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Resolvent Estimates for Magnetic Schrödinger Operators and Their Applications to Related Evolution Equations

Kiyoshi Mochizuki

Abstract. *In this article we survey some basic results for the magnetic Schrödinger operator with external potential which has a strong singularity. The following topics are treated under suitable decay conditions on the magnetic field and external potential: Selfadjointness of the operator, Growth estimates of generalized eigenfunctions, Principle of limiting absorption, Uniform resolvent estimates, and Smoothing properties for corresponding evolution equations.*

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1. Introduction and results

We consider the magnetic Schrödinger equation

$$-\sum_{j=1}^{n}\{\partial_j + ib_j(x)\}^2 u + c(x)u - \kappa^2 u = f(x), \quad x \in \mathbf{R}^n, \tag{1}$$

where $\partial_j = \partial/\partial x_j$ $(j = 1, \cdots, n)$, $i = \sqrt{-1}$, $\kappa \in \mathbf{\Pi}_\pm = \{\kappa \in \mathbf{C}; \pm\mathrm{Re}\kappa > 0,\ \mathrm{Im}\kappa > 0\}$, $b_j(x)$ are real valued C^1-functions of $x \in \mathbf{R}^n$, $c(x)$ is a real valued continuous function of $x \in \mathbf{R}^n\backslash\{0\}$ and $f(x) \in L^2 = L^2(\mathbf{R}^n)$. $b(x) = (b_1(x), \cdots, b_n(x))$ represents a magnetic potential. Thus the magnetic field is defined by its rotation $\nabla \times b(x)$. The external potential $c(x)$ may have a singularity like $O(|x|^{-2})$ at $x = 0$.

Notation: Let $a \cdot b$ and $a \times b$ respectively denote the inner product and exterior product of a, $b \in \mathbf{R}^n$. More generally, we put

$$\nabla \cdot v(x) = \partial_1 v_1(x) + \cdots + \partial_n v_n(x), \quad \nabla \times v(x) = (\partial_j v_k(x) - \partial_k v_j(x))_{1\le j<k\le n}$$

for $\nabla = (\partial_1, \cdots, \partial_n)$ and $v(x) = (v_1(x), \cdots, v_n(x))$. We also put $\nabla_b = \nabla + ib(x)$, $\Delta_b = \nabla_b \cdot \nabla_b$, $r = |x|$, $\tilde{x} = x/r$ and $\partial_r = \tilde{x} \cdot \nabla$. The inner product and norm of L^2 are defined by

$$(f,g)=\int f(x)\overline{g(x)}dx \quad \text{and} \quad \|f\|=\sqrt{(f,f)}.$$

Here we specify by $\int dx$ the integration over $\mathbf{R}^n$. For function $\xi=\xi(r)>0$ let $L^2_\xi=L^2_\xi(\mathbf{R}^n)$ be the weighted L^2-space with norm

$$\|f\|_\xi=\left\{\int \xi(r)|f(x)|^2dx\right\}^{1/2}<\infty.$$

Moreover, for $0<s<t$ we put $B_{s,t}=\{x;s<|x|<t\}$, $B_t=\{x;|x|<t\}$, $B'_t=\mathbf{R}^n\backslash B_t$ and $S_t=\{x;|x|=t\}$.

Throughout this paper we assume the existence of $c_\infty(x)\in L^\infty$ such that

$$(A1) \qquad c(x)-c_\infty(x)\ge\frac{\beta}{r^2} \quad \text{with} \quad \beta>-\frac{(n-2)^2}{4}.$$

With this condition we define the operator L acting in L^2 as follows:

$$\begin{cases} Lu=-\Delta_b u+c(x)u \quad \text{for} \quad u\in\mathcal{D}(L), \\ \mathcal{D}(L)=\left\{u\in L^2\cap H^2_{\rm loc}(\mathbf{R}^n\backslash\{0\});(-\Delta_b+c)u,\ r^{-1}u\in L^2\right\}. \end{cases} \tag{2}$$

Here $H^j=H^j(\mathbf{R}^n)$ $(j=1.2,\cdots)$ is the usual Sobolev space on $\mathbf{R}^n$ and $H^2_{\rm loc}(\Omega)$ is the H^2-space on each compact set of the domain Ω.

THEOREM 1. (i) *If* $u\in\mathcal{D}(L)$, *then we have* $\nabla_b u\in[L^2]^n$.

(ii) *L gives a lower semibounded selfadjoint operator in* L^2.

(iii) *If* $c(x)\to 0$ *as* $|x|\to\infty$, *then the essential spectrum* $\sigma_e(L)$ *of* L *is included in the half line* $[0,\infty)$.

A proof is given in Mochizuki [12, Theorems 1.1 and 1.3] under a stronger restriction on the singularity of $c(x)$ (the so called Stummel conditions are required there). On the other hand, in Kalf-Schmincke-Walter-Wist [9, Theorem 3] is treated the case with $b(x)\equiv 0$ and $c(x)$ having a strong singularity like $O(r^{-2})$. In both works (ii) is obtained, based on (i), as a Friedrichs extension of lower semibouded symmetric operators ([5]).

Theorem 1 shows that κ^2 with $\kappa\in\Pi_\pm$ is in the resolvent set of L. Thus, equation (1) has a unique solution $u=R(\kappa^2)f\in L^2$, where $R(\kappa^2)=(L-\kappa^2)^{-1}$ is the resolvent of L.

In order to study the essential spectrum of L we add the following conditions.

$(A2)$ There exist constants $R_0 > 0$ and $C_0 > 0$ such that

$$\max\left\{|\nabla \times b(x)|, \left|c(x) + \frac{(n-1)(n-3)}{4r^2}\right|\right\} \leq C_0\mu(r), \qquad r = |x| > R_0,$$

where $\mu = \mu(r)$ is a smooth, positive L^1-function of $r \in \mathbf{R}_+ = (0, \infty)$.
$(A3)$ The unique continuation property holds for $-\Delta_b + c(x)$.

THEOREM 2. *Assume* $(A1)$ *and* $(A2)$ *with* $\mu(r)$ *also satisfying*

$$\mu(r) = o(r^{-1}) \quad as \quad r \to \infty. \tag{3}$$

Let $\lambda > 0$ *and let* $u \in H^2_{\mathrm{loc}}(\mathbf{R}^n \backslash \{0\})$ *solve the homogeneous equation*

$$-\Delta_b u + c(x)u - \lambda u = 0. \tag{4}$$

If the support of u *is not compact, then*

$$\liminf_{t\to\infty} \int_{S_t} \left|\tilde{x} \cdot \nabla_b u + \frac{n-1}{2r}u - i\kappa u\right|^2 dS \neq 0,$$

where $\kappa = \sqrt{\lambda}$ *or* $= -\sqrt{\lambda}$.

THEOREM 3. *Assume* $(A1)$, $(A3)$ *and* $(A2)$ *with* $\mu(r)$ *satisfying* (3) *and also*

$$\int_r^\infty \mu(s)ds \geq r\mu(r) \quad for \quad r \geq R_0. \tag{5}$$

Then the resolvent $R(\kappa^2)$ *is continuously extended to* $\Pi_\pm \cup (0, \infty)$ *as an operator from* $L^2_{\mu^{-1}}$ *to* L^2_μ. *Thus, the positive spectrum of* L *is absolutely continuous with respect to the Lebesgue measure.*

Theorem 2 gives a real generalization of the Rellich growth estimates for the Laplace operator in exterior domain ([15]). A similar result

$$\lim_{t\to\infty} t^\epsilon \int_{S_t} \left|\tilde{x} \cdot \nabla_b u + \frac{n-1}{2r}u - i\kappa u\right|^2 dS = \infty \quad (\forall \epsilon > 0)$$

has been obtained in Ikebe-Uchiyama [7]. In this case we only use condition (3) and it is not necessary to assume $\mu \in L^1(\mathbf{R}_+)$. To show Theorem 2 we employ the methods developed in Jäger-Rejto [8] and Mochizuki [13] for non-magnetic Schrödinger operators with oscillating long range potentials. Theorem 3 is then a direct result of Theorem 2 (see e.g., Eidus [3], Mochizuki [12]).

Next, we shall show uniform resolvent estimates for $\kappa \in \Pi_\pm$. To this aim we restrict ourselves to the case $n \geq 3$ and replace $(A2)$ by some smallness conditions.

Theorem 4. *(i) Let $n \geq 3$. Assume (A1) and*

$$\text{(A4)} \qquad \max\{|\nabla \times b(x)|, |c(x)|\} \leq \epsilon_0 r^{-2} \quad in \quad \mathbf{R}^n,$$

where $0 < \epsilon_0 < 1/4\sqrt{2}$ $(n = 3)$ or $< \sqrt{(n-1)(n-3)/8}$ $(n \geq 4)$. Then we have for each $\kappa \in \Pi_\pm$,

$$\int \frac{1}{r^2}|u|^2 dx \leq C_1 \int r^2 |f|^2 dx \quad with$$

$$C_1 = \frac{8}{1 - 32\epsilon_0^2} \ (n = 3) \quad or \quad = \frac{8}{(n-1)(n-3) - 8\epsilon_0^2} \ (n \geq 4).$$

(ii) Let $n \geq 3$. Assume (A1) and

$$\text{(A5)} \qquad \max\{|\nabla \times b(x)|, |c(x)|\} \leq \epsilon_0 \min\{\mu(r), r^{-2}\}, \quad in \quad \mathbf{R}^n,$$

where $\mu(r)$ is a smooth, positive L^1-function of $r \in \mathbf{R}_+$ satisfying also

$$\mu'(r) \leq 0 \quad in \quad \mathbf{R}_+. \tag{6}$$

Then we have for each $\kappa \in \Pi_\pm$,

$$\int \left\{ \mu(|\nabla_b u|^2 + |\kappa u|^2) - \mu' \frac{n-1}{2r}|u|^2 \right\} dx \leq C_2 \int \max\{\mu^{-1}, r^2\} |f(x)|^2 dx$$

$$with \quad C_2 = 4\|\mu\|_{L^1}\left(5 + 4\epsilon_0^2 C_1\right).$$

Remark 1 The functions $(1+r)^{-1-\delta}$ and $(1+r)^{-1}[\log(e+r)]^{-1-\delta}$ $(0 < \delta \leq 1)$ are typical examples of $\mu(r)$. As is easily seen, all the conditions (3), (5) and (6) are verified by these functions.

As a corollary of Theorem 4, we are able to obtain space-time weighted estimates (smoothing properties, cf., Kato [10]) for the Schrödinger evolution equation

$$i\frac{\partial u}{\partial t} - Lu = 0, \quad u(0) = f \in L^2, \tag{7}$$

and for the relativistic Schrödinger evolution equation

$$i\frac{\partial u}{\partial t} - \sqrt{L + m^2}u = 0, \quad u(0) = f \in L^2 \tag{8}$$

with $m \geq 0$. Note that the smoothing effects for (8) give those for the Klein-Gordon $(m > 0)$ or the wave equation $(m = 0)$ in the energy space.

THEOREM 5. (i) *Under the conditions of Theorem 4 (i), we have*

$$\left|\int_0^{\pm\infty}\left\|r^{-1}\int_0^t e^{-i(t-\tau)L}h(\tau)d\tau\right\|^2 dt\right| \le C_1\left|\int_0^{\pm\infty}\|rh(t)\|^2 dt\right|$$

for $h(t)$ satisfying $r^{-1}h(t) \in L^2(\mathbf{R}\times\mathbf{R}^n)$, and

$$\left|\int_0^{\pm\infty}\|r^{-1}e^{-itL}f\|^2 dt\right| \le 2\sqrt{C_1}\|f\|^2 \quad for \quad f \in L^2.$$

(ii) *Under the conditions of Theorem 4 (ii), we have*

$$\left|\int_0^{\pm\infty}\|\min\{\mu(r)^{1/2}, r^{-1}\}e^{-it\sqrt{L+m^2}}f\|^2 dt\right| \le 4\sqrt{\max\{C_1, C_2\}}\|f\|^2.$$

Theorems 4 and 5 summarize the main results of the recent work of Mochizuki [14]. Theorem 4 (i) generalizes the corresponding results of Kato-Yajima [11], where the operator in question is restricted to the Laplace operator in $\mathbf{R}^n$ ($n \ge 3$). The Fourier transformation method employed there is not applicable in our case. We are based on the partial integration method, and the proof of Theorems 2, 3 and 4 are all reduced to one quadratic identity given in Proposition 1 of §3. Another important tools are modifications of the Hardy inequality (Lemma 3 of §2 and Lemma 9 of §5).

Results similar to Theorem 5 have been studied by many authors in connection with local smoothing properties (see, e.g., Yajima [16], Cuccagna-Schirmer [1], D'Ancona-Fanelli [2], Erdogan-Goldberg-Schlag [4] and Georgiev-Stefanov-Tarulli [6]). Note that these works are restricted to the case where the vector potential $b(x)$ itself is required to be small and to decay sufficiently fast (the smallness of $b(x)$ is not required in [4]). On the other hand, no such a requirement is in our case, and the smallness is required on $\nabla \times b(x)$. To remove it seems difficult without any decay conditions on $b(x)$.

Theorems 1 to 5 will be proved in the following §2 to §6, respectively. Finally, we give an extension of Theorem 4 (i) and add some remarks in §7. Theorem 6 there asserts that the smallness of $c(x)$ is not essential for uniform resolvent estimates.

2. Proof of Theorem 1

In this section, we give a proof of Theorem 1 following Kalf et al [7, Theorem 3] (cf., also Mochizuki [12, Theorems 1.1 and 1.3]).

For $\alpha \in \mathbf{R}$ and $u \in H^1_{\mathrm{loc}}(\mathbf{R}^n\backslash\{0\})$ we have

$$\left|\tilde{x}\cdot\nabla_b u\right|^2 = \left|\tilde{x}\cdot\nabla_b u + \frac{\alpha}{r}u - \frac{\alpha}{r}u\right|^2$$

$$= \left|\tilde{x}\cdot\nabla_b u + \frac{\alpha}{r}u\right|^2 - \nabla\cdot\left(\tilde{x}\frac{\alpha}{r}|u|^2\right) + \frac{(n-2)\alpha-\alpha^2}{r^2}|u|^2. \tag{9}$$

Integration this over $B_{\epsilon,t} = \{x\in\mathbf{R}^n; \epsilon < |x| < t\}$ gives the following

LEMMA 1. *We have*

$$\int_{B_{\epsilon,t}} |\tilde{x}\cdot\nabla_b u|^2 dx = \int_{B_{\epsilon,t}} \left|\tilde{x}\cdot\nabla_b u + \frac{\alpha}{r}u\right|^2 dx$$
$$-\left(\int_{S_t} - \int_{S_\epsilon}\right)\frac{\alpha}{r}|u|^2 dS + \int_{B_{\epsilon,t}} \frac{(n-2)\alpha-\alpha^2}{r^2}|u|^2 dx.$$

LEMMA 2. (i) *Let $r^{-1}u\in L^2$. Then we have*

$$\liminf_{\epsilon\to0}\int_{S_\epsilon} r^{-1}|u|^2 dS = 0, \quad \liminf_{\rho\to\infty}\int_{S_\rho} r^{-1}|u|^2 dS = 0.$$

(ii) *Let $u\in L^2$. Then there exist sequences $\epsilon_k\to0$, $t_k\to\infty$ $(k\to\infty)$ such that*

$$\partial_r\int_{S_1} r^n|u(r\omega)|^2 dS\Big|_{r=\epsilon_k} \geq 0, \quad \partial_r\int_{S_1} r^n|u(r\omega)|^2 dS_\omega\Big|_{r=t_k} \leq 0.$$

Proof (i) is obvious since we have

$$\int r^{-2}|u|^2 dx = \int_0^\infty r^{-1}\int_{S_r} r^{-1}|u|^2 dS dr < \infty.$$

(ii) is also verified from the inequality

$$\int |u|^2 dx = \int_0^\infty r^{-1}\int_{S_1} r^n|u(r\omega)|^2 dS_\omega dr < \infty.$$

□

Proof of Theorem 1 (i) By means of the Gauss formula we have for $u\in\mathcal{D}(L)$,

$$\mathrm{Re}\int_{B_{\epsilon,t}}(-\Delta_b u + cu)\overline{u}dx = \int_{B_{\epsilon,t}}(|\nabla_b u|^2 + c|u|^2)dx - \mathrm{Re}\left(\int_{S_t} - \int_{S_\epsilon}\right)(\tilde{x}\cdot\nabla u)\overline{u}dS. \tag{10}$$

Combine this identity and Lemma 1. Then noting

$$-\mathrm{Re}\int_{S_r}(\tilde{x}\cdot\nabla u)\overline{u}dS = -\frac{1}{2r}\partial_r\int_{S_1} r^n|u(r\omega)|^2 dS_\omega + \frac{n}{2r}\int_{S_r}|u|^2 dS,$$

we obtain

$$\mathrm{Re}\int_{B_{\epsilon,t}}(-\Delta_b u + cu)\overline{u}dx = \int_{B_{\epsilon,t}}\left|\tilde{x}\cdot\nabla_b u + \frac{\alpha}{r}u\right|^2 dx$$

$$+\int_{B_{\epsilon,t}}\left(\frac{(n-2)\alpha-\alpha^2}{r^2}+c\right)|u|^2dx$$

$$+\left(\int_{S_t}-\int_{S_\epsilon}\right)\frac{n-2\alpha}{2r}|u|^2dS-\frac{1}{2r}\left[\partial_r\int_{S_1}r^n|u(r\omega)|^2dS_\omega\right]_\epsilon^t.$$

Put $\alpha = n/2$ in this equation. Then since $r^{-1}u$, $|c|^{1/2}u \in L^2$, the first inequality of Lemma 2 (ii) shows

$$\int_{B_t}\left|\nabla_b u+\frac{n}{2r}u\right|^2dx<\infty.$$

Going back to Lemma 1 with $\alpha = n/2$ and using the first inequality of Lemma 2 (i), we conclude

$$\int_{B_t}|\nabla_b u|^2dS<\infty. \tag{11}$$

On the other hand, since (10) reduces to

$$\mathrm{Re}\int_{B_{\epsilon,t}}(-\Delta_b u+cu)\overline{u}dx\geq\int_{B_{\epsilon,t}}(|\nabla_b u|^2+c|u|^2)dx$$

$$-\frac{1}{2}\left[r^{-1}\partial_r\int_{S_1}r^n|u(r\omega)|^2dS_\omega\right]_\epsilon^t-\frac{n}{2}\int_{S_\epsilon}r^{-1}|u|^2dS,$$

the second inequality of Lemma 2 (ii) implies

$$\int_{B'_\epsilon}|\nabla_b u|^2dS<\infty. \tag{12}$$

(11) and (12) prove the assertion (i). □

Let H_b^1 be the completion of $C_0^\infty = C_0^\infty(\mathbf{R}^n)$ with respect to the norm

$$\|u\|_{H_b^1}=\left[\int\left\{|\nabla_b u|^2+|u|^2\right\}dx\right]^{1/2}<\infty. \tag{13}$$

A modified Hardy inequality is given by

Lemma 3. *Let $u \in H_b^1$. Then we have*

$$\int\frac{(n-2)^2}{4r^2}|u|^2dx\leq\int|\tilde{x}\cdot\nabla_b u|^2dx.$$

Proof Choose $\alpha = \dfrac{n-2}{2}$ in Lemma 1. Then letting $t \to \infty$, we have

$$\int_{B'_\epsilon} |\tilde{x} \cdot \nabla_b u|^2 dx \geq \int_{S_\epsilon} \frac{n-2}{2r} |u|^2 dS + \int_{B'_\epsilon} \frac{(n-2)^2}{4r^2} |u|^2 dx.$$

By assumption, we can let $\epsilon \to 0$ to conclude the desired inequality. □

Proof of Theorem 1 (ii) Let $u,\ v \in \mathcal{D}(L)$. Then with the help of (i), especially noting

$$\liminf_{\epsilon \to 0} \int_{S_\epsilon} |(\tilde{x} \cdot \nabla_b u)\overline{v}| dS = \liminf_{t \to \infty} \int_{S_t} |(\tilde{x} \cdot \nabla_b u)\overline{v}| dS = 0,$$

we easily see that

$$(Lu, v) = \int \left\{ \nabla_b u \cdot \overline{\nabla_b v} + cu\overline{v} \right\} dx.$$

Since $\mathcal{D}(L)$ is dense in L^2, this shows that L is a symmetric operator. Moreover, by means of $(A1)$ and Lemma 3,

$$(Lu, u) \geq \left(\frac{(n-2)^2}{4} + \beta \right) \|r^{-1}u\|^2 - C_\infty \|u\|^2, \quad C_\infty = \max |c_\infty(x)|, \tag{14}$$

which proves the lower semi boundedness of L.

To show that L coincides with the Friedrichs extension of the differential operator $-\Delta_b + c(x)$ in $C_0^\infty(\mathbf{R}^n \backslash \{0\})$, let $\{u_k\} \subset C_0^\infty(\mathbf{R}^n \backslash \{0\})$ satisfying

$$s - \lim_{k \to \infty} u_k = u \quad \text{in} \quad L^2,$$

$$\lim_{j,k \to \infty} ([-\Delta_b + c](u_j - u_k), u_j - u_k) = 0.$$

It then follows from (14) that $\{r^{-1}u_k\}$ forms a Cauchy sequence. Thus, $r^{-1}u \in L^2$.

This implies that $\mathcal{D}(L)$ coincides with the domain of the Friedrichs extension. □

Proof of Theorem 1 (iii) Let $L_1 = L - c_\infty(x)$ with domain $\mathcal{D}(L_1) = \mathcal{D}(L)$. Without loss of generality we can assume $c_\infty(x) \to 0$ as $r \to \infty$. Then since

$$(L_1 u, u) \geq C(\beta) \|\nabla_b u\|^2, \quad C(\beta) = 1 \ (n = 2), \quad = 1 + \frac{4\beta}{(n-2)^2} \ (n \geq 3),$$

we easily see that the multiplication operator $c_\infty(x)$ is L_1-compact. This implies $\sigma_e(L) = \sigma_e(L_1)$. L_1 being positive, we conclude (iii). □

3. Proof of Theorem 2

First we prepare a quadratic identity for solutions u of equation (1). We put $v = e^{-i\kappa r} r^{(n-1)/2} e^{\sigma(r)} u$, $g = e^{-i\kappa r} r^{(n-1)/2} e^{\sigma(r)} f$ and rewrite (1) as follows:

$$-\nabla_b \cdot \nabla_b v + \left(-2i\kappa + \frac{n-1}{r} + 2\sigma' \right) \tilde{x} \cdot \nabla_b v$$
$$+ \left(c + \frac{(n-1)(n-3)}{4r^2} + \sigma'' - \sigma'^2 + 2i\kappa\sigma' \right) v = g. \tag{15}$$

Let $\phi = \phi(r) = e^{-2\mathrm{Im}\kappa r} r^{-n+1} \varphi(r)$, where $\varphi(r)$ is a smooth, nonnegative function of $r > 0$. We multiply by $\phi \overline{\tilde{x} \cdot \nabla_b v}$ on both sides of (15) to obtain

$$-\mathrm{Re}\nabla \cdot \{ (\phi \nabla_b v) \overline{\tilde{x} \cdot \nabla_b v} \} + \phi' |(\tilde{x} \cdot \nabla_b v|^2 + \frac{\phi}{r} (|\nabla_b v|^2 - |(\tilde{x} \cdot \nabla_b v|^2)$$
$$+ \frac{1}{2} \nabla \cdot (\phi \tilde{x} |\nabla_b v|^2) - \left(\frac{\phi'}{2} + \phi \frac{n-1}{2r} \right) |\nabla_b v|^2$$
$$-\mathrm{Re}\phi \{ (\tilde{x} \times \nabla_b v) \cdot \overline{(\nabla \times ib) v} \} + \phi \left(2\mathrm{Im}\kappa + \frac{n-1}{r} + 2\sigma' \right) |\tilde{x} \cdot \nabla_b v|^2$$
$$+\mathrm{Re}\phi \left(c + \frac{(n-1)(n-3)}{4r^2} + \sigma'' - \sigma'^2 + 2i\kappa\sigma' \right) v \overline{\tilde{x} \cdot \nabla_b v} = \mathrm{Re} \left\{ \phi g \overline{\tilde{x} \cdot \nabla_b v} \right\}.$$

Integrate this over $B_{R,t}$. Then noting

$$\nabla_b v = e^{-i\kappa r} r^{(n-1)/2} \left\{ \nabla_b (e^\sigma u) + \tilde{x} \left(\frac{n-1}{2r} - i\kappa \right) (e^\sigma u) \right\},$$

$$\phi'(r) = \phi(r) \left(-2\mathrm{Im}\kappa - \frac{n-1}{r} + \frac{\varphi'}{\varphi} \right),$$

we obtain

Proposition 1. *Let $u \in H^2_{\mathrm{loc}}(\mathbf{R}^n \backslash \{0\})$ solves (1). Put $u_\sigma = e^\sigma u$, $f_\sigma = e^\sigma f$ and*

$$\theta_\sigma = \theta_\sigma(x, \kappa) = \nabla_b u_\sigma + \tilde{x} \left(\frac{n-1}{2r} - i\kappa \right) u_\sigma.$$

Then

$$\left[\int_{S_t} - \int_{S_R} \right] \varphi \left\{ -|\tilde{x} \cdot \theta_\sigma|^2 + \frac{1}{2} |\theta_\sigma|^2 \right\} dS + \int_{B_{R,t}} \varphi \left\{ \left(\frac{\varphi'}{\varphi} - \frac{1}{r} \right) |\tilde{x} \cdot \theta_b|^2 \right.$$
$$\left. + \left(\mathrm{Im}\kappa - \frac{\varphi'}{2\varphi} + \frac{1}{r} \right) |\theta_\sigma|^2 + 2\sigma' |\tilde{x} \cdot \theta_\sigma|^2 + \mathrm{Re} J_\sigma(x, \kappa) \right.$$

$$+\mathrm{Re}\left[(\sigma'' - \sigma'^2 + 2i\kappa\sigma')u_\sigma\overline{\tilde{x}\cdot\theta_\sigma}\right]\Big\}dx = \mathrm{Re}\int_{B_{R,t}} \varphi f_\sigma \overline{\tilde{x}\cdot\theta_\sigma}dx,$$

where

$$J_\sigma(x,\kappa) = -(\tilde{x}\times\theta_\sigma)\cdot\overline{(\nabla\times ib)u_\sigma} + \left(c + \frac{(n-1)(n-3)}{4r^2}\right)u_\sigma\overline{\tilde{x}\cdot\theta_\sigma}.$$

In this section we prove Theorem 2 based on this identity.

LEMMA 4. *Let u be a solution of the homogeneous equation* (4). *Then for each $\lambda > 0$ and $r > 0$ we have*

$$\mathrm{Im}\left[\int_{S_r}(\tilde{x}\cdot\nabla_b u_\sigma)\overline{u_\sigma}dS\right] = 0, \tag{16}$$

$$\int_{S_r}\left\{\left|\tilde{x}\cdot\nabla_b u_\sigma + \frac{n-1}{2r}u_\sigma\right|^2 + \lambda|u_\sigma|^2\right\}dS = \int_{S_r}|\tilde{x}\cdot\theta_\sigma|^2 dS, \tag{17}$$

where $\theta_\sigma = \theta_\sigma(x, \pm\sqrt{\lambda})$.

Proof We multiply by $\overline{u}$ on both sides of (4) and integrate by parts over B_r. Then the imaginary part gives

$$-\mathrm{Im}\int_{S_r}(\tilde{x}\cdot\nabla_b u)\overline{u}dS = 0.$$

$\sigma(r)$ being real, this implies (16). (17) is obvious from (16). □

The following lemma is a direct consequence of $(A2)$ and (17).

LEMMA 5. *Let u solve* (4). *Then there exists $C_3 > 0$ independent of $\sigma(r)$ such that*

$$\int_{S_r}|J_\sigma(x,\kappa)|dS \le C_3\mu(r)\int_{S_r}|\theta_\sigma|^2 dS \quad for \quad r > R_0.$$

Proof of Theorem 2 We define $F(r)$, $F_{\sigma,\tau}(r)$ as follows:

$$F(r) = \frac{1}{2}\int_{S_r}\{2|\tilde{x}\cdot\theta|^2 - |\theta|^2\}dS,$$

$$F_{\sigma,\tau} = \frac{1}{2}\int_{S_r}\{2|\tilde{x}\cdot\theta_\sigma|^2 - |\theta_\sigma|^2 + (\sigma'^2 - \tau)|u_\sigma|^2\}dS,$$

where θ means θ_σ with $\sigma \equiv 0$ and $\tau = \tau(r) > 0$ is another weight function.

It follows from Proposition 1 with $\sigma \equiv 0$, $\varphi \equiv 0$, $\kappa^2 = \lambda > 0$ and $f \equiv 0$ that

$$F(t) - F(R) = \int_{B_{R,t}}\left\{\frac{1}{r}(|\theta|^2 - |\tilde{x}\cdot\theta|^2) + \mathrm{Re}J(x, \pm\sqrt{\lambda})\right\}dx,$$

where J means J_σ with $\sigma \equiv 0$. We choose $R_1 \geq R_0$ such that $\dfrac{1}{r} - C_3\mu(r) \geq 0$ for $r \geq R_1$. Then differentiating both sides and using Lemma 5, we obtain

$$\frac{d}{dt}F(t) \geq -2C_3\mu(t)F(t) \quad \text{for} \quad t \geq R_1.$$

Assume here that there exists a sequence $r_k \to \infty$ such that $F(r_k) > 0$. Then we can choose $r_k \geq R_1$ to obtain

$$\frac{F(t)}{F(r_k)} \geq \exp\left\{-2C_3 \int_{r_k}^{t} \mu dr\right\}.$$

Since $\mu(r) \in L^1(\mathbf{R}_+)$, this proves the uniform positivity near infinity of $F(t)$.

Next assume the contrary that $F(r) \leq 0$ for $r \geq R_2(\geq R_1)$, and u does not have compact support. In this case we put $\varphi = r$, $\kappa^2 = \lambda$ and $f \equiv 0$ in Proposition 1, and subtract the identity

$$\frac{1}{2}\left(\int_{S_t} - \int_{S_R}\right) r\left(\sigma'^2 - \tau\right)|u_\sigma|^2 - \frac{1}{2}\int_{B_{R,t}} r\Big\{\mathrm{Re}\left[(\sigma'^2 - \tau)u_\sigma \overline{\tilde{x}\cdot\theta_\sigma}\right]$$
$$-\left(\frac{1}{r}\sigma'^2 + \sigma''\sigma' - \frac{1}{r}\tau - \frac{1}{2}\tau'\right)|u_\sigma|^2\Big\}dx = 0.$$

Then it follows that

$$\frac{d}{dt}\left[tF_{\sigma,\tau}(t)\right] = \int_{S_t} r\Big\{\frac{1}{2r}|\theta_\sigma|^2 + \mathrm{Re}J_\sigma(r) + 2\sigma'\left|\tilde{x}\cdot\nabla_b u_\sigma + \frac{n-1}{2r}u_\sigma\right|^2$$
$$+(\sigma''-\tau)u_\sigma\left(\overline{\tilde{x}\cdot\nabla_b u_\sigma} + \frac{n-1}{2r}\overline{u_\sigma}\right) + \left(\frac{1}{r}\sigma'^2 + \sigma''\sigma' - \frac{1}{r}\tau - \frac{1}{2}\tau'\right)|u_\sigma|^2\Big\}dS, \quad (18)$$

where we have used Lemma 4 to obtain

$$\int_{S_t} 2\sigma'\left\{|\tilde{x}\cdot\theta_\sigma|^2 + \mathrm{Re}[\pm i\sqrt{\lambda}u_\sigma\overline{\tilde{x}\cdot\theta_\sigma}]\right\}dS = \int_{S_t} 2\sigma'\left|\tilde{x}\cdot\nabla_b u_\sigma + \frac{n-1}{2r}u_\sigma\right|^2 dS.$$

We choose here

$$\sigma(r) = \frac{m}{1-\epsilon}r^{1-\epsilon}, \quad \tau(r) = r^{-2\epsilon}\log r$$

with $m \geq 1$ and $1/3 < \epsilon < 1/2$. Then noting (17) and assumption $\mu(r) = o(r^{-1})$, we can show (as for the details, see e.g. Mochizuki [13]) that there exists $R_3 \geq R_2$ such that for any $m \geq 1$,

$$\frac{d}{dt}[tF_{\sigma,\tau}(t)] \geq \int_{S_t} r\left(\frac{1}{2r} - o(r^{-1})\right)|\theta_\sigma|^2 dS \geq 0 \quad \text{in} \quad t \geq R_3.$$

Moreover, by assumption there exists $R_4 \geq R_3$ such that $\displaystyle\int_{S_{R_4}} |u_\sigma|^2 dS > 0$. Thus, we can choose m large to satisfy $F_{\sigma,\tau}(R_4) > 0$. Combining these properties, we conclude that $F_{\sigma,\tau}(t) > 0$ for $t \geq R_4$. Note here that

$$F_{\sigma,\tau}(t) = e^{2\sigma}\left\{F(r) + \sigma' \frac{d}{dt}\int_{S(r)} |u|^2 dS + (2\sigma'^2 - \tau)\int_{S(r)} |u|^2 dS\right\}.$$

$F(r) \leq 0$ near infinity by assumption, and the third term of the right becomes nonpositive when r goes large. Hence,

$$\frac{d}{dt}\int_{S(t)} |u|^2 dS > 0$$

for r large enough.

The desired conclusion thus holds. □

4. Proof of Theorem 3

We put

$$\varphi_1(r) = \left(\int_r^\infty \mu(s) ds\right)^{-1}.$$

Then as is easily seen

$$\varphi_1'(r) = \mu(r)\varphi_1(r)^2, \tag{19}$$

and $\mu\varphi_1$ and hence $\varphi' = \mu\varphi^2$ is not in $L^1(\mathbf{R}_+)$. Moreover, it follows from (5) that

$$\frac{\varphi_1'(s)}{\varphi_1(s)} ds = \mu(r)\varphi_1(r) \leq \frac{1}{r} \quad \text{for} \quad r \geq R_0. \tag{20}$$

We shall show that for any $0 < a < b < \infty$, the resolvent $R(\kappa^2) \in \mathcal{B}(L^2_{\mu^{-1}}, L^2_\mu)$ restricted in $\kappa \in K_\pm = \{\kappa; a \leq \pm\mathrm{Re}\kappa \leq b, 0 < \mathrm{Im}\kappa \leq 1\}$ is continuously extended to $K_\pm \cup [a, b]$.

The proof is based on Theorem 2 and the following two lemmas.

Lemma 6. *Let $u = R(\kappa^2) f$ with $\kappa \in K_\pm$ and $f \in L^2_{\mu_1^{-1}}$. Then there exists $C = C(K_\pm) > 0$ such that $u = R(\kappa^2) f$ satisfies*

$$\|\theta\|^2_{\varphi_1', B_R'} \leq C\left\{\|u\|^2_\mu + \|f\|^2_{\mu^{-1}}\right\}, \quad R \geq R_4.$$

Proof We choose $R \geq R_4$ and $t > R + 1$ in Proposition 1, and put $\sigma = 0$ and $\varphi = \chi\varphi_1$ there, where $\chi = \chi(r)$ is a smooth function such that $\chi(r) = 0$ ($r \leq R$) and $\chi(r) = 1$ ($r \geq R + 1$). Then

$$\int_{S_t} \varphi_1\left(|\tilde{x}\cdot\theta|^2 - \frac{1}{2}|\theta|^2\right) dS = \int_{B_{R,t}} \chi\varphi_1\left\{\frac{\varphi_1'}{2\varphi_1}|\theta|^2\right.$$

$$+\left(\frac{1}{r}-\frac{\varphi_1'}{\varphi_1}\right)(|\theta|^2-|\tilde{x}\cdot\theta|^2)+\frac{1}{2}\mathrm{Re}\left[J(x,\kappa)-f\tilde{x}\cdot\overline{\theta}\right]\Big\}dx$$

$$+\int_{B_{R,R+1}}\chi'\varphi_1\left(|\tilde{x}\cdot\theta|^2-\frac{1}{2}|\theta|^2\right)dx. \tag{21}$$

It follows from $(A2)$ and (19) that

$$\varphi_1|J(x,\kappa)|\leq C_0\varphi_1\mu|u\tilde{x}\cdot\bar{\theta}_\pm|\leq C_0{\varphi'}_1^{1/2}|\tilde{x}\cdot\theta|\mu_2^{1/2}|u|.$$

Moreover, by the ellipticity of equation (1), we see

$$\int_{B(R,R+1)}|\theta|^2dx\leq C\left\{\|u\|_\mu^2+\|f\|_{\mu^{-1}}^2\right\}.$$

Thus, noting (20), applying the Schwarz inequality and letting $t\to\infty$ in (21), we obtain the assertion of the lemma. □

Lemma 7. *Let $\kappa\in K_\pm$ and $f\in L^2_{\mu^{-1}}$. Then there exists $R_5\geq R_4$ such that $u=R(\kappa^2)f$ satisfies for $R\geq R_5$,*

$$a\|u\|^2_{\mu,B'_R}\leq C\varphi_1(R)^{-1}\left\{\|\tilde{x}\cdot\theta\|^2_{\varphi_1',B_R'}+\|u\|^2_\mu+\|f\|^2_{\mu^{-1}}\right\}.$$

Proof By the Gauss formula

$$\mathrm{Im}\int_{B_r}f\bar{u}dx=-\mathrm{Im}\int_{S_r}(\tilde{x}\cdot\nabla_b u)\bar{u}dS-\mathrm{Im}\kappa^2\int_{B_r}|u|^2dx.$$

It then follows that

$$\mathrm{Im}\kappa^2\int_{B_r}|u|^2dx+\mathrm{Re}\kappa\int_{S_r}|u|^2dS=-\mathrm{Im}\left[\int_{S_r}(\tilde{x}\cdot\theta)\overline{u}dS+\int_{B_r}f\overline{u}dx\right].$$

Since $\mathrm{Im}\kappa^2$ and $\mathrm{Re}\kappa$ have the same sign, and since $\pm\mathrm{Re}\kappa\geq a>0$, multiplying by $\mu(r)$ and integrating over (R,∞) with respect to r, we obtain

$$a\|u\|^2_{\mu,B'_R}\leq\int_{B'_R}\mu|\tilde{x}\cdot\theta||u|dx+\varphi_1(R)^{-1}\int|f||u|dx.$$

The inequality of the lemma then follows from the relation $\mu^{1/2}=\varphi_1^{-1}{\varphi'}_1^{1/2}$ and the Schwarz inequality. □

Proof of Theorem 3 Let $\{\kappa_k,f_k\}\subset K_\pm\times L^2_{\mu^{-1}}$ converge to $\{\kappa_0,f_0\}$ as $k\to\infty$. Since the other case is easier, we assume that $\pm\kappa_0=\sqrt{\lambda}\in[a,b]$. Let $u_k=R(\kappa_k^2)f_k$. Then since $\varphi_1(R)^{-1}\to0$ as $R\to\infty$, the Rellich compactness criterion, Lemmas 6 and 7 show that $\{u_k\}$ is compact in L^2_μ if it is bounded in the same space. Moreover, Lemma 6 shows that every accumulation point $u_0 \in L^2_\mu$ satisfies the inequality

$$\left\|\tilde{x}\cdot\nabla_b u_0 + \left(\frac{n-1}{2r} - i\kappa_0\right)u_0\right\|_{\varphi_1'} < \infty.$$

The boundedness of $\{u_k\}$ is proved by contradiction. In fact, assume that there exists a subsequence, which we also write $\{u_k\}$, such that $\|u_k\|_\mu \to \infty$ as $k \to \infty$. Put $v_k = u_k/\|u_k\|_\mu$. Then as it is explained above, $\{\kappa_k, v_k\}$ has a convergent subsequence, and if we denote the limit by $\{\kappa_0, v_0\}$, then it satisfies the homogeneous equation (4) with $\lambda = \kappa_0^2$ and also

$$\|v_0\|_\mu = 1, \quad \left\|\tilde{x}\cdot\nabla_b v_0 + \left(\frac{n-1}{2r} - i\kappa_0\right)v_0\right\|_{\varphi_1'} < \infty. \tag{22}$$

The second inequality implies

$$\liminf_{r\to\infty}\int_{S_r}\left|\tilde{x}\cdot\nabla_b v_0 + \left(\frac{n-1}{2r} - i\kappa_0\right)v_0\right|^2 dS = 0$$

since $\varphi_1'(r) \notin L^1([R_5,\infty))$. Comparing this with Theorem 2, we see that v_0 has a compact support in $x \in \mathbf{R}^n$. Hence, $v_0 \equiv 0$ by the unique continuation property for solutions to (4). But this contradicts to the first equation of (22).

We have shown that the sequence $\{u_k\}$ is precompact in L^2_μ and satisfies inequality (22). But if we apply Theorem 2 once more, then $\{u_k\}$ itself is shown to converge.

The proof of Theorem 3 is thus completed. □

5. Proof of Theorem 4

In this section we shall prove Theorem 4 by a series of lemmas for the solution $u = R(\kappa^2)f$ of (1) (the proof of this and next sections are essentially the same as in Mochizuki [14]).

LEMMA 8. *Let $\varphi = \varphi(r)$ be a positive increasing function of $r > 0$ satisfying*

$$\frac{\varphi'(r)}{\varphi(r)} \le \frac{1}{r}. \tag{23}$$

Then we have

$$\int \varphi\left(\mathrm{Im}\kappa + \frac{\varphi'}{2\varphi}\right)\left\{|\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\right\}dx$$
$$\le \int \varphi\left(|f| + \max\{|\nabla\times b|, |c|\}|u|\right)|\theta|dx.$$

Proof In the identity of Proposition 1 we put $\sigma \equiv 0$. Then letting $R \to 0$ and $t \to \infty$, we obtain

$$\int \varphi \left\{ \left(\frac{1}{r} - \frac{\varphi'}{\varphi} \right) (|\theta|^2 - |\tilde{x} \cdot \theta|^2) + \left(\mathrm{Im}\kappa + \frac{\varphi'}{2\varphi} \right) |\theta|^2 + \mathrm{Re} J(x, \kappa) \right\} dx$$

$$= \mathrm{Re} \int \varphi f \overline{\tilde{x} \cdot \theta} dx. \tag{24}$$

Here

$$\varphi \left| J(x, \kappa) - \frac{(n-1)(n-3)}{4r^2} u \overline{\tilde{x} \cdot \theta} \right| \le \varphi \max\{|\nabla \times b|, |c|\} |u||\theta|$$

and

$$\mathrm{Re} \int \varphi \frac{(n-1)(n-3)}{4r^2} u \overline{\tilde{x} \cdot \theta} dx = \int \varphi \left(\mathrm{Im}\kappa - \frac{\varphi'}{2\varphi} + \frac{1}{r} \right) \frac{(n-1)(n-3)}{4r^2} |u|^2 dx.$$

Substitute these relations to (24). Then assumption (23) on φ and the Schwarz inequality show the inequality of the lemma. □

LEMMA 9. *We have*

$$\int \frac{1}{4r^2} |u|^2 dx \le \int |\tilde{x} \cdot \theta|^2 dx.$$

Proof We begin with the identity

$$\left| \nabla_b u \right|^2 = \left| \nabla_b u + \tilde{x} \frac{\alpha}{r} u \right|^2 - \nabla \cdot \left(\tilde{x} \frac{\alpha}{r} |u|^2 \right) + \frac{(n-2)\alpha - \alpha^2}{r^2} |u|^2$$

which is similar to (9). Multiply by $\xi = \xi(r) > 0$ on both sides. Then

$$\xi |\tilde{x} \cdot \nabla_b u|^2 = \left| \tilde{x} \cdot \nabla_b (\sqrt{\xi} u) + \frac{\alpha}{r} \sqrt{\xi} u \right|^2 - \nabla \cdot \left[\tilde{x} \left(\frac{\alpha}{r} + \frac{\xi'}{2\xi} \right) |\sqrt{\xi} u|^2 \right]$$

$$+ \left\{ \frac{n-1}{r} \left(\frac{\alpha}{r} + \frac{\xi'}{2\xi} \right) + \left(\frac{\alpha}{r} + \frac{\xi'}{2\xi} \right)' - 2\frac{\alpha}{r} \left(\frac{\alpha}{r} + \frac{\xi'}{2\xi} \right) + \left(\frac{\alpha}{r} + \frac{\xi'}{2\xi} \right)^2 \right\} |\sqrt{\xi} u|^2.$$

Integrating this over $B_{\epsilon,t}$, we have

$$\int_{B_{\epsilon,t}} \xi |\tilde{x} \cdot \nabla_b u|^2 dx = \int_{B_{\epsilon,t}} \left| \tilde{x} \cdot \nabla_b (\sqrt{\xi} u) + \frac{\alpha}{r} \sqrt{\xi} u \right|^2 dx$$

$$- \left[\int_{S_t} - \int_{S_\epsilon} \right] \left(\frac{\xi'}{2\xi} + \frac{\alpha}{r} \right) |\sqrt{\xi} u|^2 dS$$

$$+ \int_{B_{\epsilon,t}} \left\{ \frac{(n-2)\alpha - \alpha^2}{r^2} + \frac{(n-1)\xi'}{2r\xi} + \frac{2\xi''\xi - \xi'^2}{4\xi^2} \right\} |\sqrt{\xi} u|^2 dx.$$

We here replace u by $v = e^{-i\kappa r}r^{(n-1)/2}u$ and choose $\xi = r^{-n+1}e^{-2\mathrm{Im}\kappa r}$ and $\alpha = \dfrac{n-2}{2}$. Then, since

$$\xi|v|^2 = |u|^2, \quad \xi|\tilde{x}\cdot\nabla_b v|^2 = |\tilde{x}\cdot\theta|^2$$

and

$$\frac{(n-2)\alpha-\alpha^2}{r^2} + \frac{(n-1)\xi'}{2r\xi} + \frac{2\xi''\xi - \xi'^2}{4\xi^2} = \frac{1}{4r^2} + (\mathrm{Im}\kappa)^2,$$

letting $\epsilon \to 0$ and $t \to \infty$, we obtain the inequality of the lemma. □

Proof of Theorem 4 (i) We choose $\varphi = r$ in Lemma 8. Then noting $(A4)$, we have for any $0 < \epsilon \le 1$,

$$\frac{1}{2}\int\left\{(1-\epsilon)|\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\right\}dx \le \frac{1}{\epsilon}\int\left(r^2|f|^2 + \epsilon_0^2 r^{-2}|u|^2\right)dx.$$

Combining this and Lemma 9 leads us to

$$\frac{-\epsilon^2 + (n-2)^2\epsilon - 8\epsilon_0^2}{8\epsilon}\int\frac{1}{r^2}|u|^2dx \le \frac{1}{\epsilon}\int r^2|f|^2dx.$$

Thus, we conclude the inequality of (ii) by choosing $\epsilon = \min\{\sqrt{8}\epsilon_0, 1\}$. □

Lemma 10. *Assume* $c(x) \ge -\dfrac{(n-2)^2}{4r^2}$. *Then for* μ *satisfying* (6) *we have*

$$\frac{1}{2}\int\left\{\mu\mathrm{Im}\kappa\frac{1}{r}|u|^2| - \mu'\frac{n-1}{r}|u|^2 + \mu(|\nabla_b u|^2 + |\kappa u|^2)\right\}dx$$

$$\le \frac{1}{2}\int\mu\left(|\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\right)dx + \|\mu\|_{L^1}\int|f(x)||i\kappa u|dx.$$

Proof We multiply by $-\overline{i\kappa u}$ on both sides of (1) and integrate the real part over B_r to obtain

$$\frac{1}{2}\int_{S_r}\left\{-|\nabla_b u - i\kappa u|^2 + |\nabla_b u|^2 + |\kappa|^2|u|^2\right\}dS$$

$$+\mathrm{Im}\kappa\int_{B_r}\left(|\nabla_b u|^2 + c|u|^2 + |\kappa u|^2\right)dx = -\mathrm{Re}\int_{B_r}f\overline{i\kappa u}dx.$$

Multiply $\mu(r)$ on both sides and integrate over $(0,\infty)$. Then noting

$$\mu|\nabla_b u - i\kappa\tilde{x}u|^2 = -\nabla\cdot\{\tilde{x}\mu\frac{n-1}{2r}|u|^2\} + \mu\left(|\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\right)$$

$$+\mu'\frac{n-1}{2r}|u|^2-\mu\mathrm{Im}\kappa\frac{n-1}{r}|u|^2,$$

we obtain

$$\begin{aligned}&\frac{1}{2}\int\left\{\mu\mathrm{Im}\kappa\frac{n-1}{r}|u|^2-\mu'\frac{n-1}{2r}|u|^2+\mu\left(|\nabla_b u|^2+|\kappa u|^2\right)\right\}dx\\&\qquad+\mathrm{Im}\kappa\int_0^\infty\mu dr\int_{B_r}\left(|\nabla_b u|^2+c|u|^2+|\kappa u|^2\right)dx\\&\quad=\frac{1}{2}\int\mu\left(|\theta|^2+\frac{(n-1)(n-3)}{4r^2}|u|^2\right)dx+\mathrm{Re}\int_0^\infty\mu dr\int_{B_r}f(x)\overline{i\kappa u}dx.\end{aligned}$$

Note here $c(x)\geq-\dfrac{(n-2)^2}{4r^2}$. Then Lemma 1 with $\alpha=\dfrac{n-2}{2}$ and $\epsilon=0$ shows

$$\begin{aligned}&\mathrm{Im}\kappa\int_0^\infty\mu dr\int_{B_r}(|\nabla_b u|^2+c(x)|u|^2)dx\\&\qquad\geq-\mathrm{Im}\kappa\int_0^\infty\mu dr\int_{S_r}\frac{n-2}{2r}|u|^2dS=-\int\mu\mathrm{Im}\kappa\frac{n-2}{2r}|u|^2dx,\end{aligned}$$

and the desired inequality holds. □

Proof of Theorem 4 (*ii*) We combine Lemmas 10 and 8 with $\varphi(r)=\displaystyle\int_0^r\mu(\sigma)d\sigma$. It is obvious that this φ satisfies (23). Then since $\varphi(r)\leq\|\mu\|_{L^1}$, it follows that

$$\frac{1}{2}\int\left\{-\mu'\frac{n-1}{2r}|u|^2+\mu\left(|\nabla_b u|^2+|\kappa u|^2\right)\right\}dx$$

$$\leq4\|\mu\|_{L^1}^2\int\mu^{-1}\left(|f|^2+|\max\{|\nabla\times b|,|c|\}u|^2\right)dx+\|\mu\|_{L^1}\int|f||i\kappa u|dx.$$

Thus, noting

$$\|\mu\|_{L^1}\int|f||i\kappa u|dx\leq\|\mu\|_{L^1}^2\int\mu^{-1}|f|^2dx+\frac{1}{4}\int\mu|\kappa u|^2dx,$$

we conclude

$$\begin{aligned}&\int\left\{\mu(|\nabla_b u|^2+|\kappa u|^2)-\mu'\frac{n-1}{2r}|u|^2\right\}dx\\&\qquad\leq4\|\mu\|_{L^1}^2\int\mu^{-1}\left(5|f(x)|^2+4|\max\{|\nabla\times b|,|c|\}u|^2\right)dx.\qquad(25)\end{aligned}$$

The use of $(A5)$ and the inequality of (i) implies that

$$\int\mu^{-1}|\max\{|\nabla\times b|,|c|\}u|^2dx\leq\epsilon_0^2C_1\int r^2|f|^2dx.$$

Thus, substituting this in (25) gives the desired inequality. □

6. Proof of Theorem 5

The following proposition summarizes abstract results which allows us to employ the resolvent estimate for a selfadjoint operator to a space-time weighted estimate for the associated evolution equation. As for the proof see, e.g., Mochizuki [14].

Let Λ be a selfadjoint operator in the Hilbert space $\mathcal{H}$, and for $z \in \mathbf{C}\backslash\mathbf{R}$ let $\mathcal{R}(z)$ be the resolvent of Λ. Suppose that A is a densely defined, closed operator from H to another Hilbert space $\mathcal{H}_1$.

PROPOSITION 2. *Assume that there exists $C > 0$ such that*

$$\sup_{z \notin \mathbf{R}} \|A\mathcal{R}(z)A^* f\|_{\mathcal{H}_1} < \sqrt{C}\|f\|_{\mathcal{H}_1} \tag{26}$$

for $f \in \mathcal{D}(A^*)$. Then we have

$$\left|\int_0^{\pm\infty} \left\|\int_0^t Ae^{-i(t-\tau)\Lambda}A^* h(\tau)d\tau\right\|_{\mathcal{H}_1}^2 dt\right| \leq C\left|\int_0^{\pm\infty} \|h(t)\|_{\mathcal{H}_1}^2 dt\right|, \tag{27}$$

$$\sup_{t \in \mathbf{R}_\pm} \left\|\int_0^t e^{i\tau\Lambda}A^* h(\tau)d\tau\right\|_{\mathcal{H}}^2 \leq 2\sqrt{C}\left|\int_0^{\pm\infty} \|h(t)\|_{\mathcal{H}_1}^2 dt\right| \tag{28}$$

for each $h(t) \in L^2(\mathbf{R}; \mathcal{D}(A^*))$, and

$$\left|\int_o^{\pm\infty} \|Ae^{-it\Lambda}f\|_{\mathcal{H}_1}^2 dt\right| \leq 2\sqrt{C}\|f\|_{\mathcal{H}}^2 \tag{29}$$

for each $f \in \mathcal{H}$.

Proof of Theorem 5 (i) Set $\Lambda = L$, $\mathcal{H} = \mathcal{H}_1 = L^2$ and $A = r^{-1}$ (multiplication operator). Then $A^* = A$ and $\mathcal{R}(z) = R(z)$, and if we let $z = \kappa^2$, then it follows from Theorem 4 (i) that

$$\|AR(z)A^* f\| = \|r^{-1}R(z)A^* f\| \leq \sqrt{C_1}\|rA^* f\| = \sqrt{C_1}\|f\|.$$

Thus, the estimates (27) and (29) can be written as

$$\left|\int_0^{\pm\infty} \|r^{-1}\int_0^t e^{-i(t-\tau)L}h(\tau)d\tau\|^2 dt\right| \leq C_1\left|\int_0^{\pm\infty} \|rh(t)\|^2 dt\right|,$$

$$\left|\int_0^{\pm\infty} \|r^{-1}e^{-itL}f\|^2 dt\right| \leq 2\sqrt{C_1}\|f\|^2.$$

These are what to be proved. $\square$

To show Theorem 5 (ii) we consider the Klein-Gordon equation

$$i\partial_t u = \Lambda u, \quad u(t) = \{w(t), \partial_t w(t)\}, \quad \Lambda = \begin{pmatrix} 0 & i \\ -i(L+m^2) & 0 \end{pmatrix}$$

in the energy space $\mathcal{H} = H_b^1 \times L^2$, where H_b^1 is the completion of $C_0^\infty(\Omega)$ in the norm

$$\|f_1\|_{H_b^1}^2 = \frac{1}{2}\int \{|\nabla_b f_1|^2 + (c(x)+m^2)|f_1|^2\}dx.$$

Then Λ with domain

$$\mathcal{D}(\Lambda) = \{f_1 \in H_b^1; \Delta_b f_1 \in L^2\} \times \{f_2 \in H_b^1 \cap L^2\}$$

forms a selfadjoint operator in $\mathcal{H}$, and its resolvent is given by

$$\mathcal{R}(z) = (L+m^2-z^2)^{-1} \begin{pmatrix} z & i \\ -i(L+m^2) & z \end{pmatrix}.$$

Let $A: \ \mathcal{H} \to \mathcal{H}_1 = L^2$ be defined by

$$Af = \min\{\sqrt{\mu(r)}, r^{-1}\}\sqrt{L+m^2} f_1 \quad \text{for} \quad f = \{f_1, f_2\} \in \mathcal{H}.$$

Then the adjoint operator A^* is given by

$$A^* g = \left\{ \sqrt{L+m^2}^{-1} \min\{\sqrt{\mu(r)}, r^{-1}\}g, 0 \right\} \quad \text{for} \quad g \in L^2.$$

Proof of Theorem 5 (ii) By definition

$$A\mathcal{R}(z)A^* g = \min\{\sqrt{\mu(r)}, r^{-1}\} z(L+m^2-z^2)^{-1} \min\{\sqrt{\mu(r)}, r^{-1}\} g \tag{30}$$

for $g \in \mathcal{D}(A^*)$. Then since

$$\int \left| \min\{\sqrt{\mu}, r^{-1}\} z(L+m^2-z-2)^{-1} f \right|^2 dx \le m^2 \int r^{-2}|(L+m^2-z^2)^{-1} f|^2 dx$$

$$+ \int \mu| - m^2 + z^2||L(L+m^2-z^2)^{-1} f|^2 dx,$$

using Theorem 4, we obtain

$$\|A\mathcal{R}(z)A^* g\| \le \sqrt{m^2 C_1 + C_2}\|g\|.$$

We return to Proposition 3 with this inequality. Then (29) shows that

$$\left| \int_0^{\pm\infty} \|Ae^{-it\Lambda} f\|^2 dt \right| = \left| \int_0^{\pm\infty} \|\min\{\sqrt{\mu(r)}, r^{-1}\}\sqrt{L+m^2} w(t)\|^2 dy \right|$$

$$\leq 2\sqrt{m^2C_1 + C_2}\|f\|^2_{\mathcal{H}}.$$

Since

$$w(t) = \cos(t\sqrt{L+m^2})f_1 + \sqrt{L+m^2}^{-1}\sin(t\sqrt{L+m^2})f_2,$$

choosing $f = \{\sqrt{L+m^2}^{-1}g, 0\}$ and $f = \{0, g\}$ for $g \in L^2$, we obtain

$$\left|\int_0^{\pm\infty} \|\min\{\sqrt{\mu(r)}, r^{-1}\}\cos(t\sqrt{L+m^2})g\|^2 dy\right| \leq \sqrt{m^2C_1 + C_2}\|g\|^2$$

and

$$\left|\int_0^{\pm\infty} \|\min\{\sqrt{\mu(r)}, r^{-1}\}\sin(t\sqrt{L+m^2})g\|^2 dy\right| \leq \sqrt{m^2C_1 + C_2}\|g\|^2,$$

respectively. These inequalities imply assertion (ii). □

7. An extension of Theorem 4 (i) and final remarks

As is proved in Theorems 3 and 4, the resolvent $R(\kappa^2)$ of L satisfies the following properties under $(A1)$ and $(A4)$.

(a) As an operator from $L^2_{\xi^{-1}}$ to L^2_{ξ}, where $\xi = (1+r)^{-2}$, $R(\kappa^2)$ is continuously extended to $\kappa \in \overline{\mathbf{C}_+} = \{\kappa \in \mathbf{C}; \mathrm{Im}\kappa \geq 0\}$.

(b) $R(\kappa^2) \in \mathcal{B}(L^2_{\xi^{-1}}, L^2_{\xi})$, $\kappa \in \overline{\mathbf{C}_+}$, is a compact operator.

(c) $(1+|\kappa|)\|R(\kappa^2)\|_{\mathcal{B}(L^2_{\xi^{-1}}, L^2_{\xi})}$ is uniformly bounded in $\kappa \in \overline{\mathbf{C}_+}$.

Now, let us consider a perturbation of L:

$$L_2 = L + c_2(x),$$

where $c_2(x)$ is a real valued L^∞-function satisfying

$$(A6) \qquad |c_2(x)| \leq C(1+r)^{-2} \quad \text{for some} \quad C > 0,$$

and the unique continuation property holds for $-\Delta_b + c(x) + c_2(x)$.

$(A7)$ L_2 has at most a finite number of negative eigenvalues, and $\kappa^2 = 0$ is neither an eigenvalue nor a resonance of L_2.

Under these conditions Theorem 4 (i) is easily extended to the operator L_2.

Theorem 6. *Let $-\lambda_0$ be the largest negative eigenvalue of L_2. Choose $\delta > 0$ to satisfy $\delta^2 < \lambda_0$, and let $\mathbf{C}_{+,\delta} = \{\kappa \in \mathbf{C} : 0 < \mathrm{Im}\kappa \leq \delta\}$. Then there exists $C > 0$ such that*

$$\int (1+r)^{-2}|R_2(\kappa^2)f|^2 dx \leq C\int (1+r)^2|f|^2 dx$$

for each $\kappa \in \mathbf{C}_{+,\delta}$ and f satisfying $(1+r)f \in L^2$.

Finally, we summarize related problems not proved in this article.

1. The finiteness of negative eigenvalues should be ascertained under condition $(A6)$.

2. It is not known, whether the essential spectrum $\sigma_e(L)$ of L fills the nonnegative real line or not.

3. The inhomogeneous smoothing property corresponding to the first estimate of Theorem 5 (i) is not obtained here for the relativistic Schrödinger equation (9).

4. What happens when the smallness of the magnetic field $\nabla \times b(x)$ like $(A4)$ is not required?

References

[1] S. Cuccagna and P. P. Schirmer, *On the wave equation with magnetic potential*, Comm. Pure Appl. Math. **54** (2001), 135–152.

[2] P. D'Ancona and L. Fanelli, *Strichartz and smoothing estimates for dispersive equations with magnetic potentials*, Comm. Partial Differential Equations **33** (2008), 1082–1112.

[3] D. M. Eidus, *The principle of limiting amplitude*, Uspekhi Math. Nauk **24** (1969), 91–156.

[4] M. B. Erdogan, M. Goldberg, and W. Schlag, *Stricharts and smoothing estimates for Schrödinger operators with large magnetic potentials in* $\mathbf{R}^3$, J. Eur. Math. Soc. **10** (2008), 507–531.

[5] K. O. Friedrichs, *Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren*, Math. Ann. **109** (1934), 469–487, 685–713; **110** (1935), 777–779.

[6] V. Georgiev, A. Stefanov, and M. Tarulli, *Smoothing-Strichartz estimates for the Schrödinger equation with small magnetic potential*, Disc. Cont. Dyn. Syst.-A **17** (2007), 771–186.

[7] T. Ikebe and J. Uchiyama, *On the asymptotic behavior of eigenfunctions of second-order elliptic operators*, J. Math. Kyoto Univ. **11** (1971), 425–448.

[8] W. Jäger and P. Rejto, *On a theorem of Mochizuki and Uchiyama about long range oscillating potentials*, Operator theory and its applications. Proceedings of the international conference (Winnipeg, Canada, October 7-11, 1998), Providence, RI. American Mathematical Society. Fields Inst. Commun. 25, 2000, pp. 305–329.

[9] H. Kalf, U.-W. Schmincke, J. Walter, and R. Wüst, *On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials.*, Spectral Theor. Differ. Equat., Proc. Symp. Dundee 1974, Lect. Notes Math. 448, 182-226, 1975.

[10] T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann. **162** (1966), 255–279.

[11] T. Kato and K. Yajima, *Some examples of smooth operators and the associated smoothing effect*, Reviews in Math. Phys. **1** (1989), 481–496.
[12] K. Mochizuki, *Spectral and scattering theory for second order elliptic differential operators in an exterior domain*, Lecture Notes Univ. Utah, Winter and Spring, 1972.
[13] K. Mochizuki, *On the spectrum of Schrödinger operators with oscillating long-range potentials*, More Progress in Analysis. Proceedings 5th International ISAAC Congress (Catania, Italy, 2005) (H.G.W. Begehr and F. Nicolosi, eds.), World Scientific, Singapore, 2009, pp. 533–542.
[14] K. Mochizuki, *Uniform resolvent estimates for magnetic Schrödinger operators and smoothing effects for related evolution equations*, Publ. Res. Inst. Math. Sci. (2010), to appear.
[15] F. Rellich, *Über das asymptotische Verhalten der Lösungen von* $\Delta u + \lambda u = 0$ *in unendlichen Gebieten*, Jahresber. Deutsch. Math. Verein **53** (1943), 57–65.
[16] K. Yajima, *Schrödinger evolution equation with magnetic fields*, J. d'Analyse Math. **56** (1991), 29–76.

Author's address:

Kiyoshi Mochizuki
Department of Mathematics
Chuo University
Kasuga, Bunnkyou, Tokyo 112-8551, Japan
E-mail: mochizuk@math.chuo-u.ac.jp

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