\frac{1}{p}}} + \frac{i-1}{(A^-)^{\frac{1}{p}}} < \frac{L}{\pi_p} < \frac{i}{(A^+)^{\frac{1}{p}}} + \frac{i}{(A^-)^{\frac{1}{p}}}$$

and for $\ell = 2i + 1$,

$$\frac{i}{(A^+)^{\frac{1}{p}}} + \frac{i}{(A^-)^{\frac{1}{p}}} < \frac{L}{\pi_p} < \frac{i+1}{(A^+)^{\frac{1}{p}}} + \frac{i}{(A^-)^{\frac{1}{p}}}$$

Then, problem (P) has at least $|k - \ell|$ solutions with $u'(0) > 0$.

Clearly, a symmetric result holds for the (NPFS).

To show an application of Corollary 4.1 which resembles the one given in the Introduction (this time for $q = 2$), we consider the following situation.

Example 3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an increasing bijection which is of class C^1 in a neighborhood of zero, with $\phi'(0) > 0$ and satisfies (9) for some $p > 1$. Suppose also that we have

$$\lim_{s\to 0^+}\frac{f(s)}{\phi(s)} = \lim_{s\to +\infty}\frac{f(s)}{\phi(s)} = a$$

$$\lim_{s\to 0^-}\frac{f(s)}{\phi(s)} = \lim_{s\to -\infty}\frac{f(s)}{\phi(s)} = b$$

Assume that (a,b) is in some region $Z^+_{j_1}(0) \cap Z^-_{j_2}(0)$. Then, for $p > 1$ sufficiently large, problem (P) for the interval $[0,\pi]$ has at least $j_1 - 2$ solutions with $u'(0) > 0$ and at least $j_2 - 2$ solutions with $u'(0) < 0$.

Indeed, $L/\pi_p = \pi/\pi_p \to \pi/2 < 2$ as $p \to +\infty$ and, at the same time, $a^{-\frac{1}{p}} + b^{-\frac{1}{p}} \to 2$, so that the pair (a,b) belongs to the second region for both the positive and the negative (PFS) at infinity, for large p.

We remark that, in any case, $|j_1 - j_2| \le 1$.

An example of a function like the ϕ considered here is the following:

$$\phi(s) = \frac{\log(1+|s|)}{\log(1+|s|^{1-p})}\,\mathrm{sgn}(s)\,.$$

Appendix

We present here some technical estimates for the comparison of the time-mappings associated to the quasilinear differential equations

$$(\phi(u'))' + g_1(u) = 0 \tag{A.1}$$

and

$$(\phi(u'))' + g_2(u) = 0 \tag{A.2}$$

where, throughout this section, we assume that $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing bijection with $\phi(0) = 0$ satisfying (ϕ_1) and $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying assumption (f_1). We also denote by $G_i(s) = \int_0^s g_i(\xi)\,d\xi$ the primitives of g_i.

According to the notation previously introduced, we consider the time-mappings

$$\mathrm{T}_{g_i}(R) := \left| \int_0^R \frac{ds}{\mathcal{L}_r^{-1}(G_i(R) - G_i(s))} \right| + \left| \int_0^R \frac{ds}{\mathcal{L}_l^{-1}(G_i(R) - G_i(s))} \right|$$

(for $i = 1,2$), which represent the distance of two consecutive zeros of a solution of $(\phi(u'))' + g_i(u) = 0$ in an interval where such a solution is positive (negative) and achieves its maximum (minimum) value R.

Lemma A.1. *Assume that there is $R_* > 0$ such that*

$$|g_1(s)| \le |g_2(s)| \quad (\textit{or}\ \ |g_1(s)| < |g_2(s)|\,),$$

holds for all $0 < s \le R_*$, or, respectively, for all $-R_* \le s < 0$. Then, $\mathrm{T}_{g_2}(R) \le \mathrm{T}_{g_1}(R)$ (or $\mathrm{T}_{g_2}(R) < \mathrm{T}_{g_1}(R)$), for all $0 < R \le R_*$ or, respectively, for all $-R_* \le R < 0$.

Proof. We consider only the case when $R > 0$, being the other one completely symmetric. Then, by the assumption, given any $R \in]0, R_*]$, we have that $g_1(s) \leq g_2(s)$ (or $g_1(s) < g_2(s)$) for all $s \in]0, R]$. Hence,

$$G_1(R) - G_1(s) = \int_s^R g_1(\xi)\, d\xi \leq \int_s^R g_2(\xi)\, d\xi = G_2(R) - G_2(s)$$

(or $G_1(R) - G_1(s) < G_2(R) - G_2(s)$) holds for each $0 < s \leq R$. From this and the definition of $\mathrm{T}_{g_i}(R)$, the result immediately follows. $\square$

LEMMA A.2. *Assume that there is $R^* > 0$ such that*

$$|g_1(s)| \leq |g_2(s)|,$$

holds for all $s \geq R^*$, or, respectively, for all $s \leq -R^*$. Then, for each $\varepsilon > 0$ there is $R_\varepsilon > R^*$ such that

$$\mathrm{T}_{g_2}(R) \leq \mathrm{T}_{g_1}(R) + \varepsilon$$

holds for all $R > R_\varepsilon$, or, respectively, for all $R < -R_\varepsilon$.

Proof. As before, we discuss only the case when $R > 0$. Let u_i be the solution of $(\phi(u'))' + g_i(u) = 0$, for $i = 1, 2$, with $u_i(0) = 0$ and $\max u_i = u(t_i^*) = R > R^*$. From the equation, we see that u_i is strictly increasing in $[0, t_i^*]$ and strictly decreasing in $[t_i^*, \mathrm{T}_{g_1}(R)]$. Hence there are uniquely determined t_i^- and t_i^+, with $0 < t_i^- < t_i^* < t_i^+ < \mathrm{T}_{g_1}(R)$, such that $u_i(t) \geq R^*$ for $t \in [0, \mathrm{T}_{g_1}(R)]$ if and only if $t \in [t_i^-, t_i^+]$. By the same argument like in the proof of Lemma A.1, it easily follows that

$$\begin{aligned}
&\int_{R^*}^R \frac{ds}{\mathcal{L}_r^{-1}(G_2(R) - G_2(s))} + \int_{R^*}^R \frac{ds}{|\mathcal{L}_l^{-1}(G_2(R) - G_2(s))|} \\
&\qquad \leq \int_{R^*}^R \frac{ds}{\mathcal{L}_r^{-1}(G_1(R) - G_1(s))} + \int_{R^*}^R \frac{ds}{|\mathcal{L}_l^{-1}(G_1(R) - G_1(s))|}\,.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\mathrm{T}_{g_2}(R) - \mathrm{T}_{g_1}(R) \\
&\quad \leq \sum_{i=1,2} ((t_i^+ - t^*) + (t^* - t_i^-)) \\
&\quad = \sum_{i=1,2} \left(\int_0^{R^*} \frac{ds}{\mathcal{L}_r^{-1}(G_i(R) - G_i(s))} + \int_0^{R^*} \frac{ds}{|\mathcal{L}_l^{-1}(G_i(R) - G_i(s))|} \right) \\
&\quad \leq \sum_{i=1,2} \left(\frac{R^*}{\mathcal{L}_r^{-1}(G_i(R) - G_i(R^*))} + \frac{R^*}{|\mathcal{L}_l^{-1}(G_i(R) - G_i(s))|} \right)
\end{aligned}$$

holds for each $R > R^*$. At this point, the result easily follows, using (f_1) and letting $R \to +\infty$. $\square$

The next result is a straightforward consequence of Lemma A.2

Corollary A.1. *Assume that there is $R^* > 0$ such that*

$$|g_1(s)| \leq |g_2(s)|,$$

holds for all $s \geq R^*$, or, respectively, for all $s \leq -R^*$. Then

$$\limsup_{R\to+\infty} \mathrm{T}_{g_2}(R) \leq \liminf_{R\to+\infty} \mathrm{T}_{g_1}(R)$$

(respectively, $\limsup_{R\to-\infty} \mathrm{T}_{g_2}(R) \leq \liminf_{R\to-\infty} \mathrm{T}_{g_1}(R)$*).*

References

[1] M. Abramowitz and I. Stegun, *Handbook of mathematical functions*, Dover, New York (1965).

[2] L. Boccardo, P. Drábek, D. Giachetti and M. Kučera, *Generalization of Fredholm alternative for nonlinear differential operators*, Nonlinear Anal. **10** (1986), 1083–1103.

[3] A. Capietto, *An existence result for a two-point superlinear boundary value problem*, Proceedings of the 2nd International Conference on Differential Equations, Marrakesh (1995).

[4] A. Capietto and W. Dambrosio, *Multiplicity results for some two-point superlinear asymmetric boundary value problem*, Nonlinear Anal. **38** (1999), 869–896.

[5] A. Capietto and W. Dambrosio, *Boundary value problems with sublinear conditions near zero*, NoDEA Nonlinear Differential Equations Appl. **6** (1999), 149–172.

[6] A. Capietto, J. Mawhin and F. Zanolin, *Boundary value problems for forced superlinear second order ordinary differential equations*, in H. Brezis, *Nonlinear partial differential equations and their applications*, Taylor & Francis (1994).

[7] A. Capietto, J. Mawhin and F. Zanolin, *On the existence of two solutions with a prescribed number of zeros for a superlinear two-point boundary value problem*, Topol. Methods Nonlinear Anal. **6** (1995), 175–188.

[8] W. Dambrosio, *Time-map techniques for some boundary value problems*, Rocky Mountain J. Math. **28** (1998), 885–926.

[9] W. Dambrosio, *Multiple solutions of weakly-coupled systems with p-Laplacian operators*, Results Math. **36** (1998), 34–54.

[10] M. Del Pino, M. Elgueta and R. Manásevich, *A homotopic deformation along p of a Leray-Schauder degree result and existence for* $(|u'|^{p-2}u')'+f(t,u) = 0,\ u(0) = u(T) = 0,\ p > 1$, J. Differential Equations **80** (1992), 1–13.

[11] M.A. Del Pino, R.F. Manásevich and A.E. Murúa, *Existence and multiplicity of solutions with prescribed period for a second order quasilinear ODE*, Nonlinear Anal. **18** (1992), 79–92.

[12] G. Dinca and L. Sanchez, *Multiple solutions of boundary value problems: an elementary approach via the shooting method*, NoDEA Nonlinear Differential Equations Appl. **1** (1994), 163–178.

[13] P. Drábek, *Solvability of boundary value problems with homogeneous ordinary differential operator*, Rend. Istit. Mat. Univ. Trieste **18** (1986), 105–124.

[14] P. Drábek and R. Manásevich, *On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian*, Differential Integral Equations **12** (1999), 773–788.

[15] S. Fučík, *Solvability of nonlinear equations and boundary value problems*, Reidel, Dordrecht (1980).

[16] M. García-Huidobro, R. Manásevich and F. Zanolin, *Strongly nonlinear second order ODE's with unilateral conditions*, Differential Integral Equations **6** (1993), 1057–1078.

[17] M. García-Huidobro, R. Manásevich and F. Zanolin, *A Fredholm-like result for strongly nonlinear second order ODE's*, J. Differential Equations **114** (1994), 132–167.

[18] M. García-Huidobro, R. Manásevich and F. Zanolin, *On a pseudo Fučík spectrum for strongly nonlinear second order ODE's and an existence result*, J. Comput. Appl. Math. **52** (1994), 219–239.

[19] M. García-Huidobro, R. Manásevich and F. Zanolin, *Infinitely many solutions for a Dirichlet problem with a nonhomogeneous p-Laplacian-like operator in a ball*, Adv. Differential Equations **2** (1997), 203–230.

[20] M. García-Huidobro and P. Ubilla, *Multiplicity of solutions for a class of second-order equations*, Nonlinear Anal. **28** (1997), 1509–1520.

[21] P. Habets, M. Ramos and L. Sanchez, *Jumping nonlinearities for Neumann BVPs with positive forcing*, Nonlinear Anal. **20** (1993), 533–549.

[22] M. Henrard, *Infinitely many solutions of weakly coupled superlinear systems*, Adv. Differential Equations **2** (1997), 753–778.

[23] R. Manásevich and F. Zanolin, *Time-mappings and multiplicity of solutions for the one-dimensional p-Laplacian*, Nonlinear Anal. **21** (1993), 269–291.

[24] Z. Opial, *Sur les périodes des solutiones de l'équation différentielle* $x'' + g(x) = 0$, Ann. Polon. Math. **10** (1961), 49–72.

[25] C. Rebelo, *A note on uniqueness of Cauchy problems associated to planar Hamiltonian systems*, Portugal. Math. **57** (2000), 415–419.

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