

Non-vanishing Theorems for Rank Two Vector Bundles on Threefolds¹

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ABSTRACT. *The paper investigates the non-vanishing of $H^1(\mathcal{E}(n))$, where \mathcal{E} is a (normalized) rank two vector bundle over any smooth irreducible threefold X with $\text{Pic}(X) \cong \mathbb{Z}$. If ϵ is defined by the equality $\omega_X = \mathcal{O}_X(\epsilon)$, and α is the least integer t such that $H^0(\mathcal{E}(t)) \neq 0$, then, for a non-stable \mathcal{E} , $H^1(\mathcal{E}(n))$ does not vanish at least between $\frac{\epsilon-c_1}{2}$ and $-\alpha - c_1 - 1$. The paper also shows that there are other non-vanishing intervals, whose endpoints depend on α and on the second Chern class of \mathcal{E} . If \mathcal{E} is stable $H^1(\mathcal{E}(n))$ does not vanish at least between $\frac{\epsilon-c_1}{2}$ and $\alpha - 2$. The paper considers also the case of a threefold X with $\text{Pic}(X) \neq \mathbb{Z}$ but $\text{Num}(X) \cong \mathbb{Z}$ and gives similar non-vanishing results.*

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1. Introduction

In 1942 G. Gherardelli ([5]) proved that, if C is a smooth irreducible curve in \mathbb{P}^3 whose canonical divisors are cut out by the surfaces of some degree e and moreover all linear series cut out by the surfaces in \mathbb{P}^3 are complete, then C is the complete intersection of two surfaces. Shortly and in the language of modern algebraic geometry: every e -subcanonical smooth curve C in \mathbb{P}^3 such that $h^1(\mathcal{I}_C(n)) = 0$ for all n is the complete intersection of two surfaces.

Thanks to the Serre correspondence between curves and vector bundles (see [7, 8, 9]) the above statement is equivalent to the following one: if \mathcal{E} is a rank two vector bundle on \mathbb{P}^3 such that $h^1(\mathcal{E}(n)) = 0$ for all n , then \mathcal{E} splits.

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There are many improvements of the above result with a variety of different approaches (see for instance [2, 3, 4, 13, 15]): it comes out that a rank two vector bundle \mathcal{E} on \mathbb{P}^3 is forced to split if $h^1(\mathcal{E}(n))$ vanishes for just one strategic n , and such a value n can be chosen arbitrarily within a suitable interval, whose endpoints depend on the Chern classes and the least number α such that $h^0(\mathcal{E}(\alpha)) \neq 0$.

When rank two vector bundles on a smooth threefold X of degree d in \mathbb{P}^4 are concerned, similar results can be obtained, with some interesting difference.

In 1998 Madonna ([11]) proved that on a smooth threefold X of degree d in \mathbb{P}^4 there are ACM rank two vector bundles (i.e. whose 1-cohomology vanishes for all twists) that do not split. And this can happen, for a normalized vector bundle \mathcal{E} ($c_1 \in \{0, -1\}$), only when $1 - \frac{d+c_1}{2} < \alpha < \frac{d-c_1}{2}$, while an ACM rank two vector bundle on X whose α lies outside of the interval is forced to split.

The following non-vanishing results for a normalized non-split rank two vector bundle on a smooth irreducible threefold of degree d in \mathbb{P}^4 are proved in [11]:

- if $\alpha \leq 1 - \frac{d+c_1}{2}$, then $h^1(\mathcal{E}(\frac{d-3-c_1}{2})) \neq 0$ if $d+c_1$ is odd, $h^1(\mathcal{E}(\frac{d-4-c_1}{2})) \neq 0$, $h^1(\mathcal{E}(\frac{d-2-c_1}{2})) \neq 0$ if $d+c_1$ is even, while $h^1(\mathcal{E}(\frac{d-c_1}{2})) \neq 0$ if $d+c_1$ is even and moreover $\alpha \leq -\frac{d+c_1}{2}$;
- if $\alpha \geq \frac{d-c_1}{2}$, then $h^1(\mathcal{E}(\frac{d-3-c_1}{2})) \neq 0$ if $d+c_1$ is odd, while $h^1(\mathcal{E}(\frac{d-4-c_1}{2})) \neq 0$ if $d+c_1$ is even.

In [11] it is also claimed that the same techniques work to obtain similar non-vanishing results on any smooth threefold X with $\text{Pic}(X) \cong \mathbb{Z}$ and $h^1(\mathcal{O}_X(n)) = 0$, for every n .

The present paper investigates the non-vanishing of $H^1(\mathcal{E}(n))$, where \mathcal{E} is a rank two vector bundle over any smooth irreducible threefold X such that $\text{Pic}(X) \cong \mathbb{Z}$ and $H^1(\mathcal{O}_X(n)) = 0$, for all n . Actually we can prove that for such an \mathcal{E} there is a wider range of non-vanishing for $h^1(\mathcal{E}(n))$, so improving the above results.

More precisely, when \mathcal{E} is (normalized and) non-stable ($\alpha \leq 0$) the first cohomology module does not vanish at least between the endpoints $\frac{\epsilon-c_1}{2}$ and $-\alpha - c_1 - 1$, where ϵ is defined by the equality $\omega(X) = \mathcal{O}_X(\epsilon)$ (and is $d - 5$ if $X \subset \mathbb{P}^4$, where $d = \text{deg}(X)$). But we can show that there are other non-vanishing intervals, whose endpoints depend on α and also on the second Chern class c_2 of \mathcal{E} .

If on the contrary \mathcal{E} is stable the first cohomology module does not vanish at least between the endpoints $\frac{\epsilon-c_1}{2}$ and $\alpha - 2$, but other ranges of non-vanishing can be produced.

We give a few examples obtained by pull-back from vector bundles on \mathbb{P}^3 .

We must remark that most of our non-vanishing results do not exclude the range for α between the endpoints $1 - \frac{d+c_1}{2}$ and $\frac{d-c_1}{2}$ (for a general threefold

it becomes $-\frac{\epsilon+3+c_1}{2} < \alpha < \frac{\epsilon+5-c_1}{2}$). Actually [11] produces some examples of non-split ACM rank two vector bundles on smooth hypersurfaces in \mathbb{P}^4 , but it can be seen that they do not conflict with our theorems.

As to threefolds with $\text{Pic}(X) \neq \mathbb{Z}$, we need to observe that a key point is a good definition of the integer α . We are able to prove, by using a boundedness argument, that α exists when $\text{Pic}(X) \neq \mathbb{Z}$ but $\text{Num}(X) \cong \mathbb{Z}$. In this event the correspondence between rank two vector bundles and two-codimensional subschemes can be proved to hold. In order to obtain non-vanishing results that are similar to the results proved when $\text{Pic}(X) \cong \mathbb{Z}$, we need also use the Kodaira vanishing theorem, which holds in characteristic 0. We can extend the results to characteristic $p > 0$ if we assume a Kodaira-type vanishing condition.

In this paper we investigate non-vanishing theorems for rank two vector bundles on *any* threefold. The problem looks quite different if the threefold is general or belongs to some family (for the case of ACM bundles see for instance [14] and [1]).

Moreover we observe that our examples of section 6 are sharp but the threefolds (except one) are quadric hypersurfaces, so that one can guess that some stronger statement holds when the degree d is large enough.

2. Notation

We work over an algebraically closed field \mathbf{k} of any characteristic.

Let X be a non-singular irreducible projective algebraic variety of dimension 3, for short a smooth threefold. We fix an ample divisor H on X , so we consider the polarized threefold (X, H) . We denote with $\mathcal{O}_X(n)$, instead of $\mathcal{O}_X(nH)$, the invertible sheaf corresponding to the divisor nH , for each $n \in \mathbb{Z}$.

For every cycle Z on X of codimension i it is defined its degree with respect to H , i.e. $\deg(Z; H) := Z \cdot H^{3-i}$, having identified a codimension 3 cycle on X , i.e. a 0-dimensional cycle, with its degree, which is an integer.

From now on (with the exception of section 7) we consider a smooth polarized threefold $(X, \mathcal{O}_X(1)) = (X, H)$ that satisfies the following conditions:

- (C1) $\text{Pic}(X) \cong \mathbb{Z}$ generated by $[H]$,
- (C2) $H^1(X, \mathcal{O}_X(n)) = 0$ for every $n \in \mathbb{Z}$,
- (C3) $H^0(X, \mathcal{O}_X(1)) \neq 0$.

By condition (C1) every divisor on X is linearly equivalent to aH for some integer $a \in \mathbb{Z}$, i.e. every invertible sheaf on X is (up to an isomorphism) of type $\mathcal{O}_X(a)$ for some $a \in \mathbb{Z}$, in particular we have for the canonical divisor $K_X \sim \epsilon H$, or equivalently $\omega_X \simeq \mathcal{O}_X(\epsilon)$, for a suitable integer ϵ . Furthermore, by Serre duality condition (C2) implies that $H^2(X, \mathcal{O}_X(n)) = 0$ for all $n \in \mathbb{Z}$.

Since by assumption $A^1(X) = \text{Pic}(X)$ is isomorphic to \mathbb{Z} through the map $[H] \mapsto 1$, where $[H] = c_1(\mathcal{O}_X(1))$, we identify the first Chern class $c_1(\mathcal{F})$ of a coherent sheaf with a whole number c_1 , where $c_1(\mathcal{F}) = c_1 H$.

The second Chern class $c_2(\mathcal{F})$ gives the integer $c_2 = c_2(\mathcal{F}) \cdot H$ and we will call this integer the second Chern number or the second Chern class of \mathcal{F} .

We set

$$d := \deg(X; H) = H^3,$$

so d is the “degree” of the threefold X with respect to the ample divisor H .

Let $c_1(X)$ and $c_2(X)$ be the first and second Chern classes of X , that is of its tangent bundle TX (which is a locally free sheaf of rank 3); then we have

$$c_1(X) = [-K_X] = -\epsilon[H],$$

so we identify the first Chern class of X with the integer $-\epsilon$. Moreover we set

$$\tau := \deg(c_2(X); H) = c_2(X) \cdot H,$$

i.e. τ is the degree of the second Chern class of the threefold X .

In the following we will call the triple of integers (d, ϵ, τ) the *characteristic numbers* of the polarized threefold $(X, \mathcal{O}_X(1))$.

We recall the well-known Riemann-Roch formula on the threefold X (e.g. see [18], Proposition 4).

THEOREM 2.1 (Riemann-Roch). *Let \mathcal{F} be a rank r coherent sheaf on X with Chern classes $c_1(\mathcal{F})$, $c_2(\mathcal{F})$ and $c_3(\mathcal{F})$. Then the Euler-Poincaré characteristic of \mathcal{F} is*

$$\begin{aligned} \chi(\mathcal{F}) = & \frac{1}{6} \left(c_1(\mathcal{F})^3 - 3c_1(\mathcal{F}) \cdot c_2(\mathcal{F}) + 3c_3(\mathcal{F}) \right) + \frac{1}{4} \left(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F}) \right) \cdot c_1(X) \\ & + \frac{1}{12} c_1(\mathcal{F}) \cdot \left(c_1(X)^2 + c_2(X) \right) + \frac{r}{24} c_1(X) \cdot c_2(X) \end{aligned}$$

where $c_1(X)$ and $c_2(X)$ are the Chern classes of X , that is the Chern classes of the tangent bundle TX of X .

So applying the Riemann-Roch Theorem to the invertible sheaf $\mathcal{O}_X(n)$, for each $n \in \mathbb{Z}$, we get the Hilbert polynomial of the sheaf $\mathcal{O}_X(1)$

$$\chi(\mathcal{O}_X(n)) = \frac{d}{6} \left(n - \frac{\epsilon}{2} \right) \left[\left(n - \frac{\epsilon}{2} \right)^2 + \frac{\tau}{2d} - \frac{\epsilon^2}{4} \right]. \quad (1)$$

Let \mathcal{E} be a rank 2 vector bundle on the threefold X with Chern classes $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$, i.e. with Chern numbers c_1 and c_2 . We assume that \mathcal{E} is normalized, i.e. that $c_1 \in \{0, -1\}$. It is defined the integer α , the so called *first relevant*

level, such that $h^0(\mathcal{E}(\alpha)) \neq 0, h^0(\mathcal{E}(\alpha - 1)) = 0$. If $\alpha > 0$, \mathcal{E} is called stable, non-stable otherwise. We set

$$\vartheta = \frac{3c_2}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} - \frac{3c_1^2}{4}, \quad \zeta_0 = \frac{\epsilon - c_1}{2}, \quad \text{and} \quad w_0 = [\zeta_0] + 1,$$

where $[\zeta_0]$ = integer part of ζ_0 , so the Hilbert polynomial of \mathcal{E} can be written as

$$\chi(\mathcal{E}(n)) = \frac{d}{3}(n - \zeta_0) \left[(n - \zeta_0)^2 - \vartheta \right]. \quad (2)$$

If $\vartheta \geq 0$ we set

$$\zeta = \zeta_0 + \sqrt{\vartheta}$$

so in this case the Hilbert polynomial of \mathcal{E} has the three real roots $\zeta' \leq \zeta_0 \leq \zeta$ where $\zeta' = \zeta_0 - \sqrt{\vartheta}$. We also define $\bar{\alpha} = [\zeta] + 1$.

The polynomial $\chi(\mathcal{E}(n))$, as a rational polynomial, has three real roots if and only if $\vartheta \geq 0$, and it has only one real root if and only if $\vartheta < 0$.

If \mathcal{E} is normalized, we set

$$\delta = c_2 + c_1 d \alpha + d \alpha^2.$$

PROPOSITION 2.2. *It holds $\delta = 0$ if and only if \mathcal{E} splits.*

Proof. (see also [17], Lemma 3.13) In fact, if $\mathcal{E} = \mathcal{O}_X(a) \otimes \mathcal{O}_X(-a + c_1)$, for some $a \geq 0$, then a direct computation shows that $\delta = 0$. Conversely, if \mathcal{E} is a non-split bundle, then $\mathcal{E}(\alpha)$ has a non-vanishing section that gives rise to a two-codimensional scheme, whose degree, by [6], Appendix A, 3, C6, is δ , which cannot be 0. \square

Unless stated otherwise, we work over the smooth polarized threefold X and \mathcal{E} is a normalized non-split rank two vector bundle on X .

3. About the Characteristic Numbers ϵ and τ

In this section we want to recall some essentially known properties of the characteristic numbers of the threefold X (see also [16] for more general statements). We start with the following remark.

REMARK 3.1. *For the fixed ample invertible sheaf $\mathcal{O}_X(1)$ we have:*

$$h^0(\mathcal{O}_X(n)) = 0 \text{ for } n < 0, \quad h^0(\mathcal{O}_X) = 1, \quad h^0(\mathcal{O}_X(n)) \neq 0 \text{ for } n > 0,$$

and also $h^0(\mathcal{O}_X(m)) - h^0(\mathcal{O}_X(n)) > 0$ for all $n, m \in \mathbb{Z}$ with $m > n \geq 0$.

Moreover it holds

$$\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^3(\mathcal{O}_X) = 1 - h^0(\mathcal{O}_X(\epsilon)),$$

so we have:

$$\chi(\mathcal{O}_X) = 1 \iff \epsilon < 0, \quad \chi(\mathcal{O}_X) = 0 \iff \epsilon = 0, \quad \chi(\mathcal{O}_X) < 0 \iff \epsilon > 0.$$

PROPOSITION 3.2. *Let $(X, \mathcal{O}_X(1))$ be a smooth polarized threefold with characteristic numbers (d, ϵ, τ) . Then it holds:*

- 1) $\epsilon \geq -4$,
- 2) $\epsilon = -4$ if and only if $X = \mathbb{P}^3$, i.e. $(d, \epsilon, \tau) = (1, -4, 6)$ and so $\frac{\tau}{2d} - \frac{\epsilon^2}{4} = -1$,
- 3) if $\epsilon = -3$, then $(d, \epsilon, \tau) = (2, -3, 8)$ and $\frac{\tau}{2d} - \frac{\epsilon^2}{4} = -\frac{1}{4}$,
- 4) $\epsilon\tau$ is a multiple of 24, in particular if $\epsilon < 0$ then $\epsilon\tau = -24$ and moreover the only possibilities for (ϵ, τ) are the following:

$$(\epsilon, \tau) \in \{(-4, 6), (-3, 8), (-2, 12), (-1, 24)\},$$

- 5) if $\epsilon \neq 0$, then $\tau > 0$,
- 6) if $\epsilon = 0$, then $\tau > -2d$,
- 7) τ is always even,
- 8) if ϵ is even, then $\frac{\tau}{2d} - \frac{\epsilon^2}{4} \geq -1$,
- 9) if ϵ is odd, then $\frac{\tau}{2d} - \frac{\epsilon^2}{4} \geq -\frac{1}{4}$.

Proof. For statements 1), 2), 3) see [16].

- 4) Observe that $\chi(\mathcal{O}_X) = -\frac{1}{24}\epsilon\tau$ is an integer, and moreover, if $\epsilon < 0$, then $\chi(\mathcal{O}_X) = 1$. If $\epsilon < 0$, then by 1) we have $\epsilon \in \{-4, -3, -2, -1\}$ and so we obtain the thesis.
- 5) By Remark 3.1 we have: if $\epsilon > 0$ then $-\frac{1}{24}\epsilon\tau < 0$, while if $\epsilon < 0$ then $-\frac{1}{24}\epsilon\tau > 0$. In both cases we deduce $\tau > 0$.
- 6) If $\epsilon = 0$, then we have

$$\chi(\mathcal{O}_X(n)) = \frac{d}{6}n\left(n^2 + \frac{\tau}{2d}\right),$$

and also

$$\chi(\mathcal{O}_X(n)) = h^0(\mathcal{O}_X(n)) > 0 \quad \forall n > 0,$$

therefore we must have $\frac{2d+\tau}{12} > 0$, so $\tau > -2d$.

- 7) Assume that ϵ is even, then we have

$$d\left(1 - \frac{\epsilon}{2}\right)\left(1 + \frac{\epsilon}{2}\right) + \frac{\tau}{2} = d\left(1 - \frac{\epsilon^2}{4} + \frac{\tau}{2d}\right) = 6\chi\left(\mathcal{O}_X\left(\frac{\epsilon}{2} + 1\right)\right) \in \mathbb{Z}$$

and moreover $d\left(1 - \frac{\epsilon}{2}\right)\left(1 + \frac{\epsilon}{2}\right) \in \mathbb{Z}$, so τ must be even.

If ϵ is odd, the proof is quite similar.

8) Let ϵ be even. If it holds

$$h^0\left(\mathcal{O}_X\left(\frac{\epsilon}{2} + 1\right)\right) - h^0\left(\mathcal{O}_X\left(\frac{\epsilon}{2} - 1\right)\right) = \chi\left(\mathcal{O}_X\left(\frac{\epsilon}{2} + 1\right)\right) < 0,$$

then we must have $h^0\left(\mathcal{O}_X\left(\frac{\epsilon}{2} - 1\right)\right) \neq 0$, which implies

$$h^0\left(\mathcal{O}_X\left(\frac{\epsilon}{2} + 1\right)\right) - h^0\left(\mathcal{O}_X\left(\frac{\epsilon}{2} - 1\right)\right) \geq 0,$$

a contradiction. So we must have

$$\chi\left(\mathcal{O}_X\left(\frac{\epsilon}{2} + 1\right)\right) = \frac{d}{6} \left(1 + \frac{\tau}{2d} - \frac{\epsilon^2}{4}\right) \geq 0,$$

therefore

$$\frac{\tau}{2d} - \frac{\epsilon^2}{4} \geq -1.$$

9) The proof is quite similar to the proof of 8). □

4. Non-stable Vector Bundles ($\alpha \leq 0$)

We make the following assumption:

\mathcal{E} is a normalized non-split rank two vector bundle with $\alpha \leq 0$.

LEMMA 4.1. For every integer n it holds:

$$\chi(\mathcal{O}_X(n - \alpha)) - \chi(\mathcal{O}_X(\epsilon - n - \alpha - c_1)) - \chi(\mathcal{E}(n)) = (n - \zeta_0)\delta.$$

Proof. It is a straightforward computation using formulas (1) and (2) for the Hilbert polynomial of $\mathcal{O}_X(1)$ and \mathcal{E} , respectively. □

PROPOSITION 4.2. Assume that $\zeta_0 < -\alpha - c_1 - 1$. Then it holds:

$$h^1(\mathcal{E}(n)) - h^2(\mathcal{E}(n)) = (n - \zeta_0)\delta$$

for every integer n such that $\zeta_0 < n \leq -\alpha - c_1 - 1$.

Proof. For each n such that $\zeta_0 < n \leq -\alpha - c_1 - 1$ it holds: $\epsilon - n + \alpha < -1$ and $n + \alpha + c_1 \leq -1$, so we have

$$\begin{aligned} h^3(\mathcal{O}_X(n - \alpha)) &= h^0(\mathcal{O}_X(\epsilon - n + \alpha)) = 0 \\ h^3(\mathcal{O}_X(\epsilon - n - \alpha - c_1)) &= h^0(\mathcal{O}_X(n + \alpha + c_1)) = 0, \end{aligned}$$

therefore we obtain:

$$\begin{aligned} h^0(\mathcal{E}(n)) &= h^0(\mathcal{O}_X(n - \alpha)) = \chi(\mathcal{O}_X(n - \alpha)) \\ h^3(\mathcal{E}(n)) &= h^0(\mathcal{E}(\epsilon - n - c_1)) = h^0(\mathcal{O}_X(\epsilon - n - \alpha - c_1)) \\ &= \chi(\mathcal{O}_X(\epsilon - n - \alpha - c_1)). \end{aligned}$$

Hence

$$\begin{aligned} h^1(\mathcal{E}(n)) - h^2(\mathcal{E}(n)) &= h^0(\mathcal{E}(n)) - h^3(\mathcal{E}(n)) - \chi(\mathcal{E}(n)) = \\ &= \chi(\mathcal{O}_X(n - \alpha)) - \chi(\mathcal{O}_X(\epsilon - n - \alpha - c_1)) - \chi(\mathcal{E}(n)), \end{aligned}$$

so using Lemma 4.1 we obtain the claim. \square

THEOREM 4.3. *Let us assume that $\zeta_0 < -\alpha - c_1 - 1$ and let n be such that $\zeta_0 < n \leq -\alpha - 1 - c_1$. Then $h^1(\mathcal{E}(n)) \geq (n - \zeta_0)\delta$. In particular $h^1(\mathcal{E}(n)) \neq 0$.*

Proof. It is enough to observe that $h^1(\mathcal{E}(n)) - h^2(\mathcal{E}(n)) = (n - \zeta_0)\delta$, by Proposition 4.2, and that the right side of this equality is strictly positive for a non-split vector bundle. \square

REMARK 4.4. *Observe that the above theorem describes a non-empty set of integers if and only if $-\alpha - c_1 - 1 > \zeta_0$; this means $\alpha < -\frac{\epsilon+2+c_1}{2}$, i.e. $\alpha \leq -\frac{\epsilon+3+c_1}{2}$. So our assumption on α agrees with the bound of [11]. Observe that the inequality on α implies that $\alpha \leq -2$ if $\epsilon \geq 1$.*

The non-vanishing result above can be improved, if other invariants both of the threefold and the bundle are considered.

Now we set $\lambda = \frac{\tau}{2d} - \frac{\epsilon^2}{4}$ and consider the following degree 3 polynomial:

$$F(X) = X^3 + \left(\lambda - \frac{6\delta}{d}\right)X + \frac{6\delta}{d}\left(\alpha + \frac{c_1}{2}\right).$$

It is easy to see that, if $\frac{6\delta}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} \leq 0$, then $F(X)$ is strictly increasing and so it has only one real root X_0 .

THEOREM 4.5. *Assume that $\frac{6\delta}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} \leq 0$. Let n be such that $\epsilon - \alpha - c_1 + 1 \leq n < -\alpha + X_0 + \zeta_0$, where $X_0 =$ unique real root of $F(X)$. Then $h^1(\mathcal{E}(n)) \geq -\frac{d}{6}F\left(n + \alpha - \zeta_0 + \frac{c_1}{2}\right) > -\frac{d}{6}F(X_0) = 0$. In particular $h^1(\mathcal{E}(n)) \neq 0$.*

Proof. For each n such that $\epsilon - \alpha - c_1 + 1 \leq n < -\alpha + X_0 + \zeta_0$ it holds: $\epsilon - n + \alpha \leq -1$ and $\epsilon - n - c_1 \leq \alpha - 1$, so we have

$$\begin{aligned} h^3(\mathcal{O}_X(n - \alpha)) &= h^0(\mathcal{O}_X(\epsilon - n + \alpha)) = 0 \\ h^3(\mathcal{E}(n)) &= h^0(\mathcal{E}(\epsilon - n - c_1)) = 0. \end{aligned}$$

Moreover, taking into account the exact sequence

$$0 \rightarrow \mathcal{O}_X(n - \alpha) \rightarrow \mathcal{E}(n) \rightarrow \mathcal{I}_Z(n + \alpha) \rightarrow 0$$

which arises from the Serre correspondence (see [18], Theorem 4), and where Z is the zero-locus of a non-zero section of $\mathcal{E}(\alpha)$, we obtain:

$$h^0(\mathcal{E}(n)) \geq h^0(\mathcal{O}_X(n - \alpha)) = \chi(\mathcal{O}_X(n - \alpha)).$$

Hence

$$\begin{aligned} h^1(\mathcal{E}(n)) &= h^0(\mathcal{E}(n)) + h^2(\mathcal{E}(n)) - h^3(\mathcal{E}(n)) - \chi(\mathcal{E}(n)) \\ &\geq \chi(\mathcal{O}_X(n - \alpha)) - \chi(\mathcal{E}(n)) = \text{(by Lemma 4.1)} \\ &= (n - \zeta_0)\delta + \chi(\mathcal{O}_X(\epsilon - n - \alpha - c_1)) \\ &= (n - \zeta_0)\delta - \frac{d}{6} \left(n + \alpha - \zeta_0 + \frac{c_1}{2} \right) \left[\left(n + \alpha - \zeta_0 + \frac{c_1}{2} \right)^2 + \lambda \right], \end{aligned}$$

so, if we put $X = n + \alpha - \zeta_0 + \frac{c_1}{2}$, then we obtain: $h^1(\mathcal{E}(n)) \geq -\frac{d}{6}F(X) > -\frac{d}{6}F(X_0) = 0$, because of the hypothesis $n < -\alpha + X_0 + \zeta_0$ and the fact that F is strictly increasing. \square

The proofs of the above theorems work perfectly without any restriction on ϵ , while for the proof of the following theorem a few more words are required if $\epsilon \leq 0$.

THEOREM 4.6. *Assume that $\frac{6\delta}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} - \frac{3c_1^2}{4} \geq 0$. Let $n > \zeta_0$ be such that $\epsilon - \alpha - c_1 + 1 \leq n < \zeta_0 + \sqrt{\frac{6\delta}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} - \frac{3c_1^2}{4}}$ and put*

$$S(n) = \frac{d}{6} \left(n - \frac{\epsilon - c_1}{2} \right) \left[\left(n - \frac{\epsilon - c_1}{2} \right)^2 - 6 \frac{c_2 + d\alpha^2 + c_1 d\alpha}{d} + \frac{\tau}{2d} - \frac{\epsilon^2}{4} + \frac{3c_1^2}{4} \right].$$

Then $h^1(\mathcal{E}(n)) \geq -S(n) > 0$. In particular $h^1(\mathcal{E}(n)) \neq 0$.

Proof. Case 1: $\epsilon \geq 1$. Assume $c_1 = 0$. Under our hypothesis $h^0(\mathcal{E}(\epsilon - n)) = 0$ and so $h^1(\mathcal{E}(n)) - h^2(\mathcal{E}(n)) \geq h^0(\mathcal{O}_X(n - \alpha)) - \chi(\mathcal{E}(n))$. Observe that $h^0(\mathcal{O}_X(n - \alpha)) - \chi(\mathcal{E}(n)) + S(n) = -\frac{1}{2}nd\alpha(-\epsilon + n + \alpha) - \frac{1}{12}d\alpha(-3\epsilon\alpha + 2\alpha^2 + \epsilon^2 + \frac{\tau}{d}) \geq 0$ (by direct computation). Therefore we have: $h^1(\mathcal{E}(n)) \geq h^2(\mathcal{E}(n)) - S(n)$. Hence $h^1(\mathcal{E}(n))$ may possibly vanish when

$$\left(n - \frac{\epsilon}{2} \right)^2 - 6 \frac{c_2 + d\alpha^2}{d} + \frac{\tau}{2d} - \frac{\epsilon^2}{4} \geq 0.$$

When $S(n) < 0$, so $-S(n) > 0$, $h^1(\mathcal{E}(n)) \geq -S(n) > 0$ and in particular it cannot vanish.

If $c_1 = -1$ the proof is quite similar.

Case 2: $\epsilon \leq 0$.

A. $\epsilon \leq -2$.

We need to know that

$$\frac{1}{2}nd\alpha(-\epsilon + n + \alpha) + \frac{1}{12}d\alpha\left(\epsilon^2 + \frac{\tau}{d} - 3\epsilon\alpha + 2\alpha^2\right) \leq 0.$$

The first term of the sum is for sure negative; as for

$$\frac{1}{12}d\alpha\left(\epsilon^2 + \frac{\tau}{d}\right) + \frac{1}{12}d\alpha^2(-3\epsilon + 2\alpha)$$

we observe that the quantity in brackets has discriminant

$$\Delta = \epsilon^2 - 8\frac{\tau}{d} = 4\left(\frac{\epsilon^2}{4} - \frac{\tau}{2d} + \frac{\tau}{2d} - 8\frac{\tau}{d}\right) \leq 4(1 - 15) < 0.$$

Therefore it is positive for all $\alpha \leq 0$ and the product is negative.

B. $\epsilon = -1$.

We need to know that

$$\frac{1}{2}nd\alpha(1 + n + \alpha) + \frac{1}{12}d\alpha\left(1 + \frac{\tau}{d}\right) + \frac{1}{12}d\alpha^2(3 + 2\alpha) \leq 0.$$

If $\alpha \leq -2$, then it is enough to observe that $\frac{\tau}{d} + 3\alpha + 2\alpha^2 \geq 0$. If $\alpha = -1$ we have to consider $-\frac{1}{2}n^2d + \frac{1}{12}d\frac{\tau}{d}$ and then we observe that $6n^2 + \frac{\tau}{d} > 0$. If $\alpha = 0$ obviously the quantity is 0.

C. $\epsilon = 0$.

In theorem 4.5 we need to know that

$$\frac{1}{2}nd\alpha(n + \alpha) + \frac{1}{12}d\alpha\left(\frac{\tau}{d}\right) + \frac{1}{12}d\alpha^2(2\alpha) \leq 0.$$

It is enough to observe that $n + \alpha \geq 1$ and that $2\alpha^2 + \frac{\tau}{d} > 0$ (by Proposition 3.2(6)), if $\alpha < 0$; otherwise we have a 0 quantity. \square

REMARK 4.7. *Observe that in Theorems 4.5 and 4.6 α can be zero.*

REMARK 4.8. *Observe that the case $\alpha = 0$ in Theorem 4.3 can occur only if $\epsilon \leq -c_1 - 3$.*

REMARK 4.9. In theorem 4.6 we do not use the hypothesis $-\frac{\epsilon+3}{2} \geq \alpha$, but we assume that $6\frac{c_2+d\alpha^2}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} - 1 \geq 0$. In theorem 4.5 we do not use the hypothesis $-\frac{\epsilon+3}{2} \geq \alpha$, but we assume that $6\frac{c_2+d\alpha^2}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} < 0$. Moreover in both theorems there is a range for n , the left endpoint being $\epsilon - \alpha - c_1 + 1$ and the right endpoint being either $\zeta_0 + \sqrt{6\frac{c_2+d\alpha^2}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} - 1}$ (4.6) or $\zeta_0 - \alpha + X_0$ (4.5).

In [11] there are examples of ACM non-split vector bundles on smooth threefolds in \mathbb{P}^4 , with $-\frac{\epsilon+3+c_1}{2} < \alpha < \frac{\epsilon+5-c_1}{2}$. We want to emphasize that our theorems do not conflict with the examples of [11]: if C is any curve described in [11] and lying on a smooth threefold of degree d , then our numerical constraints cannot be satisfied (we have checked it directly in many but not all cases).

REMARK 4.10. Let us consider a smooth degree d threefold $X \subset \mathbb{P}^4$.

We have:

$$\epsilon = d - 5, \quad \tau = d(10 - 5d + d^2), \quad \vartheta = \frac{3c_2}{d} - \frac{d^2 - 5 + 3c_1^2}{4}$$

(see [18]). As to the characteristic function of \mathcal{O}_X and \mathcal{E} , it holds:

$$\begin{aligned} \chi(\mathcal{O}_X(n)) &= \frac{d}{6} \left(n - \frac{d-5}{2} \right) \left[\left(n - \frac{d-5}{2} \right)^2 + \frac{d^2-5}{4} \right], \\ \chi(\mathcal{E}(n)) &= \frac{d}{3} \left(n - \frac{d-5-c_1}{2} \right) \left[\left(n - \frac{d-5-c_1}{2} \right)^2 + \frac{d^2}{4} - \frac{5}{4} + \frac{3c_1^2}{4} - \frac{3c_2}{d} \right]. \end{aligned}$$

Then it is easy to see that the hypothesis of Theorem 4.6, i.e. $6\frac{\delta}{d} - \frac{d^2-5+3c_1^2}{4} \geq 0$ is for sure fulfilled if $c_2 \geq 0, \alpha \leq -\frac{d-2+c_1}{2}$. In fact we have (for the sake of simplicity when $c_1 = 0$): $-6\frac{6c_2+d\alpha^2}{d} + \frac{d^2-5}{4} \leq \frac{d^2-5}{4} - 6\frac{d^2-2d+1}{4} = -\frac{5d^2-12d+11}{4} < 0$.

REMARK 4.11. Condition **(C2)** holds for sure if X is a smooth hypersurface of \mathbb{P}^4 . In general, for a characteristic 0 base field, only the Kodaira vanishing holds ([6], remark 7.15) and so, unless we work over a threefold X having some stronger vanishing, we need assume, in Theorems 4.3, 4.5, 4.6 that $n - \alpha \notin \{0, \dots, \epsilon\}$ (which implies, by duality, that also $\epsilon - n + \alpha \notin \{0, \dots, \epsilon\}$).

Observe that the first assumption ($n - \alpha \notin \{0, \dots, \epsilon\}$) in the case of Theorem 4.3 is automatically fulfilled because of the hypothesis $\zeta_0 < -\alpha - c_1 - 1$, and in Theorems 4.5 and 4.6 because of the hypothesis $\epsilon - \alpha - c_1 + 1 \leq n$. In fact $n - \alpha$ is greater than ϵ . But this implies that $\epsilon - n + \alpha < 0$ and so also the second condition is fulfilled, at least when $\epsilon \geq 0$. For the case $\epsilon < 0$ in positive characteristic see [16].

Observe that, if $\epsilon < 0$, Kodaira, and so **(C2)**, holds for every n .

For a general discussion, also in characteristic $p > 0$, of this question, see section 7, Remark 7.8.

REMARK 4.12. *In the above theorems we assume that \mathcal{E} is a non-split bundle. If \mathcal{E} splits, then (see section 2) $\delta = 0$. In Theorem 4.3 this implies $h^1(\mathcal{E}(n)) - h^2(\mathcal{E}(n)) = 0$ and so nothing can be said on the non-vanishing.*

Let us now consider Theorem 4.6. If $\delta = 0$, then we must have: $\zeta_0 < n < \zeta_0 + \sqrt{-\frac{\tau}{2d} + \frac{\epsilon^2}{4} - \frac{3c_1^2}{4}} \leq \zeta_0 + 1$ (the last inequality depending on Proposition 3.2(8) and (9)). As a consequence ζ_0 cannot be a whole number. Moreover, since we have $2\zeta_0 - \alpha + 1 \leq n < \zeta_0 + \sqrt{-\frac{\tau}{2d} + \frac{\epsilon^2}{4} - \frac{3c_1^2}{4}}$, we obtain that $\zeta_0 < \alpha \leq 0$, hence $\epsilon - c_1 \leq -1$. If $c_1 = 0$, $\epsilon \in \{-1, -3\}$. If $\epsilon = -3$, then n must satisfy the following inequalities: $-\frac{3}{2} < n < -1$ (see Proposition 3.2(8)), which is a contradiction. If $\epsilon = -1$, then, by Proposition 3.2(8), we have $-1 + \alpha + 1 < -\frac{1}{2} + \frac{1}{2} = 0$, which implies $\alpha > 0$, a contradiction. If $c_1 = -1$, then $\epsilon \in \{-2, -4\}$. If $\epsilon = -4$, we have $\sqrt{-\frac{\tau}{2d} + \frac{\epsilon^2}{4} - \frac{3c_1^2}{4}} = \frac{1}{2}$, and so we must have: $-\frac{3}{2} < n < -1$, which is impossible. If $\epsilon = -2$, then $\zeta_0 = -\frac{1}{2}$ and so $-2 - \alpha + 2 < -\frac{1}{2} + \sqrt{1 - \frac{3}{4}}$, which implies $-\alpha < 0$, hence $\alpha > 0$, a contradiction with the non-stability of \mathcal{E} .

Then we consider Theorem 4.5. The vanishing of δ on the one hand implies $\lambda > 0$ and $X_0 = 0$. But on the other hand from our hypothesis on the range of n we see that $\zeta_0 \leq -2$, hence $\epsilon = -4, c_1 = 0$. But this contradicts Proposition 3.2(2).

5. Stable Vector Bundles

We start with the following lemma which holds both in the stable and in the non-stable case but is useful only in the present section.

LEMMA 5.1. *If $h^1(\mathcal{E}(m)) = 0$ for some integer $m \leq \alpha - 2$, then $h^1(\mathcal{E}(n)) = 0$ for all $n \leq m$.*

Proof. First of all observe that, by our condition **(C3)**, from the restriction exact sequence we can obtain in cohomology the exact sequence

$$0 \rightarrow H^0(\mathcal{E}(m)) \rightarrow H^0(\mathcal{E}(m+1)) \rightarrow H^0(\mathcal{E}_H(m+1)) \rightarrow 0.$$

Since $m+1 \leq \alpha-1$ we obtain that $h^0(\mathcal{E}_H(m+1)) = 0$, and so $h^0(\mathcal{E}_H(t)) = 0$ for every $t \leq m+1$. This implies that $h^1(\mathcal{E}(t-1)) \leq h^1(\mathcal{E}(t))$ for each $t \leq m+1$, and so we prove the claim. (Our proof is quite similar to the one given in [17] for \mathbb{P}^3 , where condition **(C3)** is automatically fulfilled). \square

In the present section we assume that $\alpha \geq \frac{\epsilon - c_1 + 5}{2}$, or equivalently that $c_1 + 2\alpha \geq \epsilon + 5$. This means that $\alpha \geq 1$ in any event, so \mathcal{E} is stable.

THEOREM 5.2. *Let \mathcal{E} be a rank 2 vector bundle on the threefold X with first relevant level α . If $\alpha \geq \frac{\epsilon + 5 - c_1}{2}$, then $h^1(\mathcal{E}(n)) \neq 0$ for $w_0 \leq n \leq \alpha - 2$.*

Proof. By the hypothesis it holds $w_0 \leq \alpha - 2$, so we have $h^0(\mathcal{E}(n)) = 0$ for all $n \leq w_0 + 1$. Assume $h^1(\mathcal{E}(w_0)) = 0$, then by Lemma 5.1 it holds $h^1(\mathcal{E}(n)) = 0$ for every $n \leq w_0$. Therefore we have

$$\chi(\mathcal{E}(w_0)) = h^0(\mathcal{E}(w_0)) + h^1(\mathcal{E}(-w_0 + \epsilon - c_1)) - h^0(\mathcal{E}(-w_0 + \epsilon - c_1)) = 0.$$

Now observe that the characteristic function has at most three real roots, that are symmetric with respect to ζ_0 . Therefore, if w_0 is a root, then $w_0 = \zeta_0 + \sqrt{\vartheta}$ and the other roots are ζ_0 and $\zeta_0 - \sqrt{\vartheta}$. This implies that $\chi(\mathcal{E}(w_0 + 1)) > 0$. On the other hand

$$\chi(\mathcal{E}(w_0 + 1)) = -h^1(\mathcal{E}(w_0 + 1)) \leq 0,$$

a contradiction. So we must have $h^1(\mathcal{E}(w_0)) \neq 0$, then by Lemma 5.1 we obtain the thesis. \square

REMARK 5.3. *If \mathcal{E} is ACM, then $\alpha < \frac{\epsilon+5-c_1}{2}$.*

THEOREM 5.4. *Let \mathcal{E} be a normalized rank 2 vector bundle on the threefold X with $\vartheta \geq 0$ and $w_0 < \zeta$. Then the following hold:*

- 1) $h^1(\mathcal{E}(n)) \neq 0$ for $\zeta_0 < n < \zeta$, i.e. for $w_0 \leq n \leq \bar{\alpha} - 2$, and also for $n = \bar{\alpha} - 1$ if $\zeta \notin \mathbb{Z}$.
- 2) If $\zeta \in \mathbb{Z}$ and $\alpha < \bar{\alpha}$, then $h^1(\mathcal{E}(\bar{\alpha} - 1)) \neq 0$.

Proof.

- 1) The Hilbert polynomial of the bundle \mathcal{E} is strictly negative for each integer such that $w_0 \leq n < \zeta$, but for such an integer n we have $h^2(\mathcal{E}(n)) \geq 0$ and $h^0(\mathcal{E}(n)) - h^0(\mathcal{E}(-n + \epsilon - c_1)) \geq 0$ since $n \geq -n + \epsilon - c_1$ for every $n \geq w_0$, therefore we must have $h^1(\mathcal{E}(n)) \neq 0$. The other statements hold because $\bar{\alpha}$ is, by definition, the integral part of $\zeta + 1$.
- 2) If $\zeta \in \mathbb{Z}$, then $\zeta = \bar{\alpha} - 1$, so we have $\chi(\mathcal{E}(\bar{\alpha} - 1)) = \chi(\mathcal{E}(\zeta)) = 0$. Moreover $h^0(\mathcal{E}(\bar{\alpha} - 1)) \neq 0$ since $\alpha < \bar{\alpha}$, therefore $h^0(\mathcal{E}(\bar{\alpha} - 1)) - h^3(\mathcal{E}(\bar{\alpha} - 1)) > 0$, and $h^1(\mathcal{E}(n)) = 0$ implies $h^1(\mathcal{E}(m))$, for all $m \leq n$; hence we must have $h^1(\mathcal{E}(\bar{\alpha} - 1)) \neq 0$ to obtain the vanishing of $\chi(\mathcal{E}(\bar{\alpha} - 1))$. \square

REMARK 5.5. *Observe that in this section we assume $\alpha \geq \frac{\epsilon-c_1+5}{2}$, in order to have $w_0 \leq \alpha - 2$ and so to have a non-empty range for n in Theorem 5.2.*

REMARK 5.6. *Observe that in the stable case we need not assume any vanishing of $h^1(\mathcal{O}_X(n))$.*

REMARK 5.7. *Observe that split bundles are excluded in this section because they cannot be stable.*

6. Examples

We need the following

REMARK 6.1. *Let $X \subset \mathbb{P}^4$ be a smooth threefold of degree d and let f be the projection onto \mathbb{P}^3 from a general point of \mathbb{P}^4 not on X , and consider a normalized rank two vector bundle \mathcal{E} on \mathbb{P}^3 which gives rise to the pull-back $\mathcal{F} = f^*(\mathcal{E})$. We want to check that $f_*(\mathcal{O}_X) \cong \bigoplus_{i=0}^{d-1} \mathcal{O}_{\mathbb{P}^3}(-i)$.*

Since f is flat and $\deg(f) = d$, $f_(\mathcal{O}_X)$ is a rank d vector bundle. The projection formula and the cohomology of the hypersurface X shows that $f_*(\mathcal{O}_X)$ is ACM. Thus there are integers $a_0 \geq \dots \geq a_{d-1}$ such that $f_*(\mathcal{O}_X) \cong \bigoplus_{i=0}^{d-1} \mathcal{O}_{\mathbb{P}^3}(a_i)$. Since $h^0(X, \mathcal{O}_X) = 1$, the projection formula gives $a_0 = 0$ and $a_i < 0$ for all $i > 0$. Since $h^0(X, \mathcal{O}_X(1)) = 5 = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) + h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3})$, the projection formula gives $a_1 = -1$ and $a_i \leq -2$ for all $i \geq 2$. Fix an integer $t \leq d-2$ and assume proved $a_i = -i$ for all $i \leq t$ and $a_i < -t$ for all $i > t$. Since $h^0(X, \mathcal{O}_X(t+1)) = \binom{t+5}{4} = \sum_{i=0}^t \binom{t+4-i}{3}$, we get $a_{t+1} = -t-1$ and, if $t+1 \leq d-2$, $a_i < -t-1$ for all $i > t+1$. Since $f_*(\mathcal{O}_X) \cong \bigoplus_{i=0}^{d-1} \mathcal{O}_{\mathbb{P}^3}(-i)$, the projection formula gives the following formula for the first cohomology module:*

$$H^i(\mathcal{F}(n)) \cong H^i(\mathcal{E}(n)) \oplus H^i(\mathcal{E}(n-1)) \oplus \dots \oplus H^i(\mathcal{E}(n-d+1))$$

all i . Observe that, as a consequence of the above equality for $i = 0$, we obtain that \mathcal{F} has the same α as \mathcal{E} . Moreover the pull-back $\mathcal{F} = f^(\mathcal{E})$ and \mathcal{E} have the same Chern class c_1 , while $c_2(\mathcal{F}) = d c_2(\mathcal{E})$ and therefore $\delta(\mathcal{F}) = d \delta(\mathcal{E})$.*

Examples:

1. (a stable vector bundle with $c_1 = 0$, $c_2 = 4$ on a quadric hypersurface X).

Choose $d = 2$ and take the pull-back \mathcal{F} of the stable vector bundle \mathcal{E} on \mathbb{P}^3 of [17], example 4.1. Then the numbers of \mathcal{F} (see Notation) are: $c_1 = 0$, $c_2 = 4$, $\alpha = 1$, $\bar{\alpha} = 2$, $\zeta_0 = -\frac{3}{2}$, $w_0 = -1$, $\vartheta = \frac{25}{4}$, $\zeta = -\frac{3}{2} + \sqrt{\frac{25}{4}} = 1 \in \mathbb{Z}$. From [17], example 4.1, we know that $h^1(\mathcal{E}) \neq 0$. Since $H^1(\mathcal{F}(1)) \cong H^1(\mathcal{E}(1)) \oplus H^1(\mathcal{E})$, we have: $h^1(\mathcal{F}(1)) \neq 0$, one shift higher than it is stated in Theorem 5.4(2).

2. (a non-stable vector bundle with $c_1 = 0$, $c_2 = 45$ on a hypersurface of degree 5).

Choose $d = 5$ and take the pull-back \mathcal{F} of the stable vector bundle \mathcal{E} on \mathbb{P}^3 of [17], example 4.5. Then the numbers of \mathcal{F} (see Notation) are: $c_1 = 0$, $c_2 = 45$, $\alpha = -3$, $\delta = 90$, $\zeta_0 = 0$. From [17], Theorem 3.8, we know that $h^1(\mathcal{E}(12)) \neq 0$. Since $H^1(\mathcal{F}(16)) \cong H^1(\mathcal{E}(16)) \oplus \dots \oplus H^1(\mathcal{E}(12))$, we have: $h^1(\mathcal{F}(16)) \neq 0$ (Theorem 4.5 states that $h^1(\mathcal{F}(10)) \neq 0$).

3. (a stable vector bundle with $c_1 = -1$, $c_2 = 2$ on a quadric hypersurface).

Let \mathcal{E} be the rank two vector bundle corresponding to the union of two skew lines on a smooth quadric hypersurface $Q \subset \mathbb{P}^4$. Then its numbers are : $c_1 = -1$, $c_2 = 2$, $\alpha = 1$ and it is known that $h^1(\mathcal{E}(n)) \neq 0$ if and only if $n = 0$.

Observe that in this case $\vartheta = \frac{5}{2} \geq 0$, $\zeta_0 = -1$, $\bar{\alpha} = 1$. Therefore Theorem 5.4 states exactly that $h^1(\mathcal{E}) \neq 0$, hence this example is sharp.

4. (a non-stable vector bundle with $c_1 = 0$, $c_2 = 8$ on a quadric hypersurface).

Choose $d = 2$ and take the pull-back \mathcal{F} of the non-stable vector bundle \mathcal{E} on \mathbb{P}^3 of [17], example 4.10. Then the numbers of \mathcal{F} (see Notation) are: $c_1 = 0$, $c_2 = 8$, $\alpha = 0$, $\zeta_0 = -\frac{3}{2}$, $\delta = 8$. We know (see [17], example 4.10) that $h^1(\mathcal{E}(2)) \neq 0$, $h^1(\mathcal{E}(3)) = 0$. Since $H^1(\mathcal{F}(3)) \cong H^1(\mathcal{E}(3)) \oplus H^1(\mathcal{E}(2))$, we have: $h^1(\mathcal{F}(3)) \neq 0$, exactly the bound of Theorem 4.6.

REMARK 6.2. *The bounds for a degree d threefold in \mathbb{P}^4 agree with [17], where \mathbb{P}^3 is considered.*

7. Threefolds with $\text{Pic}(X) \neq \mathbb{Z}$

Let X be a smooth and connected projective threefold defined over an algebraically closed field \mathbf{k} . Let $\text{Num}(X)$ denote the quotient of $\text{Pic}(X)$ by numerical equivalence. Numerical classes are denoted by square brackets $[\]$. We assume $\text{Num}(X) \cong \mathbb{Z}$ and take the unique isomorphism $\eta: \text{Num}(X) \rightarrow \mathbb{Z}$ such that 1 is the image of a fixed ample line bundle. Notice that $M \in \text{Pic}(X)$ is ample if and only if $\eta([M]) > 0$.

REMARK 7.1. *Let $\eta: \text{Num}(X) \rightarrow \mathbb{Z}$ be as before. Notice that every effective divisor on X is ample and hence its η is strictly positive. For any $t \in \mathbb{Z}$ set $\text{Pic}_t(X) := \{L \in \text{Pic}(X) \mid \eta([L]) = t\}$. Hence $\text{Pic}_0(X)$ is the set of all isomorphism classes of numerically trivial line bundles on X . The set $\text{Pic}_0(X)$ is parametrized by a scheme of finite type ([10], Proposition 1.4.37). Hence for each $t \in \mathbb{Z}$ the set $\text{Pic}_t(X)$ is bounded. Let now \mathcal{E} be a rank 2 vector bundle on X . Since $\text{Pic}_1(X)$ is bounded there is a minimal integer t such that there is $B \in \text{Pic}_t(X)$ and $h^0(\mathcal{E} \otimes B) > 0$. Call it $\alpha(\mathcal{E})$ or just α . By the definition of α there is $B \in \text{Pic}_\alpha(X)$ such that $h^0(X, \mathcal{E} \otimes B) > 0$. Hence there is a non-zero map $j: B^* \rightarrow \mathcal{E}$. Since B^* is a line bundle and $j \neq 0$, j is injective. The definition of α gives the non-existence of a non-zero effective divisor D such that j factors through an inclusion $B^* \rightarrow B^*(D)$, because $\eta([D]) > 0$. Thus the inclusion j induces an exact sequence*

$$0 \rightarrow B^* \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z \otimes B \otimes \det(\mathcal{E}) \rightarrow 0 \quad (3)$$

in which Z is a closed subscheme of X of pure codimension 2.

Observe that $\eta([B]) = \alpha$, $\eta([B^*]) = -\alpha$, $\eta([B \otimes \det(\mathcal{E})]) = \alpha + c_1$, hence the exact sequence is quite similar to the usual exact sequence that holds true in the case $\text{Pic}(X) \cong \mathbb{Z}$.

NOTATION. We set $\epsilon := \eta([\omega_X])$, $\alpha := \alpha(\mathcal{E})$ and $c_1 := \eta([\det(\mathcal{E})])$. So we can speak of a normalized vector bundle \mathcal{E} , with $c_1 \in \{0, -1\}$. Moreover we say that \mathcal{E} is stable if $\alpha > 0$, non-stable if $\alpha \leq 0$. Furthermore $\zeta_0, \zeta, w_0, \bar{\alpha}, \vartheta$ are defined as in section 2.

REMARK 7.2. Fix any $L \in \text{Pic}_1(X)$ and set: $d = L^3 = \text{degree of } X$. The degree d does not depend on the numerical equivalence class. In fact, if R is numerically equivalent to 0, then $(L+R)^3 = L^3 + R^3 + 3L^2R + 3LR^2 = L^3 + 0 + 0 + 0 = L^3$. Then it is easy to see that the formulas for $\chi(\mathcal{O}_X(n))$ and $\chi(\mathcal{E}(n))$ given in section 2 still hold if we consider $\mathcal{O}_X \otimes L^{\otimes n}$ and $\mathcal{E} \otimes L^{\otimes n}$ (see [18]).

REMARK 7.3.

- (a) Assume the existence of $L \in \text{Pic}(X)$ such that $\eta([L]) = 1$ and $h^0(X, L) > 0$. Then for every integer $t > \alpha$ there is $M \in \text{Pic}(X)$ such that $\eta([M]) = t$ and $h^0(X, \mathcal{E} \otimes M) > 0$.
- (b) Assume $h^0(X, L) > 0$ for every $L \in \text{Pic}(X)$ such that $\eta([L]) = 1$. Then $h^0(X, \mathcal{E} \otimes M) > 0$ for every $M \in \text{Pic}(X)$ such that $\eta([M]) > \alpha$.

PROPOSITION 7.4. Let \mathcal{E} be a normalized rank two vector bundle and assume the existence of a spanned $R \in \text{Pic}(X)$ such that $\eta([R]) = 1$. If $\text{char}(\mathbf{k}) > 0$, assume that $|R|$ induces an embedding of X outside finitely many points. Assume

$$2\alpha \leq -\epsilon - 3 - c_1 \quad (4)$$

and $h^1(X, \mathcal{E} \otimes N) = 0$ for every $N \in \text{Pic}(X)$ such that $\eta([N]) \in \{-\alpha - c_1 - 1, \alpha + 2 + e\}$. If $h^1(X, B) = 0$ for every $B \in \text{Pic}(X)$ such that $\eta([B]) = -2\alpha - c_1$, then \mathcal{E} splits.

If moreover $h^1(X, M) = 0$ for every $M \in \text{Pic}(X)$ then it is enough to assume that $h^1(X, \mathcal{E} \otimes N) = 0$ for every $N \in \text{Pic}(X)$ such that $\eta([N]) = -\alpha - c_1 - 1$.

Proof. By assumption there is $M \in \text{Pic}(X)$ such that $\eta([M]) = \alpha$ and $h^0(X, \mathcal{E} \otimes M) > 0$. Set $A := M^*$. We have seen in remark 7.1 that \mathcal{E} fits into an extension of the following type:

$$0 \rightarrow A \rightarrow \mathcal{E} \rightarrow \mathcal{I}_C \otimes \det(\mathcal{E}) \otimes A^* \rightarrow 0 \quad (5)$$

with C a locally complete intersection closed subscheme of pure dimension 1. Let H be a general element of $|R|$ and T the intersection of H with another general element of $|R|$. Observe that T , under our assumptions, is generically

reduced by Bertini's Theorem (see [6], Theorem II, 8.18 and Remark II, 8.18.1). Since R is spanned, T is a locally complete intersection curve and $C \cap T = \emptyset$. Hence $\mathcal{E}|_T$ is an extension of $\det(\mathcal{E}) \otimes A^*|_T$ by $A|_T$. Since T is generically reduced and locally a complete intersection, it is reduced. Hence $h^0(T, M^*) = 0$ for every ample line bundle M on T . Since $\omega_T \cong (\omega_X \otimes R^{\otimes 2})|_T$, we have $\dim(\text{Ext}_T^1(\det(\mathcal{E}) \otimes A^*, A)) = h^0(T, (\det(\mathcal{E}) \otimes (A^*)^{\otimes 2} \otimes \omega_X \otimes R^{\otimes 2})|_T) = 0$ (indeed $\eta([\det(\mathcal{E}) \otimes (A^*)^{\otimes 2} \otimes \omega_X \otimes R^{\otimes 2}]) = 2\alpha + c_1 + e + 2 < 0$). Hence $\mathcal{E}|_T \cong A|_T \oplus (\det(\mathcal{E}) \otimes A^*)|_T$. Let σ be the non-zero section of $(\mathcal{E} \otimes (A \otimes \det(\mathcal{E})^*))|_T$ coming from the projection onto the second factor of the decomposition just given. The vector bundle $\mathcal{E}|_H$ is an extension of $(\det(\mathcal{E}) \otimes A^*)|_H$ by $A|_H$ if and only if $C \cap H = \emptyset$. Since R is ample, $C \cap H = \emptyset$ if and only if $C = \emptyset$. Hence we get simultaneously $C \cap H = \emptyset$ and $\mathcal{E}|_H \cong A|_H \oplus (\det(\mathcal{E}) \otimes A^*)|_H$ if we prove the existence of $\tau \in H^0(H, (\mathcal{E} \otimes (A \otimes \det(\mathcal{E})^*))|_H)$ such that $\tau|_T = \sigma$. To get τ it is sufficient to have $H^1(H, (E \otimes (A \otimes \det(\mathcal{E})^* \otimes R^*))|_H) = 0$. A standard exact sequence shows that $H^1(H, (\mathcal{E} \otimes (A \otimes \det(\mathcal{E})^* \otimes R^*))|_H) = 0$ if $h^1(X, (\mathcal{E} \otimes (A \otimes \det(\mathcal{E})^* \otimes R^*))) = 0$ and $h^2(X, (\mathcal{E} \otimes (A \otimes \det(\mathcal{E})^* \otimes R^* \otimes R^*))) = 0$. Since $\mathcal{E}^* \cong \mathcal{E} \otimes \det(\mathcal{E})^*$, Serre duality gives $h^2(X, \mathcal{E} \otimes (A \otimes \det(\mathcal{E})^* \otimes R^* \otimes R^*)) = h^1(X, \mathcal{E} \otimes A \otimes R^{\otimes 2} \otimes \omega_X)$. Since $\eta([A \otimes \det(\mathcal{E})^* \otimes R^*]) = -\alpha - c_1 - 1$ and $\eta([A \otimes R^{\otimes 2} \otimes \omega_X]) = \alpha + e + 2$, we get that $C = \emptyset$. The last sentence follows because $\eta([A^{\otimes 2} \otimes \det(\mathcal{E})^*]) = -2\alpha - c_1$. \square

REMARK 7.5. Fix integers $t < z \leq \alpha - 2$. Assume the existence of $L \in \text{Pic}(X)$ such that $\eta([L]) = z$ and $h^1(X, \mathcal{E} \otimes L) = 0$. If there is $R \in \text{Pic}(X)$ such that $\eta([R]) = 1$ and $h^0(X, R) > 0$, then there exists $M \in \text{Pic}(X)$ such that $\eta([M]) = t$ and $h^1(X, \mathcal{E} \otimes M) = 0$. If $h^0(X, R) > 0$ for every $R \in \text{Pic}(X)$ such that $\eta([R]) = 1$, then $h^1(X, \mathcal{E} \otimes M) = 0$ for every $M \in \text{Pic}(X)$ such that $\eta([M]) = t$.

The proof can follow the lines of Lemma 5.1. In fact consider a line bundle R with $\eta([R]) = 1$ and let H be the zero-locus of a non-zero section of R ; then we have the following exact sequence:

$$0 \rightarrow \mathcal{E} \otimes L \rightarrow \mathcal{E} \otimes L \otimes R \rightarrow (\mathcal{E} \otimes L \otimes R)|_H \rightarrow 0.$$

Now observe that the vanishing of $h^1(X, \mathcal{E} \otimes L)$ implies that $h^0((\mathcal{E} \otimes L \otimes R)|_H) = 0$. And now we can argue as in Lemma 5.1 (see also [17]).

REMARK 7.6.

- (a) Assume the existence of $L \in \text{Pic}(X)$ such that $\eta([L]) = 1$ and $h^0(X, L) > 0$. Then for every integer $t > \alpha$ there is $M \in \text{Pic}(X)$ such that $\eta([M]) = t$ and $h^0(X, \mathcal{E} \otimes M) > 0$.
- (b) Assume $h^0(X, L) > 0$ for every $L \in \text{Pic}(X)$ such that $\eta([L]) = 1$. Then $h^0(X, \mathcal{E} \otimes M) > 0$ for every $M \in \text{Pic}(X)$ such that $\eta([M]) > \alpha$.

REMARK 7.7. *In all our results of sections 4 and 5 we use the vanishing of $h^1(\mathcal{O}_X(n))$ for all n (and by Serre duality of $h^2(\mathcal{O}_X(n))$) (or, at least, $\forall n \notin \{0, \dots, \epsilon\}$), see Remark 4.11.*

From now on we need to use similar vanishing conditions and so we introduce the following condition:

(C4) $h^1(X, L) = 0$ for all $L \in \text{Pic}(X)$ such that either $\eta([L]) < 0$ or $\eta([L]) > \epsilon$.

Observe that (C4) is always satisfied in characteristic 0 (by the Kodaira vanishing theorem). In positive characteristic it is often satisfied. This is always the case if X is an abelian variety ([12] page 150).

Observe also that, if $\epsilon \leq -1$, the Kodaira vanishing and our condition put no restriction on n (see also Remark 4.12).

Example. If (4) holds, then $-2\alpha - c_1 > \epsilon$. Hence we may apply Proposition 7.4 to X . In particular observe that, in the case of an abelian variety with $\text{Num}(X) \cong \mathbb{Z}$ or in the case of a Calabi-Yau threefold with $\text{Num}(X) \cong \mathbb{Z}$, we have $\epsilon = 0$. Notice that Proposition 7.4 also applies to any threefold X whose ω_X has finite order.

With the assumption of condition (C4) the proofs of Theorems 4.3, 4.5, 4.6 can be easily modified in order to obtain the statements below (\mathcal{E} is normalized, i.e. $\eta([\det(\mathcal{E})]) \in \{-1, 0\}$), where, by the sake of simplicity, we assume $\epsilon \geq 0$ (if $\epsilon < 0$, (C4), which holds by [16], implies that all the vanishing of h^1 and h^2 for all $L \in \text{Pic}(X)$ hold).

THEOREM 7.8. *Assume (C4), $\alpha \leq 0$, the existence of $R \in \text{Pic}(X)$ such that $\eta([R]) = 1$ and $\zeta_0 < -\alpha - c_1 - 1$. Fix an integer n such that $\zeta_0 < n \leq -\alpha - 1 - c_1$. Fix $L \in \text{Pic}(X)$ such that $\eta([L]) = n$. Then $h^1(\mathcal{E} \otimes L) \geq (n - \zeta_0)\delta > 0$.*

REMARK 7.9. *Observe that we should require the following conditions: $n - \alpha \notin \{0, \dots, \epsilon\}$, $\epsilon - n + \alpha \notin \{0, \dots, \epsilon\}$. But they are automatically fulfilled under the assumption that $\zeta_0 < -\alpha - c_1 - 1$.*

THEOREM 7.10. *Assume (C4), $\alpha \leq 0$, the existence of $R \in \text{Pic}(X)$ such that $\eta([R]) = 1$ and the same hypotheses of Theorem 4.6. Fix $L \in \text{Pic}(X)$ such that $\eta([L]) = n$. Then $h^1(\mathcal{E} \otimes L) \geq -S(n) > 0$ ($S(n)$ being defined as in Theorem 4.6).*

THEOREM 7.11. *Assumption as in Theorem 4.5. Moreover assume (C4) and $n - \alpha \notin \{0, \dots, \epsilon\}$. Fix $L \in \text{Pic}(X)$ such that $\eta([L]) = n$. Then $h^1(\mathcal{E} \otimes L) \geq -\frac{d}{6}F(n + \alpha - \zeta_0 + \frac{c_1}{2}) > 0$ (F being defined as in Theorem 4.5).*

REMARK 7.12. *Observe that in Theorems 7.10 and 7.11 we should require $n - \alpha \notin \{0, \dots, \epsilon\}$, but the assumption $\epsilon - \alpha - c_1 + 1 \leq n$ implies that it is automatically fulfilled.*

The proofs of the above theorems are based on the existence of the exact sequence (3) and on the properties of α . They follow the lines of the proofs given in the case $\text{Pic}(X) \cong \mathbb{Z}$. Here and in section 4 we actually need only the Kodaira vanishing (true in characteristic 0 and assumed in characteristic $p > 0$) and no further vanishing of the first cohomology.

Also the stable case can be extended to a smooth threefold with $\text{Num}(X) \cong \mathbb{Z}$. Observe that the proofs can follow the lines of the proofs given in the case $\text{Pic}(X) \cong \mathbb{Z}$ and make use of Remark 7.6 (which extends Theorem 5.1).

More precisely we have:

THEOREM 7.13. *Assumptions as in Theorem 5.2 and fix $L \in \text{Pic}(X)$ such that $\eta([L]) = n$. Then, if $\alpha \geq \frac{\epsilon+5-c_1}{2}$, then $h^1(\mathcal{E} \otimes L) \neq 0$ for $w_0 \leq n \leq \alpha - 2$.*

THEOREM 7.14. *Assumptions as in Theorem 5.4 and fix $L \in \text{Pic}(X)$ such that $\eta([L]) = n$. Then the following hold:*

- 1) $h^1(\mathcal{E} \otimes L) \neq 0$ for $\zeta_0 < n < \zeta$, i.e. for $w_0 \leq n \leq \bar{\alpha} - 2$, and also for $n = \bar{\alpha} - 1$ if $\zeta \notin \mathbb{Z}$.
- 2) If $\zeta \in \mathbb{Z}$ and $\alpha < \bar{\alpha}$, then $h^1(\mathcal{E} \otimes N) \neq 0$, for every N such that $\eta([N]) = \bar{\alpha} - 1$.

REMARK 7.15. *The above theorems can be applied to any X such that $\text{Num}(X) \cong \mathbb{Z}$, $\epsilon = 0$ and $h^1(X, L) = 0$ for all $L \in \text{Pic}(X)$ such that $\eta([L]) \neq 0$, for instance to $X =$ an abelian threefold with $\text{Num}(X) \cong \mathbb{Z}$.*

REMARK 7.16. *If X is any threefold (in characteristic 0 or positive) such that $h^1(X, L) = 0$, for all $L \in \text{Pic}(X)$, then we can avoid the restriction $n - \alpha \notin \{0, \dots, \epsilon\}$. Not many threefolds, beside any $X \subset \mathbb{P}^4$, fulfill these conditions.*

REMARK 7.17. *Observe that in Theorems 7.13 and 7.14 we do not assume (C4) (see also Remark 5.6).*

REMARK 7.18. *Observe that also in the present case ($\text{Num}(X) \cong \mathbb{Z}$), we have: $\delta = 0$ if and only if \mathcal{E} splits. Therefore Remarks 4.12 and 5.7 apply here.*

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