Fundamental Solutions for Hyperbolic Operators with Variable Coefficients

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ABSTRACT. *In this article we describe a new method to construct fundamental solutions for operators with variable coefficients. That method was introduced in [11] to study the Tricomi-type equation. More precisely, the new integral operator is suggested which transforms the family of the fundamental solutions of the Cauchy problem for the equation with the constant coefficients to the fundamental solutions for the operators with variable coefficients.*

1. Introduction

In this article we describe a new method to construct fundamental solutions for operators with variable coefficients. We also give a brief survey of some results obtained by that method, which was introduced in [11] to solve the Cauchy problem for the Tricomi-type equation. Later on it was applied to several partial differential equations with variable coefficients containing some equations arising in the mathematical cosmology. More precisely, a new integral operator is suggested which transforms the family of the fundamental solutions of the Cauchy problem for the equation with the constant coefficients to the fundamental solutions for the operators with variable coefficients. The kernel of that transformation contains Gauss's hypergeometric function.

This method was used in [5, 6],[11]-[15] to investigate in a unified way several equations such as the linear and semilinear Tricomi and Tricomi-type equations, Gellerstedt equation, the wave equation in Einstein-de Sitter spacetime, the wave and the Klein-Gordon equations in the de Sitter and anti-de Sitter spacetimes. The listed equations play an important role in the gas dynamics, elementary particle physics, quantum field theory in the curved spaces, and cosmology. For all above mentioned equations, we have obtained among other things, fundamental solutions, the representation formulas for the initial-value problem, the $L_p - L_q$-estimates, local and global solutions for the semilinear equations, blow up phenomena, self-similar solutions and number of other re-

sults.

The starting point of our approach is the Duhamel's principle, which we revise in order to prepare the ground for generalizations. It is well-known that the solution of the Cauchy problem for the string equation with the source term f, $f(x,t) \in C^\infty(\mathbf{R}^2)$,

$$u_{tt} - u_{xx} = f(x,t) \quad \text{in } \mathbf{R}^2, \quad u(x,t_0) = 0, \quad u_t(x,t_0) = 0 \quad \text{in } \mathbf{R}, \qquad (1.1)$$

can be written as an integral

$$u(x,t) = \int_{t_0}^t v(x,t;\tau)\,d\tau$$

of the family of the solutions $v(x,t;\tau)$ of the problem without the source term, but with the second initial datum

$$v_{tt} - v_{xx} = 0 \quad \text{in } \mathbf{R}^2, \quad v(x,\tau;\tau) = 0, \quad v_t(x,\tau;\tau) = f(x,\tau) \quad \text{in } \mathbf{R}.$$

Our *first observation* is that we obtain the following representation of the solution of (1.1)

$$u(x,t) = \int_{t_0}^t d\tau \int_0^{t-\tau} w(x,z;\tau)\,dz, \qquad (1.2)$$

if we denote

$$w(x;t;\tau) := \frac{1}{2}\left[f(x+t,\tau) + f(x-t,\tau)\right],$$

where the function $w = w(x;t;\tau)$ is the solution of the problem

$$w_{tt} - w_{xx} = 0 \quad \text{in } \mathbf{R}^2, \quad w(x,0;\tau) = f(x,\tau), \quad w_t(x,0;\tau) = 0 \quad \text{in } \mathbf{R}. \quad (1.3)$$

This formula allows us to solve problems with the source term if we solve the problem for the *same* equation without source term but with the first initial datum. We claim that the formula (1.2) can be used also for the wave equation with $x \in \mathbf{R}^n$, for all $n \in \mathbf{N}$. (See, e.g, [11].) More precisely, it holds also for the problem

$$u_{tt} - \Delta u = f(x,t) \quad \text{in } \mathbf{R}^{n+1}, \quad u(x,t_0) = 0, \quad u_t(x,t_0) = 0 \quad \text{in } \mathbf{R}^n,$$

with the function $w = w(x;t;\tau)$ solving

$$w_{tt} - \Delta w = 0 \text{ in } \mathbf{R}^{n+1}, \quad w(x,0;\tau) = f(x,\tau),\ w_t(x,0;\tau) = 0 \text{ in } \mathbf{R}^n. \quad (1.4)$$

Note that in the last problem the initial time $t = 0$ is frozen, while in the Duhamel's principle it is varying with the parameter τ.

The *second observation* is that in (1.2) the upper limit $t - \tau$ of the inner integral is generated by the propagation phenomena with the speed which is

equals to one. In fact, that is a distance function between the points at time t and τ.
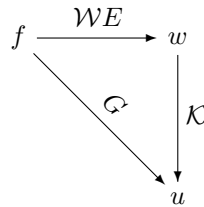
Our *third observation* is that the solution operator $G : f \longmapsto u$ can be regarded as a composition of two operators. The first one

$$\mathcal{W}E : f \longmapsto w$$

is a Fourier Integral Operator (FIO), which is a solution operator of the Cauchy problem with the first initial datum for wave equation in the Minkowski spacetime. The second operator

$$\mathcal{K} : w \longmapsto u$$

is the integral operator given by (1.2). We regard the variable z in (1.2) as a "subsidiary time". Thus, $G = \mathcal{K} \circ \mathcal{W}E$ and we arrive at the diagram:

$$
\begin{array}{ccc}
f & \xrightarrow{\mathcal{W}E} & w \\
 & \searrow{\scriptstyle G} & \downarrow{\scriptstyle \mathcal{K}} \\
 & & u
\end{array}
$$

Our aim is, based on this diagram, to generate a class of operators for which we will obtain explicit representation formulas for the solutions. That means also that we will have representations for the fundamental solutions of the partial differential operator. In fact, this diagram brings into a single hierarchy several different partial differential operators. Indeed, if we take into account the propagation cone by introducing the distance function $\phi(t)$, and if we provide the integral operator with the kernel $K(t; r, b)$ as follows:

$$\mathcal{K}[w](x,t) = 2 \int_{t_0}^{t} db \int_{0}^{|\phi(t)-\phi(b)|} K(t;r,b)w(x,r;b)dr, \quad x \in \mathbf{R}^n,\ t > t_0, \ \ (1.5)$$

then we actually can generate new representations for the solutions of the different well-known equations. Below we illustrate the suggested scheme by several examples.

1.1. Klein-Gordon equation in the Minkowski spacetime

If we choose the kernel $K(t; r, b)$ as

$$K(t;r,b) \;\; = \;\; J_0\left(\sqrt{(t-b)^2 - r^2}\right), \tag{1.6}$$

where $J_0(z)$ is the Bessel function of the first kind, and if we choose the distance function as $\phi(t) = t$, then we can prove (see Theorem 1.1 below) that the function

$$u(x,t) = \int_{t_0}^{t} db \int_{0}^{t-b} J_0\left(\sqrt{(t-b)^2 - r^2}\right) w(x,r;b)dr, \ x \in \mathbf{R}, \ t > t_0,$$

solves the problem for the Klein-Gordon equation with a positive mass equals to 1 in the one-dimensional Minkowski spacetime,

$$u_{tt} - u_{xx} + u = f(x,t) \quad \text{in } \mathbf{R}^2, \quad u(x,t_0) = 0, \quad u_t(x,t_0) = 0 \quad \text{in } \mathbf{R},$$

provided that $w(x,r;b)$ is a corresponding solution of the problem for the wave equation in the Minkowski spacetime. We emphasis that the function $w = w(x,t;b)$, with b regarded as a parameter, and the function $u = u(x,t)$ solve *different* equations. This is a fundamental distinction from the Duhamel's principle.

Now if we choose the kernel $K(t;r,b)$ as

$$K(t;r,b) \quad = \quad I_0\left(\sqrt{(t-b)^2 - r^2}\right), \tag{1.7}$$

where $I_0(z)$ is the modified Bessel function of the first kind, and the distance function as $\phi(t) = t$, then the function

$$u(x,t) = \int_{t_0}^{t} db \int_{0}^{t-b} I_0\left(\sqrt{(t-b)^2 - r^2}\right) w(x,r;b)dr, \ x \in \mathbf{R}^n, \ t > t_0,$$

solves the problem for the Klein-Gordon equation with an imaginary mass in the one-dimensional Minkowski spacetime,

$$u_{tt} - u_{xx} - u = f(x,t) \quad \text{in } \mathbf{R}^2, \quad u(x,t_0) = 0, \quad u_t(x,t_0) = 0 \quad \text{in } \mathbf{R},$$

provided that $w(x,r;b)$ is a corresponding solution of the problem (1.3) for the wave equation in the one-dimensional Minkowski spacetime. According to the next theorem the representation formulas are valid also for the higher dimensional equations. The proof is by straightforward substitution.

THEOREM 1.1. *The functions $u = u_{\mathcal{R}e}(x,t)$ and $u_{\mathcal{I}m}(x,t)$ defined by*

$$u_{\mathcal{R}e}(x,t) = \int_{t_0}^{t} db \int_{0}^{t-b} J_0\left(m\sqrt{(t-b)^2 - r^2}\right) w(x,r;b)dr, \ x \in \mathbf{R}^n, \tag{1.8}$$

$$u_{\mathcal{I}m}(x,t) = \int_{t_0}^{t} db \int_{0}^{t-b} I_0\left(m\sqrt{(t-b)^2 - r^2}\right) w(x,r;b)dr, \ x \in \mathbf{R}^n, \tag{1.9}$$

$m = |M|$, are solutions of the problems

$$u_{tt} - \Delta u + M^2 u = f(x,t) \text{ in } \mathbf{R}^{n+1}, \quad u(x,t_0) = 0, \ u_t(x,t_0) = 0 \text{ in } \mathbf{R}^n,$$

with $M^2 > 0$ and $M^2 < 0$, respectively. Here $w(x,t;b)$ is a solution of (1.4).

DEFINITION 1.2. *The integral operator (1.5) is said to be a generator of the solution operator for some equation if the operator $G = \mathcal{K} \circ \mathcal{W}E$ gives a solution operator for that equation.*

1.2. Tricomi-type equations

The first example linking to the operator with the variable coefficient is generated by the kernel $K(t; r, b) = E(0, t; r, b)$, where the function $E(x, t; r, b)$ [11] is defined by

$$
\begin{aligned}
E(x, t; r, b) \quad &:= \quad c_k \left((\phi(t) + \phi(b))^2 - (x - r)^2 \right)^{-\gamma} \\
&\times F \left(\gamma, \gamma; 1; \frac{(\phi(t) - \phi(b))^2 - (x - r)^2}{(\phi(t) + \phi(b))^2 - (x - r)^2} \right),
\end{aligned} \tag{1.10}
$$

with $\gamma := \frac{k}{2k+2}$, $c_k = (k+1)^{-k/(k+1)} 2^{-1/(k+1)}$, $2k = l \in \mathbf{N} \cup \{0\}$, where $\mathbf{N}$ is the set of natural numbers, and the distance function $\phi = \phi(t)$ is

$$
\phi(t) = \frac{1}{k+1} t^{k+1}, \tag{1.11}
$$

while $F(a, b; c; \zeta)$ is the Gauss's hypergeometric function. Here we assume that $2k \in \mathbf{N} \cup \{0\}$ but later on we consider the case of $l \in \mathbf{R}$. It is proved in [11] that for an integer non-negative l, for the smooth function $f = f(x, t)$, the function

$$
\begin{aligned}
u(x, t) \quad &= \quad 2c_l \int_0^t db \int_0^{\phi(t) - \phi(b)} \left((\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma} \\
&\times F \left(\gamma, \gamma; 1; \frac{(\phi(t) - \phi(b))^2 - r^2}{(\phi(t) + \phi(b))^2 - r^2} \right) w(x, r; b) dr, \quad t > 0,
\end{aligned}
$$

solves the Tricomi-type equation

$$
u_{tt} - t^l \Delta u = f(x, t) \quad \text{in} \quad \mathbf{R}_+^{n+1} := \{ (x, t) \mid x \in \mathbf{R}^n, \, t > 0 \}, \tag{1.12}
$$

and takes vanishing initial values

$$
u(x, 0) = 0, \quad u_t(x, 0) = 0 \quad \text{in } \mathbf{R}^n. \tag{1.13}
$$

1.3. The wave equation in the Robertson-Walker spacetime: de Sitter spacetime

The next interesting example we obtain if we set $K(t; r, b) = E(0, t; r, b)$, where the function $E(x, t; r, b)$ [17] is defined by

$$
\begin{aligned}
E(x, t; r, b) \quad &:= \quad \left((e^{-b} + e^{-t})^2 - (x - r)^2 \right)^{-\frac{1}{2}} \\
&\times F \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{-t} - e^{-b})^2 - (x - r)^2}{(e^{-t} + e^{-b})^2 - (x - r)^2} \right),
\end{aligned} \tag{1.14}
$$

and $\phi(t) := e^{-t}$. For the sake of simplicity, in (1.14) we use the notation $x^2 = x \cdot x = |x|^2$ for $x \in \mathbf{R}^n$. It is proved in [17] that defined by the integral transform (1.5) with the kernel (1.14) the function

$$
\begin{aligned}
u(x,t) \;=\; & 2 \int_0^t db \int_0^{e^{-b}-e^{-t}} \left((e^{-b} + e^{-t})^2 - r^2 \right)^{-\frac{1}{2}} \\
& \times F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{-t} - e^{-b})^2 - r^2}{(e^{-t} + e^{-b})^2 - r^2} \right) w(x,r;\tau)\, dr
\end{aligned}
$$

solves the wave equation in the Robertson-Walker spaces arising in the de Sitter model of the universe (see, e.g. [8]),

$$
u_{tt} - e^{-2t}\Delta u = f(x,t) \quad \text{in } \mathbf{R}_+^{n+1},
$$

and takes vanishing initial data (1.13).

1.4. The wave equation in the Robertson-Walker spacetime: anti-de Sitter spacetime

The third example we obtain if we set $K(t;r,b) = E(0,t;r,b)$, where the function $E(x,t;r,b)$ is defined by (see [18])

$$
\begin{aligned}
E(x,t;r,b) \;:=\; & \left((e^b + e^t)^2 - (x - r)^2 \right)^{-\frac{1}{2}} \\
& \times F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - (x - r)^2}{(e^t + e^b)^2 - (x - r)^2} \right), \quad\quad (1.15)
\end{aligned}
$$

while the distance function is $\phi(t) := e^t$. In that case the function $u = u(x,t)$ produced by the integral transform (1.5) with $t_0 = 0$ and the kernel (1.15), solves the wave equation in the Robertson-Walker space arising in the anti-de Sitter model of the universe (see, e.g. [8]),

$$
u_{tt} - e^{2t}\Delta u = f(x,t) \quad \text{in } \mathbf{R}_+^{n+1}.
$$

Moreover, it takes vanishing initial values (1.13).

1.5. The wave equation in the Einstein-de Sitter spacetime

If we allow negative $l \in \mathbf{R}$ in (1.10) and, in that way, simplify the Gauss's hypergeometric function of the kernel of the integral transform, then we obtain another way to get new operators of the above described hierarchy. In fact, in the hierarchy of the hypergeometric functions $F(a,b;c;\zeta)$ there are functions

which are polynomials. This is a case, in particular, of the parameter $a = -m$, where $m \in \mathbf{N}$. More precisely, if $m \in \mathbf{N}$, then $l = -4m/(2m+1) > -2$ and $F(-m, -m; 1; \zeta) = \sum_{n=0}^{m} \left(\frac{m(m-1)\cdots(m+1-n)}{n!} \right)^2 z^n$. In that case we choose the distance function $\phi(t) = (2m+1)t^{\frac{1}{2m+1}}$ and the kernel $K(t; r, b)$ as follows

$$
\begin{aligned}
K(t; r, b) \;=\; & c_m \sum_{n=0}^{m} \left(\frac{m(m-1)\cdots(m+1-n)}{n!} \right)^2 \\
& \times \left((2m+1)^2 (t^{\frac{1}{2m+1}} + b^{\frac{1}{2m+1}})^2 - r^2 \right)^{-m-n} \\
& \times \left((2m+1)^2 (t^{\frac{1}{2m+1}} - b^{\frac{1}{2m+1}})^2 - r^2 \right)^{n}.
\end{aligned} \tag{1.16}
$$

Thus the integral transform $\mathcal{K}$ allows us to write the representation for the solution of the equation

$$
u_{tt} - t^{-\frac{4m}{2m+1}} \Delta u = f \quad \text{in} \quad \mathbf{R}_+^{n+1}.
$$

Moreover, in the hierarchy of the hypergeometric functions the simplest non-constant function is $F(-1, -1; 1; \zeta) = 1 + \zeta$. The exponent l leading to the function $F(-1, -1; 1; \zeta)$ is exactly the exponent $l = -4/3$ of the *wave equation* (and of the *metric tensor*) *in the Einstein & de Sitter spacetime*. In that case of $m = 1$ the kernel $K(t; r, b)$ of (1.16) is $K(t; r, b) = \frac{1}{18} \left(9t^{2/3} + 9b^{2/3} - r^2 \right)$. Consequently, the function

$$
u(x, t) = \int_0^t db \int_0^{3t^{1/3} - 3b^{1/3}} \frac{1}{18} \left((3t^{1/3})^2 + (3b^{1/3})^2 - r^2 \right) w(x, r; b) dr, \tag{1.17}
$$

$x \in \mathbf{R}^n$, $t > 0$, solves (see [5]) the equation

$$
u_{tt} - t^{-4/3} \Delta u = f \quad \text{in} \quad \mathbf{R}_+^{n+1}, \tag{1.18}
$$

and takes vanishing initial data (1.13) provided that the function w is the image of f, that is $w = \mathcal{WE}(f)$. Because of the singularity in the coefficient of equation (1.18), the Cauchy problem is not well-posed. In order to obtain a well-posed problem the initial conditions must be modified to the weighted initial value conditions.

In fact, the operator of equation (1.18) coincides with the principal part of the wave equation in the Einstein & de Sitter spacetime. We remind that the Einstein & de Sitter model (EdeS model) of the universe was first proposed jointly by Einstein and de Sitter in 1932. It is the simplest non-empty expanding model with the line-element $ds^2 = -dt^2 + a_0^2 t^{4/3} \left(dx^2 + dy^2 + dz^2 \right)$. The covariant linear wave equation with the source term f written in these

coordinates is

$$\left(\frac{\partial}{\partial t}\right)^2 \psi - t^{-4/3} \sum_{i=1,2,3} \left(\frac{\partial}{\partial x^i}\right)^2 \psi + \frac{2}{t}\frac{\partial}{\partial t}\psi = f.$$

The last equation belongs to the family of the non-Fuchsian partial differential equations. There is very advanced theory of such equations (see, e.g., [7, 10]), but according to our knowledge the weighted initial value problem suggested in [5] (see (1.19) below) is the original one. Assume that $f(x,t) \in C^\infty(\mathbf{R}^n \times (0,\infty))$, and that with some $\varepsilon > 0$ one has

$$|\partial_x^\alpha f(x,t)| + |t\partial_t\partial_x^\beta f(x,t)| \le C_\alpha t^{\varepsilon - 2} \quad \text{for all } x \in \mathbf{R}^n, \quad \text{and for small } t > 0,$$

and for every α, β, $|\alpha| \le [\frac{n}{2}] + 2$, $|\beta| \le [\frac{n}{2}] + 1$. It is proved in [5] that the function

$$\psi(x,t) \quad = \quad \frac{1}{18t}\int_0^t db \int_0^{3t^{1/3}-3b^{1/3}} b\big(9t^{2/3} + 9b^{2/3} - r^2\big)w(x,r;b)\,dr$$

solves the problem

$$\begin{cases} \psi_{tt} - t^{-4/3}\triangle\psi + 2t^{-1}\psi_t = f(x,t), & t > 0, \ x \in \mathbf{R}^n, \\ \lim_{t\to 0} t\psi(x,t) = 0, & \lim_{t\to 0}(t\psi_t(x,t) + \psi(x,t)) = 0, \ x \in \mathbf{R}^n, \end{cases} \tag{1.19}$$

provided that the function w is the image of f, that is $w = \mathcal{WE}(f)$.

2. The fundamental solutions of the operators

In this section we apply the method from the previous section to construct the fundamental solutions of the operators for the above listed equations. They are hyperbolic equations and therefore they have the fundamental solutions with the support in the forward or backward light cones. First we consider the string equation. In order to write the fundamental solution with the support in the forward light cone we look for $\mathcal{E} \in \mathcal{D}'(\mathbf{R}^2)$ such that

$$\mathcal{E}_{tt} - \mathcal{E}_{xx} = \delta(x - x_0)\delta(t - t_0) \quad \text{in } \mathbf{R}^2, \quad \text{supp}\,\mathcal{E} \subseteq \{(x,t)\,|\,t \ge t_0, \ x \in \mathbf{R}\}.$$

Then $\mathcal{E}(x,t;x_0,t_0) = \mathcal{E}(x-x_0,t-t_0;0,0)$. For all $n \in \mathbf{N}$ we denote by $D(x_0,t_0)$ the forward light cone $D(x_0,t_0) := \{(x,t) \in \mathbf{R}^{n+1}\,|\,|x - x_0| \le (t - t_0)\}$. It is well known that $\mathcal{E}(x,t;x_0,t_0) = 1/2$ if $(x,t) \in D(x_0;t_0)$ and $\mathcal{E}(x,t;x_0,t_0) = 0$ otherwise. We follow the approach of the previous section and rewrite this fundamental solution in the following way:

$$\mathcal{E}(x,t;x_0,t_0) \quad = \quad H(t - t_0)\int_0^{t-t_0} \mathcal{E}^{string}(x - x_0, z)\,dz,$$

where $H(t - t_0)$ is the Heaviside step function. The distribution

$$\mathcal{E}^{string}(x,t) = \frac{1}{2} \left\{ \delta(x+t) + \delta(x-t) \right\}$$

is the fundamental solution of the Cauchy problem for the string equation:

$$\mathcal{E}^{string}_{tt} - \mathcal{E}^{string}_{xx} = 0, \qquad \mathcal{E}^{string}(x,0) = \delta(x), \ \mathcal{E}^{string}_t(x,0) = 0.$$

The string equation is partially, in direction of time, hypoelliptic, that implies $\mathcal{E}^{string} \in C^\infty(\mathbf{R}_t; \mathcal{D}'(\mathbf{R}_x))$. Hence, for every test function $\varphi \in C_0^\infty(\mathbf{R})$, we have

$$< \mathcal{E}(x,t;\cdot,t_0), \varphi(\cdot) > \quad = \quad H(t-t_0) \int_0^{t-t_0} < \mathcal{E}^{string}(x-\cdot,z), \varphi(\cdot) > \ dz.$$

Thus, the forward fundamental solution of the operator is given by the integral transform of the fundamental solution of the Cauchy problem for the wave equation corresponding to the first datum.

We can generate a class of equations which allows explicit representation formulas for the fundamental solutions. Indeed, if we provide the integral transform with a kernel as follows:

$$\mathcal{E}(x,t;x_0,t_0) = H(t-t_0) \int_0^{|\phi(t)-\phi(t_0)|} K(t;r,b)\mathcal{E}^{wave}(x-x_0,r)dr,$$

$x \in \mathbf{R}^n, \ t \in \mathbf{R}$, then we get the representations for the fundamental solutions of the wide class of partial differential equations equations. In particular, if we plug in the integral transform the kernels used in the previous examples, then we obtain the corresponding fundamental solutions with the support in the forward light cone.

2.1. Klein-Gordon equation in the Minkowski spacetime

If we choose the kernel $K(t;r,b)$ (1.6) and choose the distance function as $\phi(t) = t$, then it can be easily verified (see Theorem 2.1 below) that the distribution

$$\mathcal{E}(x,t;x_0,t_0) = H(t-t_0) \int_0^{t-b} J_0\left(\sqrt{(t-b)^2 - r^2}\right) \mathcal{E}^{wave}(x-x_0,r)dr,$$

$x \in \mathbf{R}^n, \ t \in \mathbf{R}$, is the forward fundamental solutions for the Klein-Gordon operator with a positive mass equals to 1 in the Minkowski spacetime,

$$\left(\partial_t^2 - \Delta + 1\right)\mathcal{E}(x,t;x_0,t_0) = \delta(x-x_0)\delta(t-t_0) \text{ in } \mathbf{R}^{n+1}, \quad \operatorname{supp}\mathcal{E} \subseteq D(x_0,t_0),$$

provided that $\mathcal{E}^{wave}(x,t)$ is the fundamental solution of the Cauchy problem corresponding to the first datum with the support at the origin, for the wave

equation in the Minkowski spacetime. We emphasis that the distributions $\mathcal{E}(x,t;x_0,t_0)$ and $\mathcal{E}^{wave}(x,t)$ solve *different* equations.

If we now choose the kernel $K(t;r,b)$ (1.7) and the distance function as $\phi(t) = t$, then the distribution

$$\mathcal{E}(x,t;x_0,t_0) = H(t-t_0) \int_0^{t-t_0} I_0\left(\sqrt{(t-b)^2 - r^2}\right) \mathcal{E}^{wave}(x-x_0,r)dr,$$

$x \in \mathbf{R}^n$, $t \in \mathbf{R}$, is the forward fundamental solutions for the Klein-Gordon operator with an imaginary mass in the Minkowski spacetime,

$$\left(\partial_t^2 - \Delta - 1\right)\mathcal{E}(x,t;x_0,t_0) = \delta(x-x_0)\delta(t-t_0) \text{ in } \mathbf{R}^{n+1}, \operatorname{supp}\mathcal{E} \subseteq D(x_0,t_0).$$

The following theorem can be easily proved by direct substitution.

THEOREM 2.1. *The distributions $\mathcal{E}_{\mathcal{R}e}(x,t;x_0,t_0)$ and $\mathcal{E}_{\mathcal{I}m}(x,t;x_0,t_0)$ defined by*

$$\mathcal{E}_{\mathcal{R}e}(x,t;x_0,t_0) = H(t-t_0) \int_0^{t-t_0} J_0\left(m\sqrt{(t-b)^2 - r^2}\right) \mathcal{E}^{wave}(x-x_0,r)dr,$$

$$\mathcal{E}_{\mathcal{I}m}(x,t;x_0,t_0) = H(t-t_0) \int_0^{t-t_0} I_0\left(m\sqrt{(t-b)^2 - r^2}\right) \mathcal{E}^{wave}(x-x_0,r)dr,$$

$x \in \mathbf{R}^n$, $t \in \mathbf{R}$, *are forward fundamental solutions for the Klein-Gordon operators with a real and an imaginary mass*

$$\partial_t^2 - \Delta + M^2 \quad \text{in } \mathbf{R}^{n+1},$$

with $M^2 > 0$ and $M^2 < 0$, respectively. Here $m = |M| \geq 0$ and $\mathcal{E}^{wave}(x,t)$ is the fundamental solution of the Cauchy problem corresponding to the first datum with the support at the origin, for the wave equation in the Minkowski spacetime.

2.2. Tricomi-type equations

If we now choose the kernel $K(t;r,b)$ (1.10) and the distance function as (1.11), then it is proved in [11] that $\mathcal{E}(x,t;x_0,t_0)$ is the forward fundamental solution for the Tricomi-type equation (1.12):

$$\begin{aligned}
\mathcal{E}(x,t;x_0,t_0) &= 2c_l H(t-t_0) \int_0^{\phi(t)-\phi(t_0)} \left((\phi(t) + \phi(t_0))^2 - r^2\right)^{-\gamma} \\
&\quad \times F\left(\gamma,\gamma;1; \frac{(\phi(t)-\phi(t_0))^2 - r^2}{(\phi(t)+\phi(t_0))^2 - r^2}\right) \mathcal{E}^{wave}(x-x_0,r)dr,
\end{aligned}$$

$x \in \mathbf{R}^n$, $t_0 \geq 0$, with the support in the forward light cone

$$D(x_0,t_0) := \{(x,t) \in \mathbf{R}^{n+1} \,|\, |x-x_0| \leq \phi(t) - \phi(t_0)\}.$$

2.3. Klein-Gordon equations in the Robertson-Walker spacetime

The integral transform and, in particular, its kernel and the Gauss's hypergeometric function, open a way to establish a bridge between the wave equation (massless equation) and the Klein-Gordon equation (massive equation) in the curved spacetime. Indeed, if we allow the parameter γ of the function $F(\gamma, \gamma; 1; z)$ to be a complex number, $\gamma \in \mathbf{C}$, then this continuation into the complex plane produces the fundamental solutions $\mathcal{E}_+(x, t; x_0, t_0)$ for the Klein-Gordon operator in the de Sitter spacetime as follows

$$\mathcal{E}_+(x, t; x_0, t_0) = 2H(t - t_0) \int_0^{e^{-t_0} - e^{-t}} (4e^{-t_0 - t})^{iM} \left((e^{-t_0} + e^{-t})^2 - r^2 \right)^{-\frac{1}{2} - iM}$$

$$\times F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^{-t_0} - e^{-t})^2 - r^2}{(e^{-t_0} + e^{-t})^2 - r^2} \right) \mathcal{E}^{wave}(x - x_0, r)\, dr,$$

where the distribution $\mathcal{E}^{wave}(x, t)$ is the fundamental solution of the Cauchy problem for the wave equation, while the non-negative *curved mass* $M \geq 0$ is defined as follows: $M^2 := \frac{n^2}{4} - m^2 \geq 0$. The parameter m is mass of particle. The fundamental solution $\mathcal{E}_-(x, t; x_0, t_0)$ with the support in the backward light cone admits a similar representation. The fundamental solutions $\mathcal{E}_+(x, t; x_0, t_0)$ and $\mathcal{E}_-(x, t; x_0, t_0)$ are constructed in [17] for the case of the large masses $m \geq n/2$. The integral makes sense in the topology of the space of distributions. The fundamental solutions for the Klein-Gordon operator in the anti-de Sitter spacetime can be obtained by time inversion, $t \to -t$, from the fundamental solutions for the Klein-Gordon operator in the de Sitter spacetime.

Moreover, the analytic continuation of this distribution in parameter M into $\mathbf{C}$ allows us to use it also in the case of small mass $0 \leq m \leq n/2$. The corresponding equation

$$u_{tt} - e^{-2t} \triangle u - M^2 u = 0,$$

can be regarded as a Klein-Gordon equation with an imaginary mass. Equations with imaginary mass appear in several physical models such as ϕ^4 field model, tachion (super-light) fields, Landau-Ginzburg-Higgs equation and others.

More precisely, for small mass $0 \leq m \leq n/2$ we define the distribution $\mathcal{E}_+(x, t; x_0, t_0)$ by

$$\mathcal{E}_+(x, t; x_0, t_0)$$

$$= 2H(t - t_0) \int_0^{e^{-t_0} - e^{-t}} (4e^{-t_0 - t})^{-M} \left((e^{-t_0} + e^{-t})^2 - r^2 \right)^{-\frac{1}{2} + M}$$

$$\times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-t_0} - e^{-t})^2 - r^2}{(e^{-t_0} + e^{-t})^2 - r^2} \right) \mathcal{E}^{wave}(x - x_0, r)\, dr.$$

2.4.

The above listed examples hint at some necessary condition on the pair ϕ, $K(t; r, b)$ in order for that pair to produce a generator of the solution operator for some partial differential equation.

THEOREM 2.2. *Assume that the integral transform (1.5) with the distance function ϕ and the kernel $K(t; r, b)$ generate the fundamental solution*

$$\mathcal{E}(x, t; x_0, t_0) = H(t - t_0) \int_0^{|\phi(t) - \phi(t_0)|} K(t; r, b) \mathcal{E}^{wave}(x - x_0, r) dr,$$

$x \in \mathbf{R}^n$, $t \in \mathbf{R}$, *of the partial differential equation*

$$u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(x, t) u_{x_i})_{x_j} + \sum_{i=1}^{n} (b_i(x, t) u)_{x_i} + b(t) u_t + c(t) u = f \qquad (2.1)$$

with the real-analytic coefficients. Denote by $V_1 = V_1(t)$ and $V_2 = V_2(t)$ two linearly independent solutions of the ordinary differential equation

$$V'' + b(t)V' + c(t)V = 0, \qquad V_1(0) = 1 = V_2'(0), \quad V_1'(0) = 0 = V_2(0).$$

Then the function $K(t; r, b)$ satisfies the identity

$$2 \int_0^{|\phi(t) - \phi(b)|} K(t; r, b) dr = \frac{V_1(b) V_2(t) - V_1(t) V_2(b)}{V_1(b) V_2'(b) - V_1'(b) V_2(b)} \qquad (2.2)$$

for all $t > b > 0$.

Proof. For every function $f \in C^\infty(\mathbf{R} \times [0, \infty))$, which for any given instant $t \geq 0$ has a compact support in x, the function

$$v(x, t) \quad = \quad \int_0^t db \int_0^{|\phi(t) - \phi(b)|} K(t; r, b) w(x, r; b) dr,$$

where $w(x, r; b) = < \mathcal{E}^{wave}(x - \cdot, r), f(\cdot, b) >$, solves the equation (2.1) and takes vanishing initial data. It follows

$$V(t) \quad := \quad \int_{\mathbf{R}^n} v(x, t) dx = \int_0^t db \int_0^{|\phi(t) - \phi(b)|} K(t; r, b) \left(\int_{\mathbf{R}^n} w(x, r; b) dx \right) dr.$$

On the other hand,

$$\int_{\mathbf{R}^n} w(x, r; b) \, dx = F(b), \quad F(b) := \int_{\mathbf{R}^n} f(x, b) \, dx \,.$$

Hence,

$$V(t) \quad = \quad \int_0^t F(b) \left(\int_0^{|\phi(t)-\phi(b)|} K(t;r,b)\, dr \right) db.$$

At the same time from the equation (2.1) we obtain

$$\frac{d^2}{dt^2} V(t) + b(t) \frac{d}{dt} V(t) + c(t) V(t) = F(t).$$

Hence,

$$V(t) = \int_0^t F(b) \frac{V_1(b) V_2(t) - V_1(t) V_2(b)}{V_1(b) V_2'(b) - V_1'(b) V_2(b)}\, db.$$

Thus, for the arbitrary function $f \in C^\infty(\mathbf{R} \times [0, \infty))$ for all t one has

$$\int_0^t \left(\int_{\mathbf{R}^n} f(x,b)dx \right) \left(\int_0^{|\phi(t)-\phi(b)|} K(t;r,b)dr - \frac{V_1(b)V_2(t) - V_1(t)V_2(b)}{V_1(b)V_2'(b) - V_1'(b)V_2(b)} \right) db = 0.$$

The theorem is proven. □

COROLLARY 2.3. *1) For the Tricomi-type equation and for the wave equations in the de Sitter and anti-de Sitter spacetime the following identities hold:*

$$t - b = 2c_k \int_0^{\frac{t^{k+1}}{k+1} - \frac{b^{k+1}}{k+1}} \left(\left(\frac{t^{k+1}}{k+1} + \frac{b^{k+1}}{k+1} \right)^2 - r^2 \right)^{-\frac{k}{2k+2}}$$

$$\times F\left(\frac{k}{2k+2}, \frac{k}{2k+2}; 1; \frac{\left(\frac{t^{k+1}}{k+1} - \frac{b^{k+1}}{k+1} \right)^2 - r^2}{\left(\frac{t^{k+1}}{k+1} + \frac{b^{k+1}}{k+1} \right)^2 - r^2} \right) dr, \quad t > b \geq 0,$$

$$t - b = 2 \int_0^{e^{-b} - e^{-t}} \left((e^{-b} + e^{-t})^2 - r^2 \right)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{-t} - e^{-b})^2 - r^2}{(e^{-t} + e^{-b})^2 - r^2} \right) dr,$$

$$t - b = 2 \int_0^{e^t - e^b} \left((e^b + e^t)^2 - r^2 \right)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - r^2}{(e^t + e^b)^2 - r^2} \right) dr,$$

$t > b$, *respectively.*
2) For the Klein-Gordon equation in the de Sitter spacetime we have $b(t) = 0$, $c(t) = M^2$ *and the following identities hold. If the mass term* M^2 *is positive, then*

$$\frac{1}{M} \sin M(t - b) \quad = \quad 2 \int_0^{e^{-b} - e^{-t}} (4e^{-b-t})^{iM} \left((e^{-b} + e^{-t})^2 - r^2 \right)^{-\frac{1}{2} - iM}$$

$$\times F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2} \right) dr.$$

If the mass term M^2 is negative, then

$$\frac{1}{|M|}\sinh|M|(t-b) \;=\; 2\int_0^{e^{-b}-e^{-t}} (4e^{-b-t})^{-|M|}\Big((e^{-b}+e^{-t})^2 - r^2\Big)^{-\frac{1}{2}+|M|}$$

$$\times F\Big(\frac{1}{2}-|M|, \frac{1}{2}-|M|; 1; \frac{(e^{-b}-e^{-t})^2 - r^2}{(e^{-b}+e^{-t})^2 - r^2}\Big)dr.$$

These identities were used in [12] and [15] to prove a blow up phenomenon for the semilinear Tricomi-type equation and the Klein-Gordon equation in the de Sitter spacetime.

3. Estimate of the tail inside of the light cone. De Sitter spacetime

In this section we consider the equation

$$u_{tt} - e^{-2t}\Delta u = 0. \tag{3.1}$$

The forward and backward fundamental solutions for the operator of this equation are constructed in [17]. By means of those fundamental solutions, the fundamental solutions of the Cauchy problem are given as the Fourier integral operators in the domain of hyperbolicity, $t > 0$, with the data prescribed at $t = 0$,

$$u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x), \quad x \in \mathbf{R}^n. \tag{3.2}$$

The formula for the solution of this problem is given by Theorem 0.6 [17] with $M = 0$. More precisely, the solution of equation (3.1) with the initial data φ_0, $\varphi_1 \in C_0^\infty(\mathbf{R}^n)$, prescribed at $t = 0$ is:

$$u(x,t) \;=\; e^{\frac{t}{2}} v_{\varphi_0}(x, \phi(t)) + 2\int_0^1 v_{\varphi_0}(x, \phi(t)s) K_0(\phi(t)s, t)\phi(t)\, ds$$

$$+ 2\int_0^1 v_{\varphi_1}(x, \phi(t)s) K_1(\phi(t)s, t)\phi(t)\, ds, \quad x \in \mathbf{R}^n,\ t > 0, \tag{3.3}$$

where $\phi(t) := 1 - e^{-t}$. Here the function $v_\varphi(x,t)$ is a solution of the Cauchy problem $v_{tt} - \triangle v = 0$, $v(x,0) = \varphi(x)$, $v_t(x,0) = 0$. The kernels $K_0(z,t)$ and $K_1(z,t)$ are defined by $K_0(z,t) := -\big[\frac{\partial}{\partial b}E(z,t;0,b)\big]_{b=0}$ and $K_1(z,t) := E(z,t;0,0)$, respectively, where $E(x,t;r,b)$ is given by (1.14). Thus,

$$K_0(z,t)$$

$$= \frac{1}{[(1-e^{-t})^2 - z^2]\sqrt{(1+e^{-t})^2 - z^2}}\Big[(e^{-t}-1)F\Big(\frac{1}{2},\frac{1}{2};1;\frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\Big)$$

$$+ \big(1 - e^{-2t} + z^2\big)\frac{1}{2}F\Big(-\frac{1}{2},\frac{1}{2};1;\frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\Big)\Big], \quad 0 \le z < 1 - e^{-t},$$

and

$$K_1(z,t) = \left((1+e^{-t})^2 - z^2\right)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\right), \; 0 \le z \le 1 - e^{-t}.$$

It is important that the formula (3.3) regarded in the topology of the continuous functions of variable t with the values in the distributions space $\mathcal{D}'(\mathbf{R}^n)$, is applicable to the distributions $\varphi_0, \varphi_1 \in \mathcal{D}'(\mathbf{R}^n)$ as well.

Recall that a wave equation is said to satisfy Huygens' principle if the solution vanishes at all points which cannot be reached from the initial data by a null geodesic, that is there is no tail. An exemplar equation satisfying Huygens' principle is the wave equation in $n+1$ dimensional Minkowski spacetime for odd $n \ge 3$. According to Hadamard's conjecture this is the only (modulo transformations of coordinates and unknown function) huygensian linear second-order hyperbolic equation. Counterexamples to Hadamard's conjecture, which have been found, do not change the fact that Huygens' property is a very rare and unstable, with respect to the perturbations, phenomenon. It is natural, therefore, to ask if there are other hyperbolic second-order equations which preserve Huygens' property approximately, in the sense that the tail which is left behind the wave front is comparatively small. For the equation in the de Sitter spacetime Huygens' principle is not valid [17]. In this section we show the way how our approach can be applied to derive the pointwise estimates of the tail.

We call the "tail" the part containing the integrals of (3.3), that is

$$\begin{aligned} T(x,t) \;\; := \;\; & 2\int_0^1 v_{\varphi_0}(x, \phi(t)s) K_0(\phi(t)s, t)\phi(t)\, ds \\ & + 2\int_0^1 v_{\varphi_1}(x, \phi(t)s) K_1(\phi(t)s, t)\phi(t)\, ds, \quad x \in \mathbf{R}^n, \; t > 0. \end{aligned}$$

Hence, $T(x,t) = u(x,t) - e^{\frac{t}{2}} v_{\varphi_0}(x, \phi(t))$. The tail is of considerable interest in many aspects in the physics, and in particular, in the General Relativity [4], [9].

In this section we restrict ourselves to the case of one-dimensional x. For the one-dimensional wave equation in the Minkowski spacetime Huygens' principle is not valid, and, consequently, one can not anticipate it for one-dimensional equation in the de Sitter spacetime. But the last one reveals all difficulties and technical details allowing to overcome those difficulties in the case of $x \in \mathbf{R}^n$ with $n \ge 2$. The results for the case of $n \ge 2$ will be published in a forthcoming paper. We start with a simple example.

EXAMPLE 3.1. *Let $\varphi_0(x) = H(x)$, $\varphi_1(x) = 0$, where H is the Heaviside step function. Then $u(x,t) = \frac{1}{2}e^{\frac{t}{2}}\left[H(x+1-e^{-t}) + H(x-1+e^{-t})\right] + \int_0^{1-e^{-t}}\left[H(x-z) + H(x+z)\right] K_0(z,t)\, dz$ and, consequently, $T(x,t) = \int_0^{1-e^{-t}} H(x-z)K_0(z,t)\, dz+$*

$\int_0^{1-e^{-t}} H(x+z)K_0(z,t)\,dz$. *Consider the point x_0 such that $0 \le x_0 < 1 - e^{-t}$. Then we have*

$$u(x_0,t) = \frac{1}{2}e^{\frac{t}{2}} + \int_0^{x_0} K_0(z,t)\,dz + \int_0^{1-e^{-t}} K_0(z,t)\,dz = \frac{1}{2} + \int_0^{x_0} K_0(z,t)\,dz$$

while $T(x_0,t) = \frac{1}{2} - \frac{1}{2}e^{\frac{t}{2}} + \int_0^{x_0} K_0(z,t)\,dz = u(x_0,t) - \frac{1}{2}e^{\frac{t}{2}}$. In particular,

$$\frac{|T(x_0,t)|}{|u(x_0,t) - T(x_0,t)|} = 1 - e^{-t/2} - 2e^{-t/2}\int_0^{x_0} K_0(z,t)\,dz \le 2(1 - e^{-t/2}),$$

$$\lim_{t\to\infty}\lim_{x\to(1-e^{-t})^-}\frac{|T(x,t)|}{|u(x,t) - T(x,t)|} = 2.$$

Thus, the tail dominates the huygensian part of the solution. If $x_0 > 1$, then $u(x_0,t) = 1$, while $T(x_0,t) = 1 - e^{t/2}$.

Suppose now that the initial data φ_0 and φ_1 are the homogeneous functions

$$\varphi_0(x) = C_0|x|^{-a}, \quad \varphi_1(x) = C_1|x|^{-b}. \tag{3.4}$$

The next theorem gives a pointwise estimate for the tail.

THEOREM 3.2. *Consider the Cauchy problem for the equation (3.1), (3.2) with $n = 1$ and the data (3.4), where $a, b \in (1/2, 1)$. Then in the light cone emitted from the origin, that is on the set $\{(x,t)\,|\,|x| < 1 - e^{-t},\ t \ge 0\}$, the solution, and, consequently, the tail, satisfy*

$$
\begin{aligned}
|T(x,t)| &= \left|u(x,t) - \frac{1}{2}C_0 e^{\frac{t}{2}}\Big[|x + 1 - e^{-t}|^{-a} + |x - 1 + e^{-t}|^{-a}\Big]\right| \\
&\le |C_0|C\,(1+t)\,e^{\frac{1}{2}t}e^{at}(1 + e^t(1 - |x|))^{1/2-a} \\
&\quad + |C_1|C\,(1+t)\,e^{-\frac{1}{2}t}e^{bt}(1 + e^t(1 - |x|))^{1/2-b}.
\end{aligned}
$$

Proof. The representation of the solution of the Cauchy problem for the one-dimensional case ($n = 1$) of equation (3.1) is given by Theorem 0.4 from [17] with $M = 0$. More precisely, the solution $u = u(x,t)$ of the Cauchy problem

$$u_{tt} - e^{-2t}u_{xx} = 0, \qquad u(x,0) = \varphi_0(x), \qquad u_t(x,0) = \varphi_1(x),$$

with φ_0, φ_1 of (3.4), can be represented as follows:

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}e^{\frac{t}{2}}\Big[\varphi_0(x + 1 - e^{-t}) + \varphi_0(x - 1 + e^{-t})\Big] \\
&\quad + \int_0^{1-e^{-t}}\Big[\varphi_0(x - z) + \varphi_0(x + z)\Big]K_0(z,t)\,dz \\
&\quad + \int_0^{1-e^{-t}}\Big[\varphi_1(x - z) + \varphi_1(x + z)\Big]K_1(z,t)\,dz.
\end{aligned}
$$

Then, it is evident that the solution of the Cauchy problem with $C_1 = 0$ is

$$u(x,t) = C_0 e^{\frac{t}{2}}\left[|x+1-e^{-t}|^{-a} + |x-1+e^{-t}|^{-a}\right]$$

$$+C_0 \int_0^{1-e^{-t}} \left[|x-z|^{-a} + |x+z|^{-a}\right] K_0(z,t)\, dz.$$

Consider for $x \geq 0$ the first term in the last integral. It can be estimated as follows:

$$\left|\int_0^{1-e^{-t}} |x-y|^{-a} K_0(y,t)\, dy\right| \leq \int_0^{1-e^{-t}} |x-y|^{-a}|K_0(y,t)|\, dy$$

$$= \int_0^{1-e^{-t}} |x-y|^{-a} \frac{1}{[(1-e^{-t})^2 - y^2]\sqrt{(1+e^{-t})^2 - y^2}}$$

$$\times \left|\left[(e^{-t}-1)F\left(\frac{1}{2},\frac{1}{2};1;\frac{(1-e^{-t})^2 - y^2}{(1+e^{-t})^2 - y^2}\right)\right.\right.$$

$$\left.\left.+(1-e^{-2t}+y^2)\frac{1}{2}F\left(-\frac{1}{2},\frac{1}{2};1;\frac{(1-e^{-t})^2 - y^2}{(1+e^{-t})^2 - y^2}\right)\right]\right|\, dy.$$

If we denote $z = e^t$ and make a change $y = e^{-t}r$ in the last integral, then

$$\int_0^{1-e^{-t}} |x-y|^{-a}|K_0(y,t)|\, dy$$

$$= z^a \int_0^{z-1} |zx-r|^{-a} \frac{1}{[(z-1)^2 - r^2]\sqrt{(z+1)^2 - r^2}}$$

$$\left|\left[(z-z^2)F\left(\frac{1}{2},\frac{1}{2};1;\frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right)\right.\right.$$

$$\left.\left.+(z^2-1+r^2)\frac{1}{2}F\left(-\frac{1}{2},\frac{1}{2};1;\frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right)\right]\right|\, dr. \qquad (3.5)$$

Define two zones $Z_1(\varepsilon, z)$ and $Z_2(\varepsilon, z)$ as follows:

$$Z_1(\varepsilon, z) := \left\{(z,r)\,\Big|\, \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \leq \varepsilon,\ 0 \leq r \leq z-1\right\}, \qquad (3.6)$$

$$Z_2(\varepsilon, z) := \left\{(z,r)\,\Big|\, \varepsilon \leq \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2},\ 0 \leq r \leq z-1\right\}, \qquad (3.7)$$

respectively. Then we split the integral of (3.5) into the sum $I_1 + I_2$, where

$$
\begin{aligned}
I_k \quad := \quad z^a \int_{Z_k(\varepsilon, z)} |zx - r|^{-a} \frac{1}{[(z-1)^2 - r^2]\sqrt{(z+1)^2 - r^2}} \\
\times \left| \left[(z - z^2) F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right. \right. \\
\left. \left. + (z^2 - 1 + r^2) \frac{1}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right] \right| dr, \ k = 1, 2.
\end{aligned}
$$

First, we restrict ourselves to the first zone $Z_1(\varepsilon, z)$. We follow the arguments have been used in the proofs of Lemma 7.4 [17] and Proposition 10.2 [17]. In the first zone we have

$$
F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) = 1 + \frac{1}{4} \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} + O\left(\left(\frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right)^2\right),
$$

$$
F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) = 1 - \frac{1}{4} \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} + O\left(\left(\frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right)^2\right).
$$

We use the last formulas to estimate the term containing the hypergeometric functions

$$
\left| (z - z^2) F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right.
$$

$$
\left. + (z^2 - 1 + r^2) \frac{1}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right|
$$

$$
\leq \quad \frac{1}{2}\left[(z-1)^2 - r^2\right] + \frac{1}{8} \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \left|-(r^2 + z^2 - 1) + 2(-z^2 + z)\right|
$$

$$
+ \left(\left|z - z^2\right| + \left|z^2 - 1 + r^2\right|\right) O\left(\left(\frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right)^2\right).
$$

In that zone the integral can be estimated by

$$
I_1 \quad \leq \quad C \int_0^{z-1} |zx - y|^{-a} \left[\frac{1}{\sqrt{(z+1)^2 - y^2}} \left\{1 + z^2 \frac{1}{(z+1)^2 - y^2}\right\}\right] dy.
$$

The last integral can be written as follows:

$$
\int_0^{z-1} |zx - y|^{-a} \left[\frac{1}{\sqrt{z}\sqrt{z+1-y}} \left\{1 + z\frac{1}{z+1-y}\right\}\right] dy
$$

$$
= \int_1^z |z(1-x) - s|^{-a} \left\{\frac{1}{\sqrt{z}}(1+s)^{-1/2} + \sqrt{z}(1+s)^{-3/2}\right\} ds. \quad (3.8)
$$

Consider the first term of the last sum. Inside of the light cone $|x| < 1 - e^{-t}$, that is, $|zx| < z - 1$, one has

$$\frac{1}{\sqrt{z}} \int_1^z |z(1-x) - s|^{-a} (1+s)^{-1/2} \, ds$$

$$= \frac{1}{\sqrt{z}} \int_1^{z(1-x)} (z(1-x) - s)^{-a} (1+s)^{-\frac{1}{2}} \, ds$$

$$+ \frac{1}{\sqrt{z}} \int_{z(1-x)}^z (s - z(1-x))^{-a} (1+s)^{-\frac{1}{2}} \, ds.$$

On the other hand,

$$\frac{1}{\sqrt{z}} \int_1^{z(1-x)} (z(1-x) - s)^{-a} (1+s)^{-\frac{1}{2}} \, ds = \frac{1}{\sqrt{z}} (1 + z(1-x))^{-a}$$

$$\times \left(\frac{\sqrt{\pi}\Gamma(1-a)}{\Gamma\left(\frac{3}{2} - a\right)} (1 + z(1-x))^{1/2} - 2\sqrt{2} F\left(\frac{1}{2}, a; \frac{3}{2}; \frac{2}{1 + z(1-x)} \right) \right),$$

while

$$\frac{1}{\sqrt{z}} \int_{z(1-x)}^z (s - z(1-x))^{-a} (1+s)^{-\frac{1}{2}} \, ds$$

$$= -\frac{1}{\sqrt{z}} \frac{e^{ia\pi} \sqrt{\pi}\Gamma(1-a)}{\Gamma\left(\frac{3}{2} - a\right)} (1 + z(1-x))^{\frac{1}{2} - a}$$

$$+ \frac{2e^{ia\pi}}{\sqrt{z}} \sqrt{1+z} \, (1 + z(1-x))^{-a} F\left(\frac{1}{2}, a, \frac{3}{2}, \frac{1+z}{1 + z(1-x)} \right).$$

Thus,

$$\frac{1}{\sqrt{z}} \int_1^z |z(1-x) - s|^{-a} \frac{1}{\sqrt{1+s}} \, ds$$

$$\leq z^{-\frac{1}{2}} (1 + z(1-x))^{-a} \left(-\sqrt{\pi} \left(e^{ia\pi} - 1 \right) (1 + z(1-x))^{\frac{1}{2}} \frac{\Gamma(1-a)}{\Gamma\left(\frac{3}{2} - a\right)} \right.$$

$$\left. -2\sqrt{2} F\left(\frac{1}{2}, a, \frac{3}{2}, \frac{2}{1 + z(1-x)} \right) \right)$$

$$+ 2z^{-\frac{1}{2}} (1+z)^{\frac{1}{2}} (1 + z(1-x))^{-a} e^{ia\pi} F\left(\frac{1}{2}, a, \frac{3}{2}, \frac{1+z}{1 + z(1-x)} \right). \quad (3.9)$$

Here the arguments of the hypergeometric functions satisfy the inequalities $0 < \frac{2}{1 + z(1-x)} < 1$ and $1 < \frac{1+z}{1 + z(1-x)}$ for all $z(1-x) > 1$, $x > 0$, $z > 1$. Since $\Re(\frac{3}{2} - \frac{1}{2} - a) > 0$ we have

$$\left| F\left(\frac{1}{2}, a, \frac{3}{2}, \frac{2}{1 + z(1-x)} \right) \right| \leq const \quad \text{for all} \quad z(1-x) > 1.$$

Then, to estimate the last term of (3.9) we use (4) of Sec.2.10 [2]: for all $z(1-x) \geq 1$, $z > 1$, $x > 0$,

$$F\left(\frac{1}{2}, a; \frac{3}{2}; \frac{1+z}{1+z(1-x)}\right) = A_1\left(\frac{1+z}{1+z(1-x)}\right)^{-\frac{1}{2}} + A_2\left(\frac{1+z}{1+z(1-x)}\right)^{-1}$$
$$\times \left(1 - \frac{1+z}{1+z(1-x)}\right)^{1-a} F\left(1, \frac{1}{2}; 2-a; 1 - \frac{1+z(1-x)}{1+z}\right),$$

where $A_1 = \frac{\sqrt{\pi}\Gamma(1-a)}{2\Gamma\left(\frac{3}{2}-a\right)}$, $A_2 = \frac{\sqrt{\pi}\Gamma(a-1)}{2\Gamma(a)}$. Hence,

$$z^{-\frac{1}{2}}(1+z)^{\frac{1}{2}}(1+z(1-x))^{-a}F\left(\frac{1}{2}, a; \frac{3}{2}; \frac{1+z}{1+z(1-x)}\right)$$
$$= A_1 z^{-\frac{1}{2}}(1+z)^{\frac{1}{2}}(1+z(1-x))^{-a}\left(\frac{1+z}{1+z(1-x)}\right)^{-\frac{1}{2}}$$
$$+ A_2 z^{-\frac{1}{2}}(1+z)^{\frac{1}{2}}(1+z(1-x))^{-a}\left(\frac{1+z}{1+z(1-x)}\right)^{-1}$$
$$\times \left(1 - \frac{1+z}{1+z(1-x)}\right)^{1-a} F\left(1, \frac{1}{2}; 2-a; 1 - \frac{1+z(1-x)}{1+z}\right).$$

Since $a > 1/2$ one has

$$\left|F\left(1, \frac{1}{2}; 2-a; 1 - \frac{1+z(1-x)}{1+z}\right)\right| \leq const\left(\frac{1+z}{1+z(1-x)}\right)^{a-1/2}$$

for all $z(1-x) \geq 1$. It follows

$$\left|z^{-\frac{1}{2}}(1+z)^{\frac{1}{2}}(1+z(1-x))^{-a}F\left(\frac{1}{2}, a; \frac{3}{2}; \frac{1+z}{1+z(1-x)}\right)\right|$$
$$\leq Cz^{-\frac{1}{2}}(1+z(1-x))^{1/2-a}.$$

Finally,

$$\frac{1}{\sqrt{z}}\int_1^z |z(1-x)-s|^{-a}\frac{1}{\sqrt{1+s}}\,ds \leq Cz^{-\frac{1}{2}}(1+z(1-x))^{1/2-a}.$$

From this inequality now we derive an estimate for the second integral of (3.8),

$$\sqrt{z}\int_1^z |z(1-x)-s|^{-a}\frac{1}{(1+s)^{3/2}}\,ds \leq Cz^{\frac{1}{2}}(1+z(1-x))^{1/2-a}.$$

In the second zone we have

$$\varepsilon \leq \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \leq 1 \quad \text{and} \quad \frac{1}{(z-1)^2 - r^2} \leq \frac{1}{\varepsilon[(z+1)^2 - r^2]},$$

and due to the formula 15.3.10 of Chapter 15 from [1] we obtain

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right| \leq C\left(1 + \ln z\right), \quad \text{for all} \quad (z, r) \in Z_2(\varepsilon, z).$$

This allows us to estimate the integral over the second zone:

$$
\begin{aligned}
I_2 \;&\leq\; \int_{(z,r)\in Z_2(\varepsilon,z)} |zx - r|^{-a} \frac{z^2}{[(z-1)^2 - r^2]\sqrt{(z+1)^2 - r^2}} \left(1 + \ln z\right) dr \\
&\leq\; (1 + \ln z) \int_{(z,r)\in Z_2(\varepsilon,z)} |zx - r|^{-a} \frac{z^2}{((z+1)^2 - r^2)^{3/2}} \, dr \\
&\leq\; (1 + \ln z) \int_0^{z-1} |zx - r|^{-a} \frac{z^2}{((z+1)^2 - r^2)^{3/2}} \, dr \\
&\leq\; (1 + \ln z) \sqrt{z} \int_1^z |z(1-x) - s|^{-a} (1+s)^{-3/2} \, ds.
\end{aligned}
$$

Then from the last inequality and (2.2) we obtain

$$I_2 \;\leq\; C\left(1 + \ln z\right) z^{\frac{1}{2}} (1 + z(1-x))^{1/2 - a}.$$

Now consider the case of $C_0 = 0$. Then

$$T(x,t) \;=\; C_1 \int_0^{1-e^{-t}} \left[|x-z|^{-b} + |x+z|^{-b}\right] K_1(z,t) \, dz,$$

and

$$
\begin{aligned}
|T(x,t)| \;\leq\; C_1 \int_0^{1-e^{-t}} &\left[|x-z|^{-b} + |x+z|^{-b}\right] \left((1+e^{-t})^2 - z^2\right)^{-\frac{1}{2}} \\
&\times \left| F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\right) \right| \, dz.
\end{aligned}
$$

Consider the case of $x \geq 0$. Then make the change of variable $z = re^{-t}$:

$$|T(x,t)| \leq C_1 e^{bt} \int_0^{e^t - 1} |e^t x - r|^{-b} \left((e^t + 1)^2 - r^2\right)^{-\frac{1}{2}} \left| F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - 1)^2 - r^2}{(e^t + 1)^2 - r^2}\right) \right| dr.$$

Denote $z = e^t$. Then

$$|T(x,t)| \leq C z^b \int_0^{z-1} |zx - r|^{-b} \left((z+1)^2 - r^2\right)^{-\frac{1}{2}} \left| F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right| dr.$$

Since $1/2 < b < 1$, for all $z > 1$ the following estimate

$$
\begin{aligned}
&\int_0^{z-1} |zx - r|^{-b} ((z+1)^2 - r^2)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) dr \\
&\leq\; C(1 + \ln z) z^{-1/2} (1 + z(1-x))^{1/2 - b}
\end{aligned}
$$

is fulfilled. To prove the last estimate we rewrite the argument of the hypergeometric function as follows $\frac{(z-1)^2-r^2}{(z+1)^2-r^2} = 1 - \frac{4z}{(z+1)^2-r^2}$. If $r \geq \sqrt{(z+1)^2 - 8z}$, then $\frac{4z}{(z+1)^2-r^2} \geq \frac{1}{2}$ and $0 < 1 - \frac{4z}{(z+1)^2-r^2} \leq \frac{1}{2}$ for such r and z imply

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) \right| \leq C.$$

Hence, we have

$$\int_{\sqrt{(z+1)^2-8z}}^{z-1} |zx - r|^{-b}((z+1)^2 - r^2)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) dr$$

$$\leq C \int_{\sqrt{(z+1)^2-8z}}^{z-1} |zx - r|^{-b}((z+1)^2 - r^2)^{-\frac{1}{2}} dr$$

$$\leq Cz^{-1/2} \int_0^{z-1} |zx - r|^{-b}(z+1-r)^{-\frac{1}{2}} dr$$

$$\leq Cz^{-1/2}(1 + z(1-x))^{1/2-b}.$$

If $r \leq \sqrt{(z+1)^2 - 8z}$ and $z \geq 6$, then $8 < 8z \leq (z+1)^2 - r^2 \leq (z+1)^2$ implies

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) \right| \leq C \left| \ln\left(\frac{4z}{(z+1)^2 - r^2}\right) \right| \leq C(1 + \ln z).$$

Hence,

$$\int_0^{\sqrt{(z+1)^2-8z}} |zx - r|^{-b}((z+1)^2 - r^2)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) dr$$

$$\leq \int_0^{\sqrt{(z+1)^2-8z}} |zx - r|^{-b}((z+1)^2 - r^2)^{-\frac{1}{2}} C(1 + \ln z) dr$$

$$\leq C(1 + \ln z)z^{-1/2} \int_0^{z-1} |zx - r|^{-b}(z+1-r)^{-\frac{1}{2}} dr$$

$$\leq C(1 + \ln z)z^{-1/2}(1 + z(1-x))^{1/2-b}.$$

The theorem is proven.   □

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