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Vector Bundles on Elliptic Curves and Factors of Automorphy

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Abstract. *We translate Atiyah's results on classification of vector bundles on elliptic curves to the language of factors of automorphy.*

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1. Introduction

1.1. Motivation

The problem of classification of vector bundles over an elliptic curve was considered and completely solved by Atiyah in [1].

For a group Γ acting on a complex manifold Y, an r-dimensional factor of automorphy is a holomorphic function $f : \Gamma \times Y \to \mathrm{GL}_r(\mathbb{C})$ satisfying $f(\lambda\mu, y) = f(\lambda, \mu y)f(\mu, y)$. Two factors of automorphy f and f' are equivalent if there exists a holomorphic function $h : Y \to \mathrm{GL}_r(\mathbb{C})$ such that $h(\lambda y)f(\lambda, y) = f'(\lambda, y)h(y)$.

Given a complex manifold X and the universal covering $Y \xrightarrow{p} X$, let Γ be the fundamental group of X acting naturally on Y by deck transformations. Then there is a one-to-one correspondence between equivalence classes of r-dimensional factors of automorphy and isomorphism classes of vector bundles on X with trivial pull-back along p. In particular, if Y does not possess any non-trivial vector bundles, one obtains a one-to-one correspondence between equivalence classes of r-dimensional factors of automorphy and isomorphism classes of vector bundles on X. In particular this is the case for complex tori.

Since it is known that one-dimensional complex tori correspond to elliptic curves and since the classification of holomorphic vector bundles on a projective variety over $\mathbb{C}$ is equivalent to the classification of algebraic vector bundles (cf. [13]), it is possible to formulate Atiyah's results in the language of factors of automorphy. So for example in the case of vector bundles of rank 1 and 2 such a formulation using factors of automorphy was given in [4], Theorems 4.4 and 4.5.

This paper is a shortened version of the diploma thesis [7] and aims to prove some results used without any proofs by different authors, in particular in [12] and [3]. The main result of this note, Theorem 5.24, gives a classification of indecomposable vector bundles of fixed rank and degree on a complex torus in terms of factors of automorhy. Its statement coincides with the statement of Proposition 1 from [12], which was given without any proof.

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1.2. Structure of the Paper

In Section 2 we establish a correspondence between vector bundles and factors of automorphy. Section 3 deals with properties of factors of automorphy, in particular we discuss a correspondence between operations on vector bundles and operations on factors of automorphy. From Section 4 on we restrict ourselves to the case of vector bundles on complex tori. It is shown in Theorem 4.11 that to define a vector bundle of rank r on a complex one-dimensional torus is the same as to fix a holomorphic function $\mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$. In Section 5 we first present in Theorem 5.13 a classification of indecomposable vector bundles of degree zero, using this we give then in Theorem 5.24 a complete classification of indecomposable vector bundles of fixed rank and degree in terms of factors of automorphy.

1.3. Notations and Conventions

Following Atiyah's paper [1] we denote by $\mathcal{E}(r,d) = \mathcal{E}_X(r,d)$ the set of isomorphism classes of indecomposable vector bundles over X of rank r and degree d. For a vector bundle E we usually denote the corresponding locally free sheaf of its sections by $\mathcal{E}$. By Vect we denote the category of finite dimensional vector spaces. For a divisor D we denote by $[D]$ the corresponding line bundle.

2. Correspondence between Vector Bundles and Factors of Automorphy

Let X be a complex manifold and let $p : Y \to X$ be a covering of X. Let $\Gamma \subset \mathrm{Deck}(Y/X)$ be a subgroup in the group of deck transformations $\mathrm{Deck}(Y/X)$ such that for any two points y_1 and y_2 with $p(y_1) = p(y_2)$ there exists an element $\gamma \in \Gamma$ such that $\gamma(y_1) = y_2$. In other words, Γ acts transitively in each fiber. We call this property (**T**).

Remark 2.1. *Note that for any two points y_1 and y_2 there can be only one $\gamma \in \mathrm{Deck}(Y/X)$ with $\gamma(y_1) = y_2$ (see [5], Satz 4.8). Therefore, $\Gamma = \mathrm{Deck}(Y/X)$ and the property (**T**) simply means that $p : Y \to X$ is a normal (Galois) covering.*

We have an action of Γ on Y:

$$\Gamma \times Y \to Y, \quad y \mapsto \gamma(y) =: \gamma y.$$

Definition 2.2. *A holomorphic function $f : \Gamma \times Y \to \mathrm{GL}_r(\mathbb{C})$, $r \in \mathbb{N}$ is called an r-dimensional factor of automorphy if it satisfies the relation*

$$f(\lambda\mu, y) = f(\lambda, \mu y) f(\mu, y).$$

Denote by $Z^1(\Gamma, r)$ the set of all r-dimensional factors of automorphy.

We introduce the relation $\sim$ on $Z^1(\Gamma, r)$. We say that f is equivalent to f' if there exists a holomorphic function $h : Y \to \mathrm{GL}_r(\mathbb{C})$ such that

$$h(\lambda y) f(\lambda, y) = f'(\lambda, y) h(y).$$

We write in this case $f \sim f'$.

Lemma 2.3. *The relation $\sim$ is an equivalence relation on $Z^1(\Gamma, r)$.*

Proof. Straightforward verifications. □

We denote the set of equivalence classes of $Z^1(\Gamma, r)$ with respect to $\sim$ by $H^1(\Gamma, r)$.

Consider $f \in Z^1(\Gamma, r)$ and a trivial vector bundle $Y \times \mathbb{C}^r \to Y$. Define a holomorphic action of Γ on $Y \times \mathbb{C}^r$:

$$\Gamma \times Y \times \mathbb{C}^r \to Y \times \mathbb{C}^r, \quad (\lambda, y, v) \mapsto (\lambda y, f(\lambda, y)v) =: \lambda(y, v).$$

Denote $E(f) = Y \times \mathbb{C}^r/\Gamma$ and note that for two equivalent points $(y, v) \sim_\Gamma (y', v')$ with respect to the action of Γ on $Y \times \mathbb{C}^r$ it follows that $p(y) = p(y')$. In fact, $(y, v) \sim_\Gamma (y', v')$ implies in particular that $y = \gamma y'$ for some $\gamma \in \Gamma$ and by the definition of deck transformations $p(y) = p(\gamma y') = p(y')$. Hence the projection $Y \times \mathbb{C}^r \to Y$ induces the map

$$\pi : E(f) \to X, \quad [y, v] \mapsto p(y).$$

We equip $E(f)$ with the quotient topology.

Theorem 2.4. *$E(f)$ inherits a complex structure from $Y \times \mathbb{C}^r$ and the map $\pi : E(f) \to X$ is a holomorphic vector bundle on X.*

Proof. First we prove that π is a topological vector bundle. Clearly π is a continuous map. Consider the commutative diagram

$$\begin{array}{ccc} Y \times \mathbb{C}^r & \longrightarrow & E(f) \\ \big\downarrow & & \big\downarrow \pi \\ Y & \xrightarrow{\ p\ } & X. \end{array}$$

Let x be a point of X. Since p is a covering, one can choose an open neighbourhood U of x such that its preimage is a disjoint union of open sets biholomorphic to U, i. e., $p^{-1}(U) = \bigsqcup_{i \in \mathcal{I}} V_i$, $p_i := p|_{V_i} : V_i \to U$ is a biholomorphism for each $i \in \mathcal{I}$. For each pair $(i, j) \in \mathcal{I} \times \mathcal{I}$ there exists a unique $\lambda_{ij} \in \Gamma$ such that $\lambda_{ij} p_j^{-1}(x) = p_i^{-1}(x)$ for all $x \in U$. This follows from the property (**T**).

We have $\pi^{-1}(U) = ((\bigsqcup_{i \in \mathcal{I}} V_i) \times \mathbb{C}^r)/\Gamma$.

Choose some $i_U \in \mathcal{I}$. Consider the holomorphic map

$$\varphi'_U : \left(\bigsqcup_{i \in \mathcal{I}} V_i \right) \times \mathbb{C}^r \to U \times \mathbb{C}^r, \quad (y_i, v) \mapsto (p(y_i), f(\lambda_{i_U i}, y_i)v), \ y_i \in V_i.$$

Suppose that $(y_i, v') \sim_\Gamma (y_j, v)$. This means

$$(y_i, v') = \lambda_{ij}(y_j, v) = (\lambda_{ij} y_j, f(\lambda_{ij}, y_j)v).$$

Therefore,

$$\begin{aligned} \varphi'_U(y_i, v') &= (p(y_i), f(\lambda_{i_U i}, y_i)v') = (p(\lambda_{ij} y_j), f(\lambda_{i_U i}, \lambda_{ij} y_j) f(\lambda_{ij}, y_j)v) \\ &= (p(y_j), f(\lambda_{i_U j}, y_j)v) = \varphi'_U(y_j, v). \end{aligned}$$

Thus φ'_U factorizes through $((\bigsqcup_{i \in \mathcal{I}} V_i) \times \mathbb{C}^r)/\Gamma$, i. e., the map

$$\varphi_U : \left(\left(\bigsqcup_{i \in \mathcal{I}} V_i \right) \times \mathbb{C}^r \right) / \Gamma \to U \times \mathbb{C}^r, \quad [(y_i, v)] \mapsto (p(y_i), f(\lambda_{i_U i}, y_i)v), \ y_i \in V_i$$

is well-defined and continuous. We claim that φ_U is bijective.

Suppose $\varphi_U([(y_i, v')]) = \varphi_U([(y_j, v)])$, where $y_i \in V_i$, $y_j \in V_j$. By definition this is equivalent to $(p(y_i), f(\lambda_{i_U i}, y_i)v') = (p(y_j), f(\lambda_{i_U j}, y_j)v)$, which means $y_i = \lambda_{ij} y_j$ and

$$\begin{aligned} f(\lambda_{i_U i}, \lambda_{ij} y_j)v' &= f(\lambda_{i_U i}, y_i)v' = f(\lambda_{i_U j}, y_j)v \\ &= f(\lambda_{i_U i}\lambda_{ij}, y_j)v = f(\lambda_{i_U i}, \lambda_{ij} y_j) f(\lambda_{ij}, y_j)v. \end{aligned}$$

We conclude $v' = f(\lambda_{ij}, y_j)v$ and $[(y_i, v')] = [(y_j, v)]$, which means injectivity of φ_U.

At the same time for each element $(y, v) \in U \times \mathbb{C}^r$ one has

$$\begin{aligned}&\varphi_U([(p_i^{-1}(y), f(\lambda_{i_U i}, p_i^{-1}(y))^{-1}v)]) \\ &\qquad = (pp_i^{-1}(y), f(\lambda_{i_U i}, p_i^{-1}(y))f(\lambda_{i_U i}, p_i^{-1}(y))^{-1}v) = (y, v),\end{aligned}$$

i.e., φ_U is surjective and we obtain that φ_U is a bijective map.

This means, that φ_U is a trivialization for U and that $\pi : E(f) \to X$ is a (continuous) vector bundle. If U and V are two neighbourhoods of X defined as above for which $E(f)|_U$, $E(f)|_V$ are trivial, then the corresponding transition function is

$$\varphi_U\varphi_V^{-1} : (U \cap V) \times \mathbb{C}^r \to (U \cap V) \times \mathbb{C}^r, \quad (x, v) \mapsto (x, g_{UV}(x)v),$$

where $g_{UV} : U \cap V \to \mathrm{GL}_r(\mathbb{C})$ is a cocycle defining $E(f)$. But from the construction of φ_U it follows that

$$g_{UV}(x) = f(\lambda_{i_U i_V}, p_{i_V}^{-1}(x)).$$

Therefore, g_{UV} is a holomorphic map, hence $\varphi_U\varphi_V^{-1}$ is also a holomorphic map. Thus the maps φ_U give $E(f)$ a complex structure. Since π is locally a projection, one sees that π is a holomorphic map. □

REMARK 2.5. *Note that $p^*E(f)$ is isomorphic to $Y \times \mathbb{C}^r$. An isomorphism can be given by the map*

$$p^*E(f) \to Y \times \mathbb{C}^r, \quad (y, [\tilde{y}, v]) \mapsto (y, f(\lambda, \tilde{y})v), \quad \lambda\tilde{y} = y.$$

Now we have the map from $Z^1(\Gamma, r)$ to the set $K_r = \{[E] \mid p^*(E) \simeq Y \times \mathbb{C}^r\}$ of isomorphism classes of vector bundles of rank r over X with trivial pull back with respect to p.

$$\phi' : Z^1(\Gamma, r) \to K_r; \quad f \mapsto [E(f)].$$

THEOREM 2.6. *Let K_r denote the set of isomorphism classes of vector bundles of rank r on X with trivial pull back with respect to p. Then the map*

$$H^1(\Gamma, r) \to K_r, \quad [f] \mapsto [E(f)]$$

is a bijection.

Proof. This proof generalizes the proof from [2, Appendix B] given only for line bundles.

Consider the map $\phi' : Z^1(\Gamma, r) \to K_r$ and let f and f' be two equivalent r-dimensional factors of automorphy. It means that there exists a holomorphic function $h : Y \to \mathrm{GL}_r(\mathbb{C})$ such that

$$f'(\lambda, y) = h(\lambda y)f(\lambda, y)h(y)^{-1}.$$

Therefore, for two neighbourhoods U, V constructed as above we have the following relation for cocycles corresponding to f and f'.

$$\begin{aligned} g'_{UV}(x) &= f'(\lambda_{UV}, p_{i_V}^{-1}(x)) = h(\lambda_{UV} p_{i_V}^{-1}(x)) f(\lambda_{UV}, p_{i_V}^{-1}(x)) h(p_{i_V}^{-1}(x))^{-1} \\ &= h(p_{i_U}^{-1}(x)) g_{UV}(x) h(p_{i_V}^{-1}(x))^{-1} = h_U(x) g_{UV}(x) h_V(x)^{-1}, \end{aligned}$$

where $\lambda_{UV} = \lambda_{i_U i_V}$, $h_U(x) = h(p_{i_U}^{-1}(x))$ and $h_V(x) = h(p_{i_V}^{-1}(x))$. We obtained

$$g'_{UV} = h_U g_{UV} h_U^{-1},$$

which is exactly the condition for two cocycles to define isomorphic vector bundles. Therefore, $E(f) \simeq E(f')$ and it means that ϕ' factorizes through $H^1(\Gamma, r)$, i.e., the map

$$\phi : H^1(\Gamma, r) \to K_r; \quad [f] \mapsto [E(f)]$$

is well-defined.

It remains to construct the inverse map. Suppose $E \in K_r$, in other words $p^*(E)$ is the trivial bundle of rank r over Y. Let $\alpha : p^*E \to Y \times \mathbb{C}^r$ be a trivialization. The action of Γ on Y induces a holomorphic action of Γ on p^*E :

$$\lambda(y, e) := (\lambda y, e) \text{ for } (y, e) \in p^*E = Y \times_X E.$$

Via α we get for every $\lambda \in \Gamma$ an automorphism ψ_λ of the trivial bundle $Y \times \mathbb{C}^r$. Clearly ψ_λ should be of the form

$$\psi_\lambda(y, v) = (\lambda y, f(\lambda, y)v),$$

where $f : \Gamma \times Y \to \mathrm{GL}_r(\mathbb{C})$ is a holomorphic map. The equation for the action $\psi_{\lambda\mu} = \psi_\lambda \psi_\mu$ implies that f should be an r-dimensional factor of automorphy.

Suppose α' is an another trivialization of p^*E. Then there exists a holomorphic map $h : Y \to \mathrm{GL}_r(\mathbb{C})$ such that $\alpha'\alpha^{-1}(y, v) = (y, h(y)v)$. Let f' be a factor of automorphy corresponding to α'. From

$$\begin{aligned} (\lambda y, f'(\lambda, y)v) &= \psi'_\lambda(y, v) = \alpha'\lambda\alpha'^{-1}(y, v) = \alpha'\alpha^{-1}\alpha\lambda\alpha^{-1}\alpha\alpha'^{-1}(y, v) \\ &= \alpha'\alpha^{-1}\psi_\lambda(\alpha'\alpha^{-1})^{-1}(y, v) = \alpha'\alpha^{-1}\psi_\lambda(y, h(y)^{-1}v) \\ &= \alpha'\alpha^{-1}(\lambda y, f(\lambda, y)h(y)^{-1}v) = (\lambda y, h(\lambda y)f(\lambda, y)h(y)^{-1}), \end{aligned}$$

we obtain $f'(\lambda, y) = h(\lambda y)f(\lambda, y)h(y)^{-1}$. The last means that $[f] = [f']$, in other words, the class of a factor of automorphy in $H^1(\Gamma, r)$ does not depend on the trivialization and we get a map $K_r \to H^1(\Gamma, r)$. This map is the inverse of ϕ. □

Let X be a connected complex manifold, let $p : \tilde{X} \to X$ be a universal covering of X, and let $\Gamma = \mathrm{Deck}(Y/X)$. Since universal coverings are normal coverings, Γ satisfies the property (**T**) (see [5, Satz 5.6]). Moreover, Γ is isomorphic to the fundamental group $\pi_1(X)$ of X (see [5, Satz 5.6]). An isomorphism is given as follows.

Fix $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ with $p(\tilde{x}_0) = x_0$. We define a map

$$\Phi : \mathrm{Deck}(\tilde{X}/X) \to \pi_1(X, x_0)$$

as follows. Let $\sigma \in \mathrm{Deck}(\tilde{X}/X)$ and $v : [0; 1] \to \tilde{X}$ be a curve with $v(0) = \tilde{x}_0$ and $v(1) = \sigma(\tilde{x}_0)$. Then a curve

$$pv : [0; 1] \to X, \quad t \mapsto pv(t)$$

is such that $pv(0) = pv(1) = x_0$. Define $\Phi(\sigma) := [pv]$, where $[pv]$ denotes a homotopy class of pv. The map Φ is well defined and is an isomorphism of groups.

So we can identify Γ with $\pi_1(X)$. Therefore, we have an action of $\pi_1(X)$ on $\tilde{X}$ by deck transformations.

Consider an element $[w] \in \pi_1(X, x_0)$ represented by a path $w : [0; 1] \to X$. We denote $\sigma = \Phi^{-1}([w])$. Consider any $\tilde{x}_0 \in X$ such that $p(\tilde{x}_0) = w(0) = w(1)$, then the path w can be uniquely lifted to the path

$$v : [0; 1] \to \tilde{X}$$

with $v(0) = \tilde{x}_0$ (see [5], Satz 4.14). Denote $\tilde{x}_1 = v(1)$. Then σ is a unique element in $\mathrm{Deck}(\tilde{X}/X)$ such that $\sigma(\tilde{x}_0) = \tilde{x}_1$. This gives a description of the action of $\pi_1(X, x_0)$ on $\tilde{X}$.

Now we have a corollary to Theorem 2.6.

Corollary 2.7. *Let X be a connected complex manifold, let $p : \tilde{X} \to X$ be the universal covering, let Γ be the fundamental group of X naturally acting on $\tilde{X}$ by deck transformations. As above, $H^1(\Gamma, r)$ denotes the set of equivalence classes of r-dimensional factors of automorphy*

$$\Gamma \times \tilde{X} \to \mathrm{GL}_r(\mathbb{C}).$$

Then there is a bijection

$$H^1(\Gamma, r) \to K_r, \quad [f] \mapsto E(f),$$

where K_r denotes the set of isomorphism classes of vector bundles of rank r on X with trivial pull back with respect to p.

3. Properties of Factors of Automorphy

Definition 3.1. *Let* $f : \Gamma \times Y \to \mathrm{GL}_r(\mathbb{C})$ *be an* r*-dimensional factor of automorphy. A holomorphic function* $s : Y \to \mathbb{C}^r$ *is called an* f*-theta function if it satisfies*

$$s(\gamma y) = f(\gamma, y)s(y) \text{ for all } \gamma \in \Gamma,\ y \in Y.$$

Theorem 3.2. *Let* $f : \Gamma \times Y \to \mathrm{GL}_r(\mathbb{C})$ *be an* r*-dimensional factor of automorphy. Then there is a one-to-one correspondence between sections of* $E(f)$ *and* f*-theta functions.*

Proof. Let $\{V_i\}_{i\in\mathcal{I}}$ be a covering of Y such that p restricted to V_i is a homeomorphism. Denote $\varphi_i := (p|_{V_i})^{-1}$, $U_i := p(V_i)$. Then $\{U_i\}$ is a covering of X such that $E(f)$ is trivial over each U_i.

Consider a section of $E(f)$ given by functions $s_i : U_i \to \mathbb{C}^r$ satisfying

$$s_i(x) = g_{ij}(x)s_j(x) \text{ for } x \in U_i \cap U_j,$$

where

$$g_{ij}(x) = f(\lambda_{U_iU_j}, \varphi_j(x)), \quad x \in U_i \cap U_j$$

is a cocycle defining $E(f)$ (see the proof of Theorem 2.6). Define $s : Y \to \mathbb{C}^r$ by $s(\varphi_i(x)) := s_i(x)$. To prove that this is well-defined we need to show that $s_i(x) = s_j(x)$ when $\varphi_i(x) = \varphi_j(x)$. But since $\varphi_i(x) = \varphi_j(x)$ we obtain $\lambda_{U_iU_j} = 1$. Therefore,

$$s_i(x) = g_{ij}(x)s_i(x) = f(\lambda_{U_iU_j}, \varphi_j(x))s_j(x) = f(1, \varphi_j(x))s_j(x) = s_j(x).$$

For any $\gamma \in \Gamma$ for any point $y \in Y$ take $i, j \in \mathcal{I}$ and $x \in X$ such that $y = \varphi_j(x)$ and $\gamma y = \gamma\varphi_j(x) = \varphi_i(x)$. Thus $\gamma = \lambda_{U_iU_j}$ and one obtains

$$\begin{aligned} s(\gamma y) &= s(\varphi_i(x)) = s_i(x) = g_{ij}(x)s_j(x) \\ &= f(\lambda_{U_iU_j}, \varphi_j(x))s_j(x) = f(\gamma, y)s(\varphi_j(x)) = f(\gamma, z)s(y). \end{aligned}$$

In other words, s is an f-theta function.

Vice versa, let $s : Y \to \mathbb{C}^r$ be an f-theta function. We define $s_i : U_i \to \mathbb{C}^r$ by $s_i(x) := s(\varphi_i(x))$. Then for a point $x \in U_i \cap U_j$ we have

$$\begin{aligned} s_i(x) &= s(\varphi_i(x)) = s(\lambda_{U_iU_j}\varphi_j(x)) \\ &= f(\lambda_{U_iU_j}, \varphi_j(x))s(\varphi_j(x)) = g_{ij}(x)s_j(x), \end{aligned}$$

which means that the functions s_i define a section of $E(f)$. The described correspondences are clearly inverse to each other. □

The following statement will be useful in the sequel.

THEOREM 3.3. *Let*

$$f(\lambda, y) = \begin{pmatrix} f'(\lambda, y) & \tilde{f}(\lambda, y) \\ 0 & f''(\lambda, y) \end{pmatrix}$$

be an $r' + r''$-dimensional factor of automorphy, where $f'(\lambda, y) \in \mathrm{GL}_{r'}(\mathbb{C})$, $f''(\lambda, y) \in \mathrm{GL}_{r''}(\mathbb{C})$. Then

(a) $f' : \Gamma \times Y \to \mathrm{GL}_{r'}(\mathbb{C})$ and $f'' : \Gamma \times Y \to \mathrm{GL}_{r''}(\mathbb{C})$ are r' and r''-dimensional factors of automorphy respectively;

(b) there is an extension of vector bundles

$$0 \longrightarrow E(f') \xrightarrow{\ i\ } E(f) \xrightarrow{\ \pi\ } E(f'') \longrightarrow 0\ .$$

Proof. The statement of (a) follows from straightforward verification. To prove (b) we define maps i and π as follows.

$$i : E(f') \to E(f), \qquad [y, v] \mapsto [y, \begin{pmatrix} v \\ 0 \end{pmatrix}], \qquad v \in \mathbb{C}^{r'}, \quad \begin{pmatrix} v \\ 0 \end{pmatrix} \in \mathbb{C}^{r'+r''}$$

$$\pi : E(f) \to E(f''), \qquad [y, \begin{pmatrix} v \\ w \end{pmatrix}] \to [y, w], \qquad v \in \mathbb{C}^{r'}, \quad w \in \mathbb{C}^{r''}$$

Since $[\lambda y, f'(\lambda, y)v]$ is mapped via i to

$$\left[\lambda y, \begin{pmatrix} f'(\lambda, y)v \\ 0 \end{pmatrix}\right] = \left[\lambda y, f(\lambda, y) \begin{pmatrix} v \\ 0 \end{pmatrix}\right],$$

one concludes that i is well-defined. Analogously, since $[\lambda y, f''(\lambda, y)w] = [y, w]$ one sees that π is well-defined. Using the charts from the proof of (2.4) one easily sees that the defined maps are holomorphic.

Notice that i and π respect fibers, i is injective and π is surjective in each fiber. This proves the statement. □

Now we recall one standard construction from linear algebra. Let A be an $m \times n$ matrix. It represents some morphism $\mathbb{C}^n \to \mathbb{C}^m$ for fixed standard bases in $\mathbb{C}^n$ and $\mathbb{C}^m$.

Let $\mathcal{F} : \mathsf{Vect}^p \to \mathsf{Vect}$ be a covariant functor. Let $A_1, \ldots, A_p$ be the matrices representing morphisms $\mathbb{C}_1^n \xrightarrow{f_1} \mathbb{C}_1^m, \ldots, \mathbb{C}_p^n \xrightarrow{f_p} \mathbb{C}_p^m$ in standard bases.

If for each object $\mathcal{F}(\mathbb{C}^m)$ we fix some basis, then the matrix corresponding to the morphism $\mathcal{F}(f_1, \ldots, f_p)$ is denoted by $\mathcal{F}(A_1, \ldots, A_p)$. Clearly it satisfies

$$\mathcal{F}(A_1 B_1, \ldots, A_p B_p) = \mathcal{F}(A_1, \ldots, A_p)\mathcal{F}(B_1, \ldots, B_p).$$

In this way $A \otimes B$, $S^q(A)$, $\Lambda^q(A)$ can be defined. As $\mathcal{F}$ one considers the functors

$$\_ \otimes \_ : \mathsf{Vect}^2 \to \mathsf{Vect}, \quad S^n : \mathsf{Vect} \to \mathsf{Vect}, \quad \Lambda : \mathsf{Vect} \to \mathsf{Vect}$$

respectively.

Recall that every holomorphic functor $\mathcal{F} : \mathsf{Vect}^n \to \mathsf{Vect}$ can be canonically extended to the category of vector bundles of finite rank over X. By abuse of notation we will denote the extended functor by $\mathcal{F}$ as well.

THEOREM 3.4. *Let $\mathcal{F} : \mathsf{Vect}^n \to \mathsf{Vect}$ be a covariant holomorphic functor. Let $f_1, \dots, f_n$ be r_i-dimensional factors of automorphy. Then $f = \mathcal{F}(f_1, \dots, f_n)$ is a factor of automorphy defining $\mathcal{F}(E(f_1), \dots, E(f_n))$.*

Proof. One clearly has

$$\begin{aligned}\mathcal{F}(f_1, \dots, f_n)(\lambda\mu, y) &= \mathcal{F}(f_1(\lambda\mu, y), \dots, f_n(\lambda\mu, y)) \\ &= \mathcal{F}(f_1(\lambda, \mu y) f_1(\mu, y), \dots, f_n(\lambda, \mu y) f_n(\mu, y)) \\ &= F(f_1(\lambda, \mu y), \dots, f_n(\lambda, \mu y)) F(f_1(\mu, y), \dots, f_n(\mu, y)) \\ &= \mathcal{F}(f_1, \dots, f_n)(\lambda, \mu y) \mathcal{F}(f_1, \dots, f_n)(\mu, y).\end{aligned}$$

Since $(f_1, \dots, f_n)$ represents an isomorphism in Vect^n, $\mathcal{F}(f_1, \dots, f_n)$ also represents an isomorphism $\mathbb{C}^r \to \mathbb{C}^r$ for some $r \in \mathbb{N}$. Therefore, f is an r-dimensional factor of automorphy.

Since $f = \mathcal{F}(f_1, \dots, f_n)$, for cocycles defining the corresponding vector bundles the equality $g_{U_1U_2} = \mathcal{F}(g_{1U_1U_2}, \dots, g_{nU_1U_2})$ holds true, where $g_{iU_1U_2}$ is a cocycle defining $E(f_i)$. This shows that $E(f) = \mathcal{F}(E(f_1), \dots, E(f_n))$ and proves the required statement. □

For example for $\mathcal{F} = \_ \otimes \_ : \mathsf{Vect}^2 \to \mathsf{Vect}$ we get the following obvious corollary.

COROLLARY 3.5. *Let $f' : \Gamma \times Y \to \mathrm{GL}_{r'}(\mathbb{C})$ and $f'' : \Gamma \times Y \to \mathrm{GL}_{r''}(\mathbb{C})$ be two factors of automorphy. Then $f = f' \otimes f'' : \Gamma \times Y \to \mathrm{GL}_{r'r''}(\mathbb{C})$ is also a factor of automorphy. Moreover, $E(f) \simeq E(f') \otimes E(f'')$.*

It is not essential that the functor in Theorem 3.4 is covariant. The following theorem is a generalization of Theorem 3.4.

THEOREM 3.6. *Let $\mathcal{F} : \mathsf{Vect}^n \to \mathsf{Vect}$ be a holomorphic functor. Let $\mathcal{F}$ be covariant in k first variables and contravariant in $n - k$ last variables. Let $f_1, \dots, f_n$ be r_i-dimensional factors of automorphy. Then*

$$f = \mathcal{F}(f_1, \dots, f_k, f_{k+1}^{-1}, \dots, f_n^{-1})$$

is a factor of automorphy defining $\mathcal{F}(E(f_1), \dots, E(f_n))$.

Proof. The proof is analogous to the proof of Theorem 3.4. □

4. Vector Bundles on Complex Tori

4.1. One Dimensional Complex Tori

Let X be a complex torus, i.e., $X = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z}\tau + \mathbb{Z}$, $\operatorname{Im}\tau > 0$. Then the universal covering is $\tilde{X} = \mathbb{C}$, namely

$$\mathrm{pr} : \mathbb{C} \to \mathbb{C}/\Gamma, \quad x \mapsto [x].$$

We have an action of Γ on $\mathbb{C}$:

$$\Gamma \times \mathbb{C} \to \mathbb{C}, \quad (\gamma, y) \mapsto \gamma + y.$$

Clearly Γ acts on $\mathbb{C}$ by deck transformations and satisfies the property (**T**).

Since $\mathbb{C}$ is a non-compact Riemann surface, by [5, Theorem 30.4, p. 204], there are only trivial bundles on $\mathbb{C}$. Therefore, we have a one-to-one correspondence between classes of isomorphism of vector bundles of rank r on X and equivalence classes of factors of automorphy

$$f : \Gamma \times \mathbb{C} \to \mathrm{GL}_r(\mathbb{C}).$$

As usually, V_a denotes the standard parallelogram constructed at point a, U_a is the image of V_a under the projection, $\varphi_a : U_a \to V_a$ is the local inverse of the projection.

Remark 4.1. *Let f be an r-dimensional factor of automorphy. Then*

$$g_{ab}(x) = f(\varphi_a(x) - \varphi_b(x), \varphi_b(x))$$

is a cocycle defining $E(f)$. This follows from the construction of the cocycle in the proof of Theorem 2.6.

Example 4.2. *There are factors of automorphy corresponding to classical theta functions. For any theta-characteristic $\xi = a\tau + b$, where $a, b \in \mathbb{R}$, there is a holomorphic function $\theta_\xi : \mathbb{C} \to \mathbb{C}$ defined by*

$$\theta_\xi(z) = \sum_{n\in\mathbb{Z}} \exp(\pi i(n + a)^2\tau)\exp(2\pi i(n + a)(z + b)),$$

which satisfies

$$\theta_\xi(\gamma + z) = \exp(2\pi ia\gamma - \pi ip^2\tau - 2\pi ip(z + \xi))\theta_\xi(z) = e_\xi(\gamma, z)\theta_\xi(z),$$

where $\gamma = p\tau + q$ and $e_\xi(\gamma, z) = \exp(2\pi ia\gamma - \pi ip^2\tau - 2\pi ip(z + \xi))$. Since

$$e_\xi(\gamma_1 + \gamma_2, z) = e_\xi(\gamma_1, \gamma_2 + z)e_\xi(\gamma_2, z),$$

we conclude that $e_\xi(\gamma, z)$ is a factor of automorphy.

By Theorem 3.2 $\theta_\xi(z)$ defines a section of $E(e_\xi(\gamma, z))$.

For more information on classical theta functions see [8, 9, 10].

THEOREM 4.3. $\deg E(e_\xi) = 1$.

Proof. We know that sections of $E(e_\xi)$ correspond to e_ξ - theta functions. The classical e_ξ-theta function $\theta_\xi(z)$ defines a section s_ξ of $E(e_\xi)$. Since θ_ξ has only simple zeros and the set of zeros of $\theta_\xi(z)$ is $\frac{1}{2}+\frac{\tau}{2}+\xi+\Gamma$, we conclude that s_ξ has exactly one zero at point $p = [\frac{1}{2}+\frac{\tau}{2}+\xi] \in X$. Hence by [6, p. 136] we get $E(e_\xi) \simeq [p]$ and thus $\deg E(e_\xi) = 1$. ☐

THEOREM 4.4. *Let ξ and η be two theta-characteristics. Then*

$$E(e_\xi) \simeq t^*_{[\eta-\xi]} E(e_\eta),$$

where $t_{[\eta-\xi]} : X \to X, \quad x \mapsto x + [\eta - \xi]$ is the translation by $[\eta - \xi]$.

Proof. As in the proof of Theorem 4.3 $E(e_\xi) \simeq [p]$ and $E(e_\eta) = [q]$ for $p = [\frac{1}{2}+\frac{\tau}{2}+\xi]$ and $q = [\frac{1}{2}+\frac{\tau}{2}+\eta]$. Since $t_{[\eta-\xi]}p = q$, we get

$$E(e_\xi) \simeq [p] \simeq t^*_{[\eta-\xi]}[q] \simeq t^*_{[\eta-\xi]} E(e_\eta),$$

which completes the proof. ☐

Now we are going to investigate the extensions of the type

$$0 \to X \times \mathbb{C} \to E \to X \times \mathbb{C} \to 0.$$

In this case the transition functions are given by matrices of the type

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

and E is isomorphic to $E(f)$ for some factor of automorphy f of the form

$$f(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}.$$

Note that the condition for f to be a factor of automorphy in this case is equivalent to the condition

$$\mu(\lambda + \lambda', \tilde{x}) = \mu(\lambda, \lambda' + \tilde{x}) + \mu(\lambda', \tilde{x}),$$

where we use the additive notation for the group operation since Γ is commutative.

THEOREM 4.5. *f defines the trivial bundle if and only if $\mu(\lambda, \tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x})$ for some holomorphic function $\xi : \mathbb{C} \to \mathbb{C}$.*

Proof. We know that E is trivial if and only if $h(\lambda\tilde{x}) = f(\lambda,\tilde{x})h(\tilde{x})$ for some holomorphic function $h : \tilde{X} \to \mathrm{GL}_2(\mathbb{C})$. Let

$$h = \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix},$$

then the last condition is

$$\begin{aligned}\begin{pmatrix} a(\lambda\tilde{x}) & b(\lambda\tilde{x}) \\ c(\lambda\tilde{x}) & d(\lambda\tilde{x}) \end{pmatrix} &= \begin{pmatrix} 1 & \mu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix} \\ &= \begin{pmatrix} a(\tilde{x}) + c(\tilde{x})\mu(\lambda,\tilde{x}) & b(\tilde{x}) + d(\tilde{x})\mu(\lambda,\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix}.\end{aligned}$$

In particular it means $c(\lambda\tilde{x}) = c(\tilde{x})$ and $d(\lambda\tilde{x}) = d(\tilde{x})$, i.e., c and d are doubly periodic functions on $\tilde{X} = \mathbb{C}$, so they should be constant, i.e., $c(\lambda,\tilde{x}) = c \in \mathbb{C}$, $d(\lambda,\tilde{x}) = d \in \mathbb{C}$.

Now we have

$$\begin{aligned} a(\tilde{x}) + c\mu(\lambda,\tilde{x}) &= a(\lambda\tilde{x}) \\ b(\tilde{x}) + d\mu(\lambda,\tilde{x}) &= b(\lambda\tilde{x}) \end{aligned}$$

which implies

$$\begin{aligned} c\mu(\lambda,\tilde{x}) &= a(\lambda\tilde{x}) - a(\tilde{x}) \\ d\mu(\lambda,\tilde{x}) &= b(\lambda\tilde{x}) - b(\tilde{x}). \end{aligned}$$

Since $\det h(\tilde{x}) \neq 0$ for all $\tilde{x} \in \tilde{X} = \mathbb{C}$ one of the numbers c and d is not equal to zero. Therefore, one concludes that $\mu(\lambda,\tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x})$ for some holomorphic function $\xi : \tilde{X} = \mathbb{C} \to \mathbb{C}$.

Now suppose $\mu(\lambda,\tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x})$ for some holomorphic function $\xi : \mathbb{C} \to \mathbb{C}$. Clearly for $h(\tilde{x}) = \begin{pmatrix} 1 & \xi(\tilde{x}) \\ 0 & 1 \end{pmatrix}$ one has that $\det h(\tilde{x}) = 1 \neq 0$ and

$$\begin{aligned} f(\lambda,\tilde{x})h(\tilde{x}) &= \begin{pmatrix} 1 & \mu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \xi(\tilde{x}) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \xi(\tilde{x}) + \mu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \xi(\lambda\tilde{x}) \\ 0 & 1 \end{pmatrix} = h(\lambda\tilde{x}). \end{aligned}$$

We have shown, that f defines the trivial bundle. This proves the statement of the theorem. □

THEOREM 4.6. *Two factors of automorphy*

$$f(\lambda,\tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad f'(\lambda,\tilde{x}) = \begin{pmatrix} 1 & \nu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix}$$

defining non-trivial bundles are equivalent if and only if

$$\mu(\lambda,\tilde{x}) - k\nu(\lambda,\tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x}), \quad k \in \mathbb{C}, \quad k \neq 0$$

for some holomorphic function $\xi : \mathbb{C} = \tilde{X} \to \mathbb{C}$.

Proof. Suppose that the factors of automorphy

$$f(\lambda,\tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix}$$

and

$$f'(\lambda,\tilde{x}) = \begin{pmatrix} 1 & \nu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix}$$

are equivalent. Then there is an equality $f(\lambda,\tilde{x})h(\tilde{x}) = h(\lambda\tilde{x})f(\lambda,\tilde{x})$ for some holomorphic function $h : \mathbb{C} = \tilde{X} \to \mathrm{GL}_2(\mathbb{C})$. Let us write h in the form

$$h(\tilde{x}) = \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix}.$$

Then the condition for equivalence of f and f' can be rewritten as follows:

$$\begin{pmatrix} 1 & \mu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix} = \begin{pmatrix} a(\lambda\tilde{x}) & b(\lambda\tilde{x}) \\ c(\lambda\tilde{x}) & d(\lambda\tilde{x}) \end{pmatrix} \begin{pmatrix} 1 & \nu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix}$$

After multiplication one obtains

$$\begin{pmatrix} a(\tilde{x}) + c(\tilde{x})\mu(\lambda,\tilde{x}) & b(\tilde{x}) + d(\tilde{x})\mu(\lambda,\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix} = \begin{pmatrix} a(\lambda\tilde{x}) & a(\lambda\tilde{x})\nu(\lambda,\tilde{x}) + b(\lambda\tilde{x}) \\ c(\lambda\tilde{x}) & c(\lambda\tilde{x})\nu(\lambda,\tilde{x}) + d(\lambda\tilde{x}) \end{pmatrix},$$

which leads to the system of equations

$$\begin{cases} a(\tilde{x}) + c(\tilde{x})\mu(\lambda,\tilde{x}) = a(\lambda\tilde{x}) \\ b(\tilde{x}) + d(\tilde{x})\mu(\lambda,\tilde{x}) = a(\lambda\tilde{x})\nu(\lambda,\tilde{x}) + b(\lambda\tilde{x}) \\ c(\tilde{x}) = c(\lambda\tilde{x}) \\ d(\tilde{x}) = c(\lambda\tilde{x})\nu(\lambda,\tilde{x}) + d(\lambda\tilde{x}). \end{cases}$$

The third equation means that c is a double periodic function. Therefore, c should be a constant function.

If $c \neq 0$ from the first and the last equations using Theorem 4.5 one concludes that f and f' define the trivial bundle.

In the case $c = 0$ one has

$$\begin{cases} a(\tilde{x}) = a(\lambda\tilde{x}) \\ b(\tilde{x}) + d(\tilde{x})\mu(\lambda,\tilde{x}) = a(\lambda\tilde{x})\nu(\lambda,\tilde{x}) + b(\lambda\tilde{x}) \\ d(\tilde{x}) = d(\lambda\tilde{x}), \end{cases}$$

i.e., as above, a and d are constant and both not equal to zero since $\det(h) \neq 0$. Finally one concludes that

$$d\mu(\lambda, \tilde{x}) - a\nu(\lambda, \tilde{x}) = b(\lambda\tilde{x}) - b(\tilde{x}), \quad a, d \in \mathbb{C}, \quad ad \neq 0 \tag{1}$$

Vice versa, if μ and ν satisfy (1) for

$$h(\tilde{x}) = \begin{pmatrix} a & b(\tilde{x}) \\ 0 & d \end{pmatrix}$$

we have

$$\begin{aligned} f(\lambda, \tilde{x})h(\tilde{x}) &= \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b(\tilde{x}) \\ 0 & d \end{pmatrix} \\ &= \begin{pmatrix} a & b(\tilde{x}) + d\mu(\lambda, \tilde{x}) \\ 0 & d \end{pmatrix} \\ &= \begin{pmatrix} a & b(\lambda\tilde{x}) + a\nu(\lambda, \tilde{x}) \\ 0 & d \end{pmatrix} \\ &= \begin{pmatrix} a & b(\lambda\tilde{x}) \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & \nu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} = h(\lambda\tilde{x})f(\lambda, \tilde{x}). \end{aligned}$$

This means that f and f' are equivalent. □

4.2. Higher Dimensional Complex Tori

One can also consider higher dimensional complex tori. Let $\Gamma \subset \mathbb{C}^g$ be a lattice,

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_g, \quad \Gamma_i = \mathbb{Z} + \mathbb{Z}\tau_i, \quad \operatorname{Im} \tau > 0.$$

Then as for one dimensional complex tori we obtain that $X = \mathbb{C}^g/\Gamma$ is a complex manifold. Clearly the map

$$\mathbb{C}^g \to \mathbb{C}^g/\Gamma = X, \quad x \mapsto [x]$$

is the universal covering of X. Since all vector bundles on $\mathbb{C}^g$ are trivial, we obtain a one-to-one correspondence between equivalence classes of r-dimensional factors of automorphy

$$f : \Gamma \times \mathbb{C}^g \to \mathrm{GL}_r(\mathbb{C})$$

and vector bundles of rank r on X.

Let $\Gamma = \mathbb{Z}^g + \Omega\mathbb{Z}^g$, where Ω is a symmetric complex $g \times g$ matrix with positive definite real part. Note that Ω is a generalization of τ from one dimensional case.

For any theta-characteristic $\xi = \Omega a + b$, where $a \in \mathbb{R}^g$, $b \in \mathbb{R}^g$ there is a holomorphic function $\theta_\xi : \mathbb{C}^g \to \mathbb{C}$ defined by

$$\theta_\xi(z) = \theta_b^a(z,\Omega) = \sum_{n\in\mathbb{Z}^g} \exp(\ \pi i(n+a)^t\Omega(n+a)\tau\ )\exp(\ 2\pi i(n+a)^t\Omega(z+b)),$$

which satisfies

$$\theta_\xi(\gamma+z) = \exp(2\pi i a^t\gamma - \pi i p^t\Omega p - 2\pi i p^t(z+\xi))\theta_\xi(z) = e_\xi(\gamma,z)\theta_\xi(z),$$

where $\gamma = \Omega p + q$ and $e_\xi(\gamma,z) = \exp(2\pi i a^t\gamma - \pi i p^t\Omega p - 2\pi i p^t(z+\xi))$. Since

$$e_\xi(\gamma_1+\gamma_2,z) = e_\xi(\gamma_1,\gamma_2+z)e_\xi(\gamma_2,z),$$

we conclude that $e_\xi(\gamma,z)$ is a factor of automorphy.

As above $\theta_\xi(z)$ defines a section of $E(e_\xi(\gamma,z))$.

For more detailed information on higher dimensional theta functions see [8, 9, 10].

4.3. Factors of Automorphy depending only on the τ-Direction of the Lattice Γ

Here X is a complex torus, $X = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z}\tau+\mathbb{Z}$, $\operatorname{Im}\tau > 0$. Denote $q = e^{2\pi i\tau}$. Consider the canonical projection

$$\mathrm{pr} : \mathbb{C}^* \to \mathbb{C}^*/<q>, \quad u \to [u] = u<q>.$$

Clearly one can equip $\mathbb{C}^*/<q>$ with the quotient topology. Therefore, there is a natural complex structure on $\mathbb{C}^*/<q>$.

Consider the homomorphism

$$\mathbb{C} \overset{\exp}{\to} \mathbb{C}^* \overset{\mathrm{pr}}{\to} \mathbb{C}^*/<q>, \quad z \mapsto e^{2\pi i z} \mapsto [e^{2\pi i z}].$$

It is clearly surjective. An element $z \in \mathbb{C}$ is in the kernel of this homomorphism if and only if $e^{2\pi i z} = q^k = e^{2\pi i k\tau}$ for some integer k. But this holds if and only if $z - k\tau \in \mathbb{Z}$ or, in other words, if $z \in \Gamma$. Therefore, the kernel of the map is exactly Γ, and we obtain an isomorphism of groups

$$iso : \mathbb{C}/\Gamma \to \mathbb{C}^*/<q>= \mathbb{C}^*/\mathbb{Z}, \quad [z] \to [e^{2\pi i z}].$$

Since the diagram

$$\begin{array}{ccccc} \mathbb{C} & \xrightarrow{\mathrm{pr}} & \mathbb{C}/\Gamma & & \\ \downarrow{\scriptstyle\exp} & & \updownarrow{\scriptstyle iso} & & \\ \mathbb{C}^* & \xrightarrow{\mathrm{pr}} & \mathbb{C}^*/\mathbb{Z} & = & \mathbb{C}^*/<q> \end{array}$$

is commutative, we conclude that the complex structure on $\mathbb{C}^*/<q>$ inherited from $\mathbb{C}/\Gamma$ by the isomorphism *iso* coincides with the natural complex structure on $\mathbb{C}^*/<q>$. Therefore, *iso* is an isomorphism of complex manifolds. Thus complex tori can be represented as $\mathbb{C}^*/<q>$, where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$.

So for any complex torus $X = \mathbb{C}^*/<q>$ we have a natural surjective holomorphic map

$$\mathbb{C}^* \to \mathbb{C}^*/<q>= X, \quad u \to [u].$$

This map is moreover a covering of X. Consider the group $\mathbb{Z}$. It acts holomorphically on $X = \mathbb{C}^*$:

$$\mathbb{Z} \times \mathbb{C}^* \to \mathbb{C}^*, \quad (n,u) \mapsto q^n u.$$

Moreover, since $\operatorname{pr}(q^n u) = \operatorname{pr}(u)$, $\mathbb{Z}$ is naturally identified with a subgroup in the group of deck transformations $\operatorname{Deck}(X/\mathbb{C}^*)$. It is easy to see that $\mathbb{Z}$ satisfies the property (**T**). We obtain that there is a one-to-one correspondence between classes of isomorphism of vector bundles over X and classes of equivalence of factors of automorphy

$$f : \mathbb{Z} \times \mathbb{C} \to \mathrm{GL}_r(\mathbb{C}).$$

Consider the following action of Γ on $\mathbb{C}^*$:

$$\Gamma \times \mathbb{C}^* \to \mathbb{C}^*; \quad (\lambda, u) \mapsto \lambda u =: e^{2\pi i\lambda} u$$

Let $A : \Gamma \times \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$ be a holomorphic function satisfying

$$A(\lambda + \lambda', u) = A(\lambda, \lambda' u) A(\lambda', u) \tag{$*$}$$

for all $\lambda, \lambda' \in \Gamma$. We call such functions $\mathbb{C}^*$-factors of automorphy. Consider the map

$$\mathrm{id}_\Gamma \times \exp : \Gamma \times \mathbb{C} \to \Gamma \times \mathbb{C}^*, \quad (\lambda, x) \to (\lambda, e^{2\pi i x})$$

Then the function

$$f_A = A \circ (\mathrm{id}_\Gamma \times \exp) : \Gamma \times \mathbb{C} \to \mathrm{GL}_r(\mathbb{C})$$

is an r-dimensional factor of automorphy, because

$$\begin{aligned} f_A(\lambda + \lambda', x) = A(\lambda + \lambda', e^{2\pi i x}) &= A(\lambda, e^{2\pi i\lambda'} e^{2\pi i x}) A(\lambda', e^{2\pi i x}) \\ &= A(\lambda, e^{2\pi i(\lambda' + x)}) A(\lambda', e^{2\pi i x}) \\ &= f_A(\lambda, \lambda' + x) f_A(\lambda', x). \end{aligned}$$

So, factors of automorphy on $\mathbb{C}^*$ define factors of automorphy on $\mathbb{C}$.

We restrict ourselves to factors of automorphy $f : \Gamma \times \mathbb{C} \to \mathrm{GL}_r(\mathbb{C})$ with the property

$$f(m\tau + n, x) = f(m\tau, x), \quad m, n \in \mathbb{Z}. \tag{2}$$

It follows from this property that $f(n, x) = f(0, x) = \mathrm{id}_{\mathbb{C}^r}$. Therefore,

$$f(\lambda + k, x) = f(\lambda, k + x)f(k, x) = f(\lambda, k + x) \text{ for all } \lambda \in \Gamma,\ k \in \mathbb{Z}$$

and it is possible to define the function

$$A_f : \Gamma \times \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C}), \quad (\lambda, e^{2\pi i x}) \mapsto f(\lambda, x),$$

which is well-defined because from $e^{2\pi i x_1} = e^{2\pi i x_2}$ follows $x_1 = x_2 + k$ for some $k \in \mathbb{Z}$ and $f(\lambda, x_1) = f(\lambda, x_2 + k) = f(\lambda, x_2)$.

Consider A with the property $A(m\tau + n, u) = A(m\tau, u) =: A(m, u)$. Then clearly $f_A(m\tau + n, u) = f_A(m\tau, u)$. So for any $\mathbb{C}^*$-factor of automorphy $A : \Gamma \times \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$ with the property $A(m\tau + n, u) = A(m\tau, u)$ one obtains the factor of automorphy f_A satisfying (2). We proved the following

THEOREM 4.7. *Factors of automorphy $f : \Gamma \times \mathbb{C} \to \mathrm{GL}_r(\mathbb{C})$ with the property (2) are in a one-to-one correspondence with $\mathbb{C}^*$-factors of automorphy with property $A(m\tau + n, u) = A(m\tau, u)$.*

Now we want to translate the conditions for factors of automorphy with the property (2) to be equivalent in the language of $\mathbb{C}^*$-factors of automorphy with the same property.

THEOREM 4.8. *Let f, f' be r-factors of automorphy with the property (2). Then $f \sim f'$ if and only if there exists a holomorphic function $B : \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$ such that*

$$A_f(m, u)B(u) = B(q^m u)A_{f'}(m, u))$$

for $q := e^{2\pi i \tau}$, where $A(m, u) := A(m\tau, u)$. In this case we also say A_f is equivalent to $A_{f'}$ and write $A_f \sim A_{f'}$.

Proof. Let $f \sim f'$. By definition it means that there exists a holomorphic function $h : \mathbb{C} \to \mathrm{GL}_r(\mathbb{C})$ such that $f(\lambda, x)h(x) = h(\lambda x)f'(\lambda, x)$. Therefore, from $f(n, x)h(x) = h(n + x)f'(n, x)$ and $f(n, x) = f'(n, x) = \mathrm{id}_{\mathbb{C}^r}$ it follows $h(x) = h(n + x)$ for all $n \in \mathbb{Z}$. Therefore, the function

$$B : \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C}), \quad e^{2\pi i x} \mapsto h(x)$$

is well-defined. We have

$$\begin{aligned} A_f(m, e^{2\pi i x})B(e^{2\pi i x}) =& f(m\tau, x)h(x) = h(m\tau + x)f'(m\tau, x) = \\ & B(e^{2\pi i(m\tau + x)})f'(m, e^{2\pi i x}) = B(q^m e^{2\pi i x})A_{f'}(m, e^{2\pi i x}). \end{aligned}$$

Vice versa, let B be such that $A_f(m,u)B(u) = B(q^m A_{f'}(m,u))$. Define $h = B \circ \exp$. We obtain

$$\begin{aligned} f(m\tau+n,x)h(x) &= A_f(m\tau+n,e^{2\pi i x})B(e^{2\pi i x}) \\ &= B(q^m e^{2\pi i x})A_{f'}(m\tau+n,e^{2\pi i x}) = B(e^{2\pi i(m\tau+x)})A_{f'}(m\tau+n,e^{2\pi i x}) \\ &= B(e^{2\pi i(m\tau+n+x)})A_{f'}(m\tau+n,e^{2\pi i x}) = h(m\tau+n+x)f'(m\tau+n,x), \end{aligned}$$

which means that $f \sim f'$ and completes the proof. □

Remark 4.9. *The last two theorems allow us to embed the set* $Z^1(\mathbb{Z},r)$ *of factors of automorphy* $\mathbb{Z}\times X \to \mathrm{GL}_r(\mathbb{C})$ *to the set* $Z^1(\Gamma,r)$. *The embedding is*

$$\Psi : Z^1(\mathbb{Z},r) \to Z^1(\Gamma,r), \quad f \mapsto g, \quad g(n\tau+m,x) := f(n,x).$$

Two factors of automorphy from $Z^1(\mathbb{Z},r)$ *are equivalent if and only if their images under* Ψ *are equivalent in* $Z^1(\Gamma,r)$. *That is why it is enough to consider only factors of automorphy*

$$\Gamma\times\mathbb{C} \to \mathrm{GL}_r(\mathbb{C})$$

satisfying (2).

Corollary 4.10. *A factor of automorphy* f *with property (2) is trivial if and only if* $A_f(m,u) = B(q^m u)B(u)^{-1}$ *for some holomorphic function* $B : \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$.

Theorem 4.11. *Let* A *be a* $\mathbb{C}^*$*-factor of automorphy.* $A(m,u)$ *is uniquely determined by* $A(u) := A(1,u)$.

$$A(m,u) = A(q^{m-1}u)\dots A(qu)A(u), \qquad m>0 \tag{3}$$

$$A(-m,u) = A(q^{-m}u)^{-1}\dots A(q^{-1}u)^{-1}, \qquad m>0. \tag{4}$$

$A(m,u)$ *is equivalent to* $A'(m,u)$ *if and only if*

$$A(u)B(u) = B(qu)A'(u) \tag{5}$$

for some holomorphic function $B : \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$. *In particular* $A(m,u)$ *is trivial iff* $A(u) = B(qu)B(u)^{-1}$.

Proof. Since $A(1,u) = A(u)$ the first formula holds for $m = 1$. Therefore,

$$A(m+1,u) = A(1,q^m u)A(m,u) = A(q^m)A(m,u)$$

and we prove the first formula by induction.

Now $\mathrm{id} = A(0,u) = A(m-m,u) = A(m,q^{-m}u)A(-m,u)$ and hence

$$\begin{aligned} A(-m,u) &= A(m,q^{-m}u)^{-1} = (A(q^{m-1}q^{-m}u)\dots A(qq^{-m}u)A(q^{-m}u))^{-1} \\ &= A(-m,u) = A(q^{-m}u)^{-1}\dots A(q^{-1}u)^{-1} \end{aligned}$$

which proves the second formula.

If $A(m,u) \sim A'(m,u)$ then clearly (5) holds.

Vice versa, suppose $A(u)B(u) = B(qu)A'(u)$. Then

$$\begin{aligned} A(m,u)B(u) &= A(q^{m-1}u)\dots A(qu)A(u)B(u) \\ &= A(q^{m-1}u)\dots A(qu)B(qu)A'(u) \\ &= \dots \\ &= B(q^m u)A'(q^{m-1}u)\dots A'(qu)A'(u) \\ &= B(q^m u)A'(m,u) \end{aligned}$$

for $m > 0$.

Since $A(-m,u) = A(m,q^{-m}u)^{-1}$ we have

$$\begin{aligned} A(-m,u)B(u) &= A(m,q^{-m}u)^{-1}B(u) = (B(u)^{-1}A(m,q^{-m}u))^{-1} \\ &= (B(u)^{-1}A(m,q^{-m}u)B(q^{-m}u)B(q^{-m}u)^{-1})^{-1} \\ &= (B(u)^{-1}B(q^m q^{-m}u)A'(m,q^{-m}u)B(q^{-m}u)^{-1})^{-1} \\ &= (B(u)^{-1}B(u)A'(m,q^{-m}u)B(q^{-m}u)^{-1})^{-1} \\ &= B(q^{-m}u)A'(m,q^{-m}u)^{-1} = B(q^{-m}u)A'(-m,u), \end{aligned}$$

which completes the proof. □

Remark 4.12. *Theorem 4.11 means that all the information about a vector bundle of rank r on a complex torus can be encoded by a holomorphic function $\mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$.*

For a holomorphic function $A : \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$, let us denote by $E(A)$ the corresponding vector bundle on X.

Theorem 4.13. *Let $A : \mathbb{C}^* \to \mathrm{GL}_n(\mathbb{C})$, $B : \mathbb{C}^* \to \mathrm{GL}_m(\mathbb{C})$ be two holomorphic maps. Then $E(A) \otimes E(B) \simeq E(A \otimes B)$.*

Proof. By theorem 3.5 we have

$$E(A) \otimes E(B) \simeq E(A(n,u)) \otimes E(B(n,u)) \simeq E(A(n,u) \otimes B(n,u)).$$

Since $A(1,u) \otimes B(1,u) = A(u) \otimes B(u)$, we obtain $E(A) \otimes E(B) \simeq E(A \otimes B)$. □

5. Classification of Vector Bundles over a Complex Torus

Here we work with factors of automorphy depending only on τ, i.e., with holomorphic functions $\mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$.

5.1. Vector Bundles of Degree Zero

We return to extensions of the type $0 \to I_1 \to E \to I_1 \to 0$, where I_1 denotes the trivial vector bundle of rank 1.

Theorem 4.5 can be rewritten as follows.

THEOREM 5.1. *A function*

$$A(u) = \begin{pmatrix} 1 & a(u) \\ 0 & 1 \end{pmatrix}$$

defines the trivial bundle if and only if $a(u) = b(qu) - b(u)$ *for some holomorphic function* $b : \mathbb{C}^* \to \mathbb{C}$.

COROLLARY 5.2.

$$A(u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

defines a non-trivial vector bundle.

Proof. Suppose A defines the trivial bundle. Then $1 = b(qu) - b(u)$ for some holomorphic function $b : \mathbb{C}^* \to \mathbb{C}$. Considering the Laurent series expansion $\sum_{-\infty}^{+\infty} b_k u^k$ of b we obtain $1 = b_0 - b_0 = 0$ which shows that our assumption was false. ☐

Let $a : \mathbb{C}^* \to \mathbb{C}$ be a holomorphic function such that

$$A_2(u) = \begin{pmatrix} 1 & a(u) \\ 0 & 1 \end{pmatrix}$$

defines non-trivial bundle, i.e., by Theorem 5.1, there is no holomorphic function $b : \mathbb{C}^* \to \mathbb{C}$ such that

$$a(u) = b(qu) - b(u).$$

Let F_2 be the bundle defined by A_2. Then by Theorem 3.3 there exists an exact sequence

$$0 \to I_1 \to F_2 \to I_1 \to 0.$$

For $n \geqslant 3$ we define $A_n : \mathbb{C}^* \to \mathrm{GL}_n(\mathbb{C})$,

$$A_n = \begin{pmatrix} 1 & a & & \\ & \ddots & \ddots & \\ & & 1 & a \\ & & & 1 \end{pmatrix},$$

where empty entries stay for zeros.

Let F_n be the bundle defined by A_n. By (3.3) one sees that A_n defines the extension

$$0 \to I_1 \to F_n \to F_{n-1} \to 0.$$

THEOREM 5.3. *F_n is not the trivial bundle. The extension*

$$0 \to I_1 \to F_n \to F_{n-1} \to 0.$$

is non-trivial for all $n \geqslant 2$.

Proof. Suppose F_n is trivial. Then $A_n(u)B(u) = B(qu)$ for some $B = (b_{ij})_{ij^n}$. In particular it means $b_{ni}(u) = b_{ni}(qu)$ for $i = \overline{1,n}$. Let $b_{ni} = \sum_{-\infty}^{+\infty} b_k^{(ni)} u^k$ be the expansion of b_{ni} in Laurent series. Then $b_{ni}(u) = b_{ni}(qu)$ implies $b_k^{(ni)} = q^k b_k^{(ni)}$ for all k.

Note that $|q| < 1$ because $\tau = \xi + i\eta$, $\eta > 0$ and

$$|q| = |e^{2\pi i\tau}| = |e^{2\pi i(\xi+i\eta)}| = |e^{2\pi i\xi}e^{-2\pi\eta}| = e^{-2\pi\eta} < 1.$$

Therefore, $b_k^{(ni)} = 0$ for $k \neq 0$ and we conclude that b_{ni} should be constant functions.

We also have

$$b_{n-1i}(u) + b_{ni}a(n) = b_{n-1i}(qu).$$

Since at least one of b_{ni} is not equal to zero because of invertibility of B, we obtain

$$a(u) = \frac{1}{b_{ni}}(b_{n-1i}(qu) - b_{n-1i}(u))$$

for some i, which contradicts the choice of a. Therefore, F_n is not trivial.

Assume now, that for some $n > 2$ the extension

$$0 \to I_1 \to F_n \to F_{n-1} \to 0$$

is trivial(for $n = 2$ it is not trivial since F_2 is not a trivial vector bundle). This means

$$A_n \sim \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix},$$

i.e., there exists a holomorphic function $B : \mathbb{C}^* \to \mathrm{GL}_n(\mathbb{C})$, $B = (b_{ij})_{i,j}^n$ such that

$$A_n(u)B(u) = B(qu) \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix}.$$

Considering the elements of the first and second columns we obtain for the first column

$$b_{n1}(u) = b_{n1}(qu),$$

$$b_{i1}(u) + b_{i+11}(u)a(u) = b_{i1}(qu), \quad i < n$$

and for the second column

$$b_{n2}(u) = b_{n2}(qu),$$
$$b_{i2}(u) + b_{i+12}(u)a(u) = b_{i2}(qu), \quad i < n.$$

For the first column as above considering Laurent series we have that b_{n1} should be a constant function. If $b_{n1} \neq 0$ it follows

$$a(u) = \frac{1}{b_{n1}}(b_{n-11}(qu) - b_{n-11}(u)),$$

which contradicts the choice of a. Therefore, $b_{n1} = 0$ and $b_{n-11}(qu) = b_{n-11}(u)$, in other words b_{n-11} is a constant function. Proceeding by induction one obtains that b_{11} is a constant function and $b_{i1} = 0$ for $i > 1$.

For the second column absolutely analogously we obtain a similar result: b_{12} is constant, $b_{i2} = 0$ for $i > 1$. This contradicts the invertibility of $B(u)$ and proves the statement. □

Corollary 5.4. *The vector bundle F_n is the only indecomposable vector bundle of rank n and degree 0 that has non-trivial sections.*

Proof. This follows from [1, Theorem 5]. □

So we have that the vector bundles $F_n = E(A_n)$ are exactly F_n's defined by Atiyah in [1].

Remark 5.5. *Note that constant matrices A and B having the same Jordan normal form are equivalent. This is clear because $A = SBS^{-1}$ for some constant invertible matrix S, which means that A and B are equivalent.*

Consider an upper triangular matrix $B = (b_{ij})_1^n$ of the following type:

$$b_{ii} = 1, \quad b_{ii+1} \neq 0. \tag{6}$$

It is easy to see that this matrix is equivalent to the upper triangular matrix A,

$$a_{ii} = a_{ii+1} = 1, \quad a_{ij} = 0, \quad j \neq i+1, \quad j \neq i. \tag{7}$$

In fact, these matrices have the same characteristic polynomial $(t-1)^n$ and the dimension of the eigenspace corresponding to the eigenvalue 1 is equal to 1 for both matrices. Therefore, A and B have the same Jordan form. By Remark above we obtain that A and B are equivalent. We proved the following:

Lemma 5.6. *A matrix satisfying (6) is equivalent to the matrix defined by (7). Moreover, two matrices of the type (6) are equivalent, i.e., they define two isomorphic vector bundles.*

THEOREM 5.7. *$F_n \simeq S^{n-1}(F_2)$.*

Proof. We know that F_2 is defined by the constant matrix

$$A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We know by Theorem 3.4 that $S^n(F_2)$ is defined by $S^n(A_2)$. We calculate $S^n(f_2)$ for $n \in \mathbb{N}_0$. Since f_2 is a constant matrix, $S^n(f_2)$ is also a constant matrix defining a map $S^n(\mathbb{C}^2) \to S^n(\mathbb{C}^2)$. Let e_1, e_2 be the standard basis of $\mathbb{C}^2$, then $S^n(\mathbb{C})$ has a basis

$$\{e_1^k e_2^{n-k} |\ k = n, n-1, \dots, 0\}.$$

Since $A_2(e_1) = e_1$ and $A_2(e_2) = e_1 + e_2$, we conclude that $e_1^k e_2^{n-k}$ is mapped to

$$\begin{aligned} A_2(e_1)^k A_2(e_1)^{n-k} &= e_1^k (e_1 + e_2)^{n-k} \\ &= e_1^k \sum_{i=0}^{n-k} \binom{n-k}{i} e_1^{n-k-i} e_2^i = \sum_{i=0}^{n-k} \binom{n-k}{i} e_1^{n-i} e_2^i. \end{aligned}$$

Therefore,

$$S^n(A_2) = \begin{pmatrix} 1 & 1 & 1 & \dots & \binom{n}{0} \\ & 1 & 2 & \dots & \binom{n}{1} \\ & & 1 & \dots & \binom{n}{2} \\ & & & \ddots & \vdots \\ & & & & \binom{n}{n} \end{pmatrix},$$

where empty entries stay for zero. In other words, the columns of $S^n(A_2)$ are columns of binomial coefficients. By Lemma 5.6 we conclude that $S^n(A_2)$ is equivalent to A_{n+1}. This proves the statement of the theorem. □

Let E be a 2-dimensional vector bundle over a topological space X. Then there exists an isomorphism

$$S^p(E) \otimes S^q(E) \simeq S^{p+q}(E) \oplus (\det E \otimes S^{p-1}(E) \otimes S^{q-1}(E)).$$

This is the Clebsch-Gordan formula. If $\det E$ is the trivial line bundle, then we have $S^p(E) \otimes S^q(E) \simeq S^{p+q}(E) \oplus S^{p-1}(E) \otimes S^{q-1}(E)$, and by iterating one gets

$$S^p(E) \otimes S^q(E) \simeq S^{p+q}(E) \oplus S^{p+q-2}(E) \oplus \dots \oplus S^{p-q}(E), \quad p \geqslant q. \tag{8}$$

THEOREM 5.8. *$F_p \otimes F_q \simeq F_{p+q-1} \oplus F_{p+q-3} \oplus \dots \oplus F_{p-q+1}$ for $p \geqslant q$.*

Proof. Using Theorem 5.7 and (8) we obtain

$$\begin{aligned} F_p \otimes F_q &\simeq S^{p-1}(F_2) \otimes S^{q-1}(F_2) \\ &\simeq S^{p+q-2}(F_2) \oplus S^{p+q-4}(F_2) \oplus \cdots \oplus S^{p-q}(F_2) \\ &\simeq F_{p+q-1} \oplus F_{p+q-3} \oplus \cdots \oplus F_{p-q+1}. \end{aligned}$$

This completes the proof. □

REMARK 5.9. *The possibility of proving the last theorem using Theorem 5.7 is exactly what Atiyah states in remark 1) after Theorem 9 (see [1, p. 439]).*

We have already given (Corollary 5.4) a description of vector bundles of degree zero with non-trivial sections. We give now a description of all vector bundles of degree zero.

Consider the function $\varphi_0(z) = \exp(-\pi i\tau - 2\pi iz) = q^{-1/2}u^{-1} = \varphi(u)$, where $u = e^{2\pi iz}$. It defines the factor of automorphy

$$e_0(p\tau + q, z) = \exp(-\pi ip^2\tau - 2\pi izp) = q^{-\frac{p^2}{2}}u^{-p}$$

corresponding to the theta-characteristic $\xi = 0$.

THEOREM 5.10. $\deg E(\varphi_0) = 1$, *where as above* $\varphi_0(z) = \exp(-\pi i\tau - 2\pi iz) = q^{-1/2}u^{-1} = \varphi(u)$.

Proof. Follows from Theorem 4.3 for $\xi = 0$. □

THEOREM 5.11. *Let* $L' \in \mathcal{E}(1, d)$. *Then there exists* $x \in X$ *such that* $L' \simeq t_x^* E(\varphi_0) \otimes E(\varphi_0)^{d-1}$.

Proof. Since $E(\varphi_0)^d$ has degree d, we obtain that there exists $\tilde{L} \in \mathcal{E}(1, 0)$ such that $L' \simeq E(\varphi_0)^d \otimes \tilde{L}$. We also know that $\tilde{L} \simeq t_x^* E(\varphi_0) \otimes E(\varphi_0)^{-1}$ (cf. proof of Theorem 4.3 and Theorem 4.4) for some $x \in X$. Combining these one obtains

$$L' \simeq E(\varphi_0)^d \otimes t_x^* E(\varphi_0) \otimes E(\varphi_0)^{-1} \simeq t_x^* E(\varphi_0) \otimes E(\varphi_0)^{d-1}.$$

This proves the required statement. □

THEOREM 5.12. *The map*

$$\mathbb{C}^* / < q > \to \mathrm{Pic}^0(X), \quad \bar{a} \mapsto E(a).$$

is well-defined and is an isomorphism of groups.

Proof. Let $\varphi_0(z) = \exp(-\pi i\tau - 2\pi iz)$ as above. For $x \in X$ consider $t_x^* E(\varphi_0)$, where the map

$$t_x : X \to X, \quad y \mapsto y + x$$

is the translation by x. Let $\xi \in \mathbb{C}$ be a representative of x. Clearly, $t_x^* E(\varphi_0)$ is defined by

$$\varphi_{0\xi}(z) = t_\xi \varphi_0(z) = \varphi_0(z+\xi) = \exp(-\pi i\tau - 2\pi iz - 2\pi i\xi) = \varphi_0(z)\exp(-2\pi i\xi).$$

(Note that if η is another representative of x, then $\varphi_{0\xi}$ and $\varphi_{0\eta}$ are equivalent.) Therefore, the bundle $t_x^* E(\varphi_0) \otimes E(\varphi_0)^{-1}$ is defined by

$$(\varphi_{0\xi}{\varphi_0}^{-1})(z) = \varphi_0(z)\exp(-2\pi i\xi){\varphi_0}^{-1}(z) = \exp(-2\pi i\xi).$$

Since for any $L \in \mathcal{E}(1,0)$ there exists $x \in X$ such that $L \simeq t_x^* E(\varphi_0) \otimes E(\varphi_0)^{-1}$, we obtain $L \simeq E(a)$ for $a = \exp(-2\pi i\xi) \in \mathbb{C}^*$, where $\xi \in \mathbb{C}$ is a representative of x. We proved that any line bundle of degree zero is defined by a constant function $a \in \mathbb{C}^*$.

Vice versa, let $L = E(a)$ for $a \in \mathbb{C}^*$. Clearly, there exists $\xi \in \mathbb{C}$ such that $a = \exp(-2\pi i\xi)$. Therefore,

$$L \simeq E(a) \simeq L(\varphi_{0\xi}{\varphi_0}^{-1}) \simeq t_x^* E(\varphi_0) \otimes E(\varphi_0)^{-1},$$

where x is the class of ξ in X, which implies that $E(a)$ has degree zero. So we obtained that the line bundles of degree zero are exactly the line bundles defined by constant functions.

We have the map

$$\phi : \mathbb{C}^* \to \mathrm{Pic}^0(X), \quad a \mapsto E(a),$$

which is surjective. By Theorem 4.13 it is moreover a homomorphism of groups. We are looking now for the kernel of this map.

Suppose $E(a)$ is a trivial bundle. Then there exists a holomorphic function $f : \mathbb{C}^* \to \mathbb{C}^*$ such that $f(qu) = af(u)$. Let $f = \sum f_\nu a^\nu$ be the Laurent series expansion of f. Then from $f(qu) = af(u)$ one obtains

$$af_\nu = f_\nu q^\nu \text{ for all } \nu \in \mathbb{Z}.$$

Therefore, $f_\nu(a - q^\nu) = 0$ for all $\nu \in \mathbb{Z}$.

Since $f \not\equiv 0$, we obtain that there exists $\nu \in \mathbb{Z}$ with $f_\nu \neq 0$. Hence $a = q^\nu$ for some $\nu \in \mathbb{Z}$.

Vice versa, if $a = q^\nu$, for $f(u) = u^\nu$ we get

$$f(qu) = q^\nu u^\nu = af(u).$$

This means that $E(a)$ is the trivial bundle, which proves $\operatorname{Ker}\phi = < q >$. We obtain the required isomorphism

$$\mathbb{C}^*/ < q > \to \operatorname{Pic}^0(X), \quad \bar{a} \mapsto E(a).$$

This completes the proof. $\square$

THEOREM 5.13. *For any $F \in \mathcal{E}(r,0)$ there exists a unique $\bar{a} \in \mathbb{C}^*/ < q >$ such that $F \simeq E(A_r(a))$, where*

$$A_r(a) = \begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & a & 1 \\ & & & a \end{pmatrix}.$$

Proof. By [1, Theorem 5] $F \simeq F_r \otimes L$ for a unique $L \in \mathcal{E}(1,0)$. Since $F_r \simeq E(A_r)$ and $L \simeq E(a)$ for a unique $\bar{a} \in \mathbb{C}^*/ < q >$ we get $F \simeq E(A_r \otimes a)$. So F is defined by the matrix

$$\begin{pmatrix} a & a & & \\ & \ddots & \ddots & \\ & & a & a \\ & & & a \end{pmatrix},$$

where empty entries stay for zeros. It is easy to see that the Jordan normal form of this matrix is

$$\begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & a & 1 \\ & & & a \end{pmatrix}.$$

This proves the statement of the theorem. $\square$

5.2. Vector Bundles of Arbitrary Degree

Denote by $E_\tau = \mathbb{C}/\Gamma_\tau$, where $\Gamma_\tau = \mathbb{Z}\tau + \mathbb{Z}$. Consider the r-covering

$$\pi_r : E_{r\tau} \to E_\tau, \quad [x] \mapsto [x].$$

THEOREM 5.14. *Let F be a vector bundle of rank n on E_τ defined by $A(u) = A(1,u) = A(\tau,u)$. Then $\pi_r^*(F)$ is defined by*

$$\tilde{A}(r\tau,u) = \tilde{A}(u) = \tilde{A}(1,u) := A(r\tau,u) = A(q^{r-1}u)\ldots A(qu)A(u).$$

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} & \mathbb{C} & \\ {\scriptstyle p_{r\tau}}\swarrow & & \searrow{\scriptstyle p_{\tau}} \\ E_{r\tau} & \xrightarrow{\pi_r} & E_\tau \end{array}$$

Consider the map

$$E(\tilde{A}) \to \pi_r^*(E(A)) = E_{r\tau} \times_{E_\tau} E(A) = \{([z]_{r\tau},\ [z,v]_\tau) \in E_{r\tau} \times E(A)\},$$
$$[z,v]_{r\tau} \mapsto ([z]_{r\tau},\ [z,v]_\tau).$$

It is clearly bijective. It remains to prove that it is biholomorphic. From the construction of $E(A)$ and $E(\tilde{A})$ it follows that the diagram

$$\begin{array}{ccc} E(\tilde{A}) & & \\ & \searrow & \\ & \pi_r^*(E(A)) & \longrightarrow E(A) \\ & \downarrow & \downarrow \\ & E_{r\tau} & \longrightarrow E_\tau \end{array}$$

locally looks as

$$\begin{array}{ccc} U \times \mathbb{C}^n & & \\ \searrow & & \\ \Delta(U \times U) \times \mathbb{C}^n & \longrightarrow & U \times \mathbb{C}^n \\ \downarrow & & \downarrow \\ U & = & U \end{array}, \qquad \begin{array}{ccc} (z,v) & & \\ \searrow & & \\ ((z,z),v) & \longmapsto & (z,v), \\ \downarrow & & \downarrow \\ z & = & z \end{array}$$

where $\Delta(U \times U)$ denotes the diagonal of $U \times U$.

This proves the required statement. □

THEOREM 5.15. *Let F be a vector bundle of rank n on $E_{r\tau}$ defined by $\tilde{A}(u) = \tilde{A}(r\tau, u)$. Then $\pi_{r*}(F)$ is defined by*

$$A(u) = \begin{pmatrix} 0 & I_{(r-1)n} \\ \tilde{A}(u) & 0 \end{pmatrix}.$$

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} & \mathbb{C} & \\ {\scriptstyle p_{r\tau}}\swarrow & & \searrow{\scriptstyle p_{\tau}} \\ E_{r\tau} & \xrightarrow{\pi_r} & E_\tau \end{array}$$

Let $z \in \mathbb{C}$. Consider $y = p_{r\tau}(z) \in E_{r\tau}$ and $x = p_\tau(z) = \pi_r p_{r\tau}(z) \in E_\tau$.

Choose a point $b \in \mathbb{C}$ such that $z \in V_b$, where V_b is the standard parallelogram at point b. Clearly $x \in U_b = p_r(V_b)$ and we have the isomorphism $\varphi_b : U_b \to V_b$ with $\varphi_b(x) = z$.

Consider $\pi_r^{-1}(U_b) = W_b \bigsqcup \cdots \bigsqcup W_{b+(r-1)\tau}$, where $y \in W_b$ and $\pi_r|_{W_{b+i\tau}} : W_{b+i\tau} \to U_b$ is an isomorphism for each $0 \leqslant i < r$.

We have

$$\begin{aligned}\pi_{r*}(\mathcal{E}(\tilde{A}))(U_b) = \mathcal{E}(\tilde{A})(\pi_r^{-1}(U_b)) &= \mathcal{E}(\tilde{A})\left(W_b \bigsqcup \cdots \bigsqcup W_{b+(r-1)\tau}\right)\\ &= \mathcal{E}(\tilde{A})(W_b) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{b+(r-1)\tau}),\end{aligned}$$

where $\mathcal{E}(\tilde{A})$ is the sheaf of sections of $E(\tilde{A})$.

Choose $a \in \mathbb{C}$ such that $z \notin V_a$, $z \in V_{a+\tau}$. We have $\varphi_a(x) = z + \tau$. As above, $\pi_r^{-1}(U_a) = W_a \bigsqcup \cdots \bigsqcup W_{a+(r-1)\tau}$ and

$$\begin{aligned}\pi_{r*}(\mathcal{E}(\tilde{A}))(U_a) =;\mathcal{E}(\tilde{A})(\pi_r^{-1}(U_a)) &= \mathcal{E}(\tilde{A})\left(W_a \bigsqcup \cdots \bigsqcup W_{a+(r-1)\tau}\right)\\ &= \mathcal{E}(\tilde{A})(W_a) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{a+(r-1)\tau}).\end{aligned}$$

Since $g_{ab}(x) = A(\varphi_a(x) - \varphi_b(x), \varphi_b(x))$, we obtain

$$g_{ab}(x) = A(\varphi_a(x) - \varphi_b(x), \varphi_b(x)) = A(z + \tau - z, z) = A(\tau, z).$$

Therefore, to obtain $A(\tau, z)$ it is enough to compute $g_{ab}(x)$.

Note that $\pi_{r*}(\mathcal{E}(\tilde{A}))_x = \mathcal{E}(\tilde{A})_y \oplus \cdots \oplus \mathcal{E}(\tilde{A})_{y+(r-1)\tau}$. Note also that g_{ab} is a map from

$$\pi_{r*}(\mathcal{E}(\tilde{A}))(U_b) = \mathcal{E}(\tilde{A})(W_b) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{b+(r-1)\tau})$$

to

$$\pi_{r*}(\mathcal{E}(\tilde{A}))(U_a) = \mathcal{E}(\tilde{A})(W_a) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{a+(r-1)\tau}).$$

One easily sees that $y \in W_b$, $y \in W_{a+(r-1)\tau}$ and $y + i\tau \in W_{b+i\tau}$, $y + i\tau \in W_{a+(i-1)\tau}$ for $0 < i < r$. Therefore, $g_{ab}(x)$ equals

$$\begin{pmatrix} 0 & \tilde{g}_{a\ b+\tau}(y+\tau) & & \\ \vdots & & \ddots & \\ 0 & & & \tilde{g}_{a+(r-2)\tau\ b+(r-a)\tau}(y+(r-1)\tau) \\ \tilde{g}_{a+(r-1)\tau\ b}(y) & 0 & \dots & 0 \end{pmatrix}.$$

It remains to compute the entries of this matrix. Since

$$\begin{aligned}\tilde{g}_{a+(r-1)\tau\ b}(y) &= \tilde{A}(\tilde{\varphi}_{a+(r-1)\tau}(y) - \tilde{\varphi}_b(y), \tilde{\varphi}_b(y))\\ &= \tilde{A}(z + r\tau - z, z) = \tilde{A}(r\tau, z)\end{aligned}$$

and

$$\begin{aligned}\tilde{g}_{a+(i-1)\tau\ b+i\tau}(y+i\tau) &= \tilde{A}(\tilde{\varphi}_{a+(i-1)\tau}(y+i\tau)-\tilde{\varphi}_{b+i\tau}(y+i\tau),\tilde{\varphi}_{b+i\tau}(y+i\tau))\\ &= \tilde{A}(z+i\tau-(z+i\tau)=\tilde{A}(0,z+i\tau)=I_n,\end{aligned}$$

one obtains

$$g_{ab}(x)=\begin{pmatrix}0 & I_n & & \\ \vdots & & \ddots & \\ 0 & & & I_n\\ \tilde{A}(z) & 0 & \dots & 0\end{pmatrix}.$$

Therefore,

$$A(z)=\begin{pmatrix}0 & I_n & & \\ \vdots & & \ddots & \\ 0 & & & I_n\\ \tilde{A}(z) & 0 & \dots & 0\end{pmatrix}=\begin{pmatrix}0 & I_{(r-1)n}\\ \tilde{A}(u) & 0\end{pmatrix},$$

which proves the required statement. □

LEMMA 5.16. *Let $A_i \in \mathrm{GL}_n(\mathbb{R})$, $i=1,\dots,n$. Then*

$$\prod_{i=1}^{r}\begin{pmatrix}0 & I_{(r-1)n}\\ A_i & 0\end{pmatrix}=\mathrm{diag}(A_r,\dots,A_1)$$

Proof. Straightforward calculation. □

From Theorem 5.14 and Theorem 5.15 one obtains the following:

COROLLARY 5.17. *Let $E(A)$ be a vector bundle of rank n on $E_{r\tau}$, where $A:\mathbb{C}^*\to\mathrm{GL}_n(\mathbb{C}V)$ is a holomorphic function. Then $\pi_r^*\pi_{r*}E(A)$ is defined by*

$$\mathrm{diag}(A(q^{r-1}u),\dots,A(qu),A(u)).$$

In other words $\pi_r^*\pi_{r*}E(A)$ is isomorphic to the direct sum

$$\bigoplus_{i=0}^{r-1}E(A(q^iu)).$$

Proof. We know that $\pi_r^*\pi_{r*}E(A)$ is defined by $B(r,u)$, where

$$B(1,u)=\begin{pmatrix}0 & I_{(r-1)n}\\ A & 0\end{pmatrix}.$$

Therefore, using Lemma 5.16, one obtains

$$\begin{aligned} B(r,u) &= \begin{pmatrix} 0 & I_{(r-1)n} \\ A(q^{r-1}u) & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & I_{(r-1)n} \\ A(qu) & 0 \end{pmatrix} \begin{pmatrix} 0 & I_{(r-1)n} \\ A(u) & 0 \end{pmatrix} \\ &= \operatorname{diag}(A(q^{r-1}u), \dots, A(qu), A(u)), \end{aligned}$$

which completes the proof. □

Corollary 5.18. *Let* $L \in \mathcal{E}(r,0)$, *then* $\pi_r^* \pi_{r*} L = \bigoplus_1^r L$.

Proof. Clear, since $L = E(A)$ for a constant matrix A by Theorem 5.13. □

Note that for a covering $\pi_r : E_{r\tau} \to E_\tau$ the group of deck transformations $\operatorname{Deck}(E_{r\tau}/E_\tau)$ can be identified with the kernel $\operatorname{Ker}(\pi_r)$. But $\operatorname{Ker}\pi_r$ is cyclic and equals $\{1, [q], \dots [q]^{r-1}\}$, where $[q]$ is a class of $q = e^{2\pi i\tau}$ in $E_{r\tau}$. Clearly

$$[q]^*(E(A(u))) = E(A(qu)).$$

Therefore, we get one more corollary.

Corollary 5.19. *Let* ϵ *be a generator of* $\operatorname{Deck}(E_{r\tau}/E_\tau)$. *Then for a vector bundle* E *on* $E_{r\tau}$ *we have*

$$\pi_r^* \pi_{r*} E = E \oplus \epsilon^* E \oplus \cdots \oplus (\epsilon^{r-1})^* E.$$

To proceed we need the following result.

Theorem 5.20 (Oda, [11, Theorem 1.2, (i)]). *Let* $\varphi : Y \to X$ *be an isogeny of* g*-dimensional abelian varieties over a field* k*, and let* L *be a line bundle on* Y *such that the restriction of the map*

$$\Lambda(L) : Y \to \operatorname{Pic}^0(Y), \quad y \mapsto t_y^* L \otimes L^{-1},$$

to the kernel of φ *is an isomorphism. Then* $\operatorname{End}(\varphi_* L) = k$ *and* $\varphi_* L$ *is an indecomposable vector bundle on* X.

Theorem 5.21. *Let* $L \in \mathcal{E}(1,d)$ *and let* $(r,d) = 1$. *Then* $\pi_{r*}(L) \in \mathcal{E}(r,d)$.

Proof. It is clear that $\pi_{r*}L$ has rank r and degree d. It remains to prove that $\pi_{r*}L$ is indecomposable.

We have the isogeny $\pi_r : E_{r\tau} \to E_r$. Since $Y = E_{r\tau}$ is a complex torus (elliptic curve), $Y \simeq \operatorname{Pic}^0(Y)$ with the identification $y \leftrightarrow t_y^* E(\varphi_0) \otimes E(\varphi_0)^{-1}$.

We know that $L = E(\varphi_0)^d \otimes \tilde{L}$ for some $\tilde{L} = E(a) \in \mathcal{E}(1,0)$, $a \in \mathbb{C}^*$. Since $t_y^*(\tilde{L}) = t_y^*(E(a)) = E(a) = \tilde{L}$, as in the proof of Theorem 5.12 one gets

$$\begin{aligned}\Lambda(L)(y) &= t_y^*(L) \otimes L^{-1} = t_y^*(E(\varphi_0)^d \otimes \tilde{L}) \otimes (E(\varphi_0)^d \otimes \tilde{L})^{-1} \\ &= t_y^*(E(\varphi_0)^d) \otimes t_y^*(\tilde{L}) \otimes E(\varphi_0)^{-d} \otimes \tilde{L}^{-1} = t_y^*(E(\varphi_0)^d) \otimes E(\varphi_0)^{-d} \\ &= t_y^*(E(\varphi_0^d)(z)) \otimes E(\varphi_0^{-d}) = E(\varphi_0^d(z+\eta)) \otimes E(\varphi_0^{-d}) \\ &= E(\varphi_0^d(z+\eta)\varphi_0^{-d}(z)) = E(\exp(-2\pi i d\eta)) = t_{dy}^*(E(\varphi_0)) \otimes E(\varphi_0)^{-1},\end{aligned}$$

where $\eta \in \mathbb{C}$ is a representative of y. This means that the map $\Lambda(L)$ corresponds to the map

$$d_Y : E_{r\tau} \to E_{r\tau}, \quad y \mapsto dy.$$

Since $\operatorname{Ker} \pi_r$ is isomorphic to $\mathbb{Z}/r\mathbb{Z}$, we conclude that the restriction of d_Y to $\operatorname{Ker} \pi_r$ is an isomorphism if and only if $(r,d) = 1$. Therefore, using Theorem 5.20, we obtain the required statement. □

Now we are able to prove the following main theorem:

Theorem 5.22.

(i) *Every indecomposable vector bundle $F \in \mathcal{E}_{E_\tau}(r,d)$ is of the form $\pi_{r'*}(L' \otimes F_h)$, where $(r,d) = h$, $r = r'h$, $d = d'h$, $L' \in \mathcal{E}_{E_{r'\tau}}(1,d')$.*

(ii) *Every vector bundle of the form $\pi_{r'*}(L' \otimes F_h)$, where L' and r' are as above, is an element of $\mathcal{E}_{E_\tau}(r,d)$.*

Proof[1].

(i) By [1, Lemma 26] we obtain $F \simeq E_A(r,d) \otimes L$ for some line bundle $L \in \mathcal{E}(1,0)$. By [1, Lemma 24] we have $E_A(r,d) \simeq E_A(r',d') \otimes F_h$, hence $F \simeq E_A(r',d') \otimes F_h \otimes L$.

Consider any line bundle $\tilde{L} \in \mathcal{E}_{E_{r'\tau}}(1,d')$. Since by Theorem 5.21 $\pi_{r'*}(\tilde{L}) \in \mathcal{E}(r',d')$, it follows from [1, Lemma 26] that there exists a line bundle L'' such that $E_A(r',d') \otimes L \simeq \pi_{r'*}(\tilde{L}) \otimes L''$.

Using the projection formula, we get

$$\begin{aligned}F &\simeq \pi_{r'*}(\tilde{L}) \otimes L'' \otimes F_h \\ &\simeq \pi_{r'*}(\tilde{L} \otimes \pi_{r'}^*(L'') \otimes \pi_{r'}^*(F_h)) \\ &\simeq \pi_{r'*}(L' \otimes \pi_{r'}^*(F_h))\end{aligned}$$

for $L' = \tilde{L} \otimes \pi_{r'}^*(L'')$.

Since F_h is defined by a constant matrix we obtain by Theorem 5.14 that $\pi_{r'}^*(F_h)$ is defined by $f_h^{r'}$, which is has the same Jordan normal form as f_h. Therefore, $\pi_{r'}^*(F_h) \simeq F_h$ and finally one gets $F \simeq \pi_{r'*}(L' \otimes F_h)$.

(ii) Consider $F = \pi_{r'*}(L' \otimes F_h)$. As above $F_h = \pi^*_{r'}(F_h)$. Using the projection formula we get

$$F = \pi_{r'*}(L' \otimes F_h) = \pi_{r'*}(L' \otimes \pi^*_{r'}(F_h)) = \pi_{r'*}(L') \otimes F_h.$$

By Theorem 5.21 $\pi_{r'*}(L')$ is an element from $\mathcal{E}_{E_\tau}(r', d')$. Therefore, $\pi_{r'*}(L') = E_A(r', d') \otimes L$ for some line bundle $L \in \mathcal{E}_{E_\tau}(1, 0)$. Finally we obtain

$$\begin{aligned} F &= \pi_{r'*}(L') \otimes F_h \\ &= E_A(r', d') \otimes L \otimes F_h \\ &= E_A(r'h, d'h) \otimes L \\ &= E_A(r, d) \otimes L, \end{aligned}$$

which means that F is an element of $\mathcal{E}_{E_\tau}(r, d)$. □

REMARK 5.23. *Since any line bundle of degree d' is of the form $t^*_x E(\varphi_0) \otimes E(\varphi_0)^{d'-1}$, Theorem 5.22(i) takes exactly the form of Proposition 1 from [12], which was given without any proof.*

Any line bundle of degree d' over $E_{r\tau}$ is of the form $E(a) \otimes E(\varphi^{d'})$, where $a \in \mathbb{C}^*$. Therefore, $L' \otimes F_h = E(a) \otimes E(\varphi_0^{d'}) \otimes E(A_h) = E(\varphi_0^{d'} A_h(a))$. Using Theorem 5.15 we obtain the following:

THEOREM 5.24. *Indecomposable vector bundles of rank r and degree d on E_τ are exactly those defined by the matrices*

$$\begin{pmatrix} 0 & I_{(r'-1)h} \\ \varphi_0^{d'} A_h(a) & 0 \end{pmatrix},$$

where $(r, d) = h$, $r' = r/h$, $d' = d/h$, $\varphi_0(u) = q^{-\frac{r}{2}} u^{-1}$, $q = e^{2\pi i \tau}$, $a \in \mathbb{C}^*$, and

$$A_h(a) = \begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & a & 1 \\ & & & a \end{pmatrix} \in \mathrm{GL}_h(\mathbb{C}).$$

Note that if $d = 0$, we get $h = r$, $r' = 1$, and $d' = 0$. In this case the statement of Theorem 5.24 is exactly Theorem 5.13.

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