Rend. Istit. Mat. Univ. Trieste
Volume 42 Suppl. (2010), 27–50.

$L^p - L^q$-Decay Estimates for the Klein-Gordon Equation in the Anti-de Sitter Space-Time

Anahit Galstian

Abstract. *We derive $L^p - L^q$ - decay estimates for the solutions of the Cauchy problem for the Klein-Gordon equation in the anti-de Sitter spacetime, that is, for $\Box_g u - m^2 u = f$ in models of mathematical cosmology. The obtained L^p–L^q estimates imply exponential decay of the solutions for large times.*

Keywords: anti de Sitter spacetime; Klein-Gordon equation; decay estimates.
MS Classification 2000: 35L05; 35Q75; 35L15.

1. Introduction.

In this article we prove decay estimates for the solutions of the Cauchy problem for the Klein-Gordon equation $\Box_g \phi - m^2\phi = f$ in the anti-de Sitter spacetime.

In the model of the universe proposed by de Sitter the line element has the form

$$ds^2 \quad = \quad -\left(1-\frac{\Lambda r^2}{3}\right)c^2dt^2 + \left(1-\frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2).$$

The constant $\Lambda > 0$ is the *cosmological constant*. The corresponding metric with this line element is called the de Sitter metric. If $\Lambda < 0$, it is called the anti-de Sitter metric. In the de Sitter and anti-de Sitter spacetimes the equation for the scalar field with mass m is the covariant Klein-Gordon equation

$$\Box_g\phi - m^2\phi = f \quad \text{or} \quad \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ik}\frac{\partial\phi}{\partial x^k}\right) - m^2\phi = f\,,$$

with the usual summation convention, where $x=(x^0,x^1,\dots,x^n)$ and g^{ik} is a metric tensor. Written explicitly after Lamaître-Robertson transformation [7], in coordinates in the de Sitter spacetime this equation has the form

$$\phi_{tt} + n\phi_t - e^{-2t}\Delta\phi + m^2\phi = f\,. \tag{1}$$

Here t is x^0, while $\triangle$ is the Laplace operator on the flat metric in $\mathbb{R}^n$. If we introduce the new unknown function $u = e^{\frac{n}{2}t}\phi$, then the equation (1) takes the form of the linear Klein-Gordon equation for u on de Sitter spacetime

$$u_{tt} - e^{-2t} \triangle u + M^2 u = f, \tag{2}$$

where the "curved mass" M is defined as follows: $M^2 := m^2 - n^2/4$. In [13] the fundamental solutions for the Klein-Gordon operator in de Sitter spacetime are given. The fundamental solution with the support in the forward light cone has been used in [13] to represent solutions of the Cauchy problem and to prove $L^p - L^q$ estimates for the solutions of the equation with and without a source term.

The time inversion transformation $t \to -t$ reduces the equation (2) to the equation

$$\partial_t^2 u - e^{2t} \triangle u + M^2 u = f \tag{3}$$

that can be regarded as an equation in the anti-de Sitter spacetime. The anti-de Sitter spacetime certainly deserves mathematical attention in its own right, moreover, there is a considerable interest from high energy physics. Recently, in [10] the forward Dirichlet problem is studied and it is proved in [10] that the problem is globally well-posed under a global condition on the generalized broken bicharacteristic.

In the present paper we consider the Klein-Gordon operator in anti-de Sitter spacetime, that is $\mathcal{S} := \partial_t^2 - e^{2t} \triangle + M^2$, where M is the curved mass, and $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. In this article we restrict ourselves to nonnegative curved mass $M \geq 0$. The Cauchy problem for the strictly hyperbolic equation (3) is well-posed in some different functional spaces. Consequently, the solution operator is well-defined in those functional spaces. The equation (3) possesses two fundamental solutions resolving the Cauchy problem without source term f. They can be written in terms of Fourier integral operators. Unlike to (2) the equation (3) does not possesses the so-called horizon (cf. [13]).

The wave equation without source, that is (3) with $M = 0$ and $f = 0$, was investigated in [5]. More precisely, in [5] the resolving operator for the Cauchy problem

$$\partial_t^2 u - e^{2t} \triangle u = 0, \qquad u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x)\,, \tag{4}$$

is written as a sum of Fourier integral operators with amplitudes given in terms of the Bessel functions and in terms of confluent hypergeometric functions. One important tool to prove global existence for nonlinear equations is a L^p–L^q decay estimate (see, e.g. [9]). The typical $L^p - L^q$ decay estimates obtained in [5] by dyadic decomposition of the phase space, contain some loss of regularity. More precisely, it is proved that for the solution $u = u(x,t)$ to the Cauchy problem (4) with $n \geq 2$, $\varphi_0(x) \in C_0^\infty(\mathbb{R}^n)$ and $\varphi_1(x) = 0$, for all large $t \geq T > 0$, the following estimate is satisfied:

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq C(1+e^t)^{-\frac{1}{2}(n-1)(\frac{1}{p}-\frac{1}{q})}\|\varphi_0\|_{W_p^N(\mathbb{R}^n)}, \tag{5}$$

where $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, and $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q}) \leq N < \frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})+1$ and $W_p^N(\mathbb{R}^n)$ is the Sobolev space. In particular, the loss of regularity, N, is positive, unless $p = q = 2$. This loss of regularity phenomenon exists for the wave equation in Minkowski spacetime as well.

According to Theorem 1 [5], for the solution $u = u(x,t)$ to the Cauchy problem (4) with $n \geq 2$, $\varphi_0(x) = 0$ and $\varphi_1(x) \in C_0^\infty(\mathbb{R}^n)$, for all large $t \geq T > 0$ and for any small $\varepsilon > 0$, the following estimate is satisfied:

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq C_\varepsilon(1+t)(1+e^t)^{r_0-n(\frac{1}{p}-\frac{1}{q})}\|\varphi_1\|_{W_p^N(\mathbb{R}^n)},$$

where $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, $r_0 = \max\{\varepsilon; \frac{(n+1)}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{q}\}$, $\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{q} \leq N < \frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{p}$. See also [11] , [6] for decay estimates with some loss of regularity.

For the Klein-Gordon equation in the Minkowski spacetime the following $L^p - L^q$ estimate is well-known. Let A denote $-\Delta + m^2$, $m \neq 0$. Then according to Theorem 2.2 [8] for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, $1 < p \leq 2 \leq q < \infty$, the following estimate holds:

$$\left\|A^{-\frac{1}{2}}\sin\left(A^{\frac{1}{2}}t\right)\varphi\right\|_{L_q(\mathbb{R}^n)} \leq K(t)\|\varphi\|_{W_p^{\frac{n-1+\theta}{2}-\frac{n+1+\theta}{q}}(\mathbb{R}^n)},$$

where

$$K(t) = c\begin{cases} t^{-(n-1-\theta)\left(\frac{1}{2}-\frac{1}{q}\right)} & 0 < t \leq 1\,, \\ t^{-(n-1+\theta)\left(\frac{1}{2}-\frac{1}{q}\right)} & t \geq 1\,. \end{cases}$$

In [2] a family of Strichartz estimates is demonstrated for a particular Klein-Gordon equation on a class of asymptotically de Sitter spaces with C^2 metrics.

We use representations of the solutions obtained in [14] to derive $L^p - L^q$-decay estimates for the Klein-Gordon equation in anti-de Sitter space-time. In particular, we obtain in Sections 4-5 for $n \geq 2$ and for the curved mass $M \geq 0$, the following estimate:

$$\begin{aligned} & \|(-\triangle)^{-s}u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \qquad (6)\\ & \leq Ce^{t\left(2s-n(\frac{1}{p}-\frac{1}{q})\right)}\int_0^t \|f(\cdot,b)\|_{L^p(\mathbb{R}^n)}(1+t-b)^{1-\operatorname{sgn} M}\,db \\ & +C_M(e^t-1)^{2s-n(\frac{1}{p}-\frac{1}{q})}\left\{\|\varphi_0\|_{L^p(\mathbb{R}^n)}+(1-e^{-t})(1+t)^{1-\operatorname{sgn} M}\|\varphi_1\|_{L^p(\mathbb{R}^n)}\right\} \end{aligned}$$

provided that $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)\left(\frac{1}{p} - \frac{1}{q}\right) \le 2s \le n\left(\frac{1}{p} - \frac{1}{q}\right) < 2s+1$. Moreover, according to Theorem 2.2 the estimate (6) with $\varphi_0 = 0$ and $\varphi_1 = 0$ is valid for $n = 1$ and $s = 0$ as well. The case of $n = 1$, $f(x,t) = 0$, and non-vanishing φ_0 and φ_1 is discussed in Section 3. It is essentially different from the decay estimate obtained in [12] for the wave equation in the de Sitter spacetime. This difference is caused by the striking difference between the global geometries of the forward and backward light cones of the equation (3).

The paper is organized as follows. In Section 2 we obtain $L^p - L^q$ and $L^q - L^q$ estimates for the solutions of the one-dimensional equation. In Section 3 we derive $L^p - L^q$ estimates for the solutions of the one-dimensional equation without source term and we prove some estimates for the kernels K_0 and K_1. In Section 4 we establish the $L^p - L^q$ decay estimates for the equation with source in higher dimensional space $n \ge 2$. Using the estimates for the kernels K_0 and K_1 obtained in Section 3 we derive $L^p - L^q$ decay estimates for the equation without source, $n \ge 2$. Applications of all these results to the nonlinear equations will be done in a forthcoming paper.

2. $L^p - L^q$ and $L^q - L^q$ estimates in 1D

We define the "forward light cone" $D_+(x_0,t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and the "backward light cone" $D_-(x_0,t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, as follows:

$$D_\pm(x_0,t_0) \quad := \quad \Big\{(x,t) \in \mathbb{R}^{n+1}\,;\, |x - x_0| \le \pm(e^t - e^{t_0})\Big\}. \tag{7}$$

For $t_0 \in \mathbb{R}$ in the domain $D_+(x_0,t_0) \cup D_-(x_0,t_0)$ let the function

$$\begin{aligned} E(x,t;x_0,t_0) \quad &= \quad (4e^{t_0+t})^{iM}\Big((e^t + e^{t_0})^2 - (x-x_0)^2\Big)^{-\frac{1}{2}-iM} \\ &\qquad F\Big(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(e^{t_0}-e^t)^2 - (x-x_0)^2}{(e^{t_0}+e^t)^2 - (x-x_0)^2}\Big), \end{aligned} \tag{8}$$

where $F\big(a,b;c;\zeta\big)$ is the hypergeometric function. Note that $E(x,t;x_0,t_0) = E(x-x_0,t;0,t_0)$. Here and in what follows we use the notation $x^2 = |x|^2$ for $x \in \mathbb{R}^n$. Let $E(x,t;0,t_0)$ be the function (8), and set

$$\mathcal{E}_\pm(x,t;0,t_0) := \begin{cases} E(x,t;0,t_0) & \text{in } D_\pm(0,t_0), \\ 0 & \text{elsewhere}. \end{cases}$$

Since the function $E = E(x,t;0,t_0)$ is smooth in $D_\pm(0,t_0)$ and is locally integrable, it follows that $\mathcal{E}_+(x,t;0,\, t_0)$ and $\mathcal{E}_-(x,t;0,\, t_0)$ are distributions whose supports are in $D_+(0,t_0)$ and $D_-(0,t_0)$, respectively.

THEOREM 2.1. [13] *Suppose that* $M \in \mathbb{C}$. *The distributions* $\mathcal{E}_+(x,t;0,t_0)$ *and* $\mathcal{E}_-(x,t;0,t_0)$ *are the fundamental solutions for the operator* $\mathcal{S} = \partial_t^2 - e^{2t}\partial_x^2 + M^2$ *relative to the point* $(0,t_0)$, *that is* $\mathcal{S}\mathcal{E}_\pm(x,t;0,t_0) = \delta(x,t-t_0)$.

Assume that $f \in C^\infty$ and that for every fixed t it has compact support, $\mathrm{supp} f(\cdot,t) \subset \mathbb{R}$, $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R})$. Then, according to Theorems 3, 4 [14], the solution $u = u(x,t)$ of the Cauchy problem

$$u_{tt} - e^{2t}u_{xx} + M^2 u = f\,, \qquad u(x,0) = \varphi_0(x)\,, \qquad u_t(x,0) = \varphi_1(x)\,, \tag{9}$$

is given by

$$\begin{aligned} u(x,t) \quad = \quad & \int_0^t db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy\, f(y,b)(4e^{b+t})^{iM}\Big((e^t+e^b)^2-(x-y)^2\Big)^{-\frac{1}{2}-iM} \\ & \qquad\qquad \times F\Big(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(e^b-e^t)^2-(x-y)^2}{(e^b+e^t)^2-(x-y)^2}\Big) \\ & + \frac{1}{2}e^{-\frac{t}{2}}\Big[\varphi_0(x+e^t-1)+\varphi_0(x-e^t+1)\Big] \\ & + \sum_{i=0}^{1}\int_0^{e^t-1}\big[\varphi_i(x-z)+\varphi_i(x+z)\big]K_i(z,t)\,dz\,, \end{aligned} \tag{10}$$

where the kernels $K_0(z,t)$ and $K_1(z,t)$ are defined by (See [14, Proposition 1, (26)] for details.)

$$\begin{aligned} K_0(z,t) := & -\Big[\frac{\partial}{\partial b}E(z,t;0,b)\Big]_{b=0} \qquad\qquad (11) \\ = \quad & -(4e^t)^{iM}\big((1+e^t)^2-z^2\big)^{-iM}\frac{1}{[(1-e^t)^2-z^2]\sqrt{(1+e^t)^2-z^2}} \\ \times & \Bigg[\big(e^t-1-iM(e^{2t}-1-z^2)\big)F\Big(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(1-e^t)^2-z^2}{(1+e^t)^2-z^2}\Big) \\ + & \big(1-e^{2t}+z^2\big)\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(1-e^t)^2-z^2}{(1+e^t)^2-z^2}\Big)\Bigg], \end{aligned}$$

with $0 \le z < e^t - 1$ and

$$\begin{aligned} K_1(z,t) \quad := \quad & E(z,t;0,0) = (4e^t)^{iM}\big((e^t+1)^2-z^2\big)^{-\frac{1}{2}-iM} \\ & \times F\left(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(e^t-1)^2-z^2}{(e^t+1)^2-z^2}\right)\,,\ 0 \le z \le e^t-1\,. \end{aligned} \tag{12}$$

The kernels $K_0(z,t)$ and $K_1(z,t)$ play leading roles in the derivation of $L^p - L^q$ estimates.

From now we restrict ourselves to nonnegative curved mass $M \geq 0$. First we consider the Cauchy problem (9) with the source term and with vanishing initial data.

THEOREM 2.2. *For every function $f \in C^2(\mathbb{R} \times [0,\infty))$ such that $f(\cdot,t) \in C_0^\infty(\mathbb{R})$, the solution $u = u(x,t)$ of the Cauchy problem (9) with $\varphi_0 = 0$, $\varphi_1 = 0$, satisfies the inequality*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \quad \leq \quad Ce^{-t(1-\frac{1}{\rho})} \int_0^t (1+t-b)^{1-\operatorname{sgn} M} \|f(\cdot,b)\|_{L^p(\mathbb{R})}\, db$$

for all $t > 0$, where $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\rho < 2$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$.

Proof. Using the fundamental solution from Theorem 2.1 one can write the convolution

$$u(x,t) = \int_0^t db \int_{-\infty}^{\infty} \mathcal{E}_+(x-y,t;0,b) f(y,b)\, dy\,.$$

Due to Young's inequality, we have

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq c\int_0^t db \left(\int_{-(\phi(t)-\phi(b))}^{\phi(t)-\phi(b)} |E(x,t;0,b)|^\rho dx \right)^{1/\rho} \|f(\cdot,b)\|_{L^p(\mathbb{R})},$$

where $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$, $\phi(t) = e^t - 1$. The integral in parentheses can be transformed as follows:

$$\int_{-(\phi(t)-\phi(b))}^{\phi(t)-\phi(b)} |E(x,t;0,b)|^\rho dx = 2e^{b-b\rho} \int_0^{e^{t-b}-1} \Big((e^{t-b}+1)^2 - y^2 \Big)^{-\frac{\rho}{2}}$$
$$\times \left| F\Big(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^{t-b}-1)^2 - y^2}{(e^{t-b}+1)^2 - y^2} \Big) \right|^\rho dy.$$

Denote $z := e^{t-b}$, where $t \geq b$ and $z \in [1,\infty)$, and consider the integral

$$\int_0^{z-1} \Big((z+1)^2 - y^2 \Big)^{-\frac{\rho}{2}} \left| F\Big(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} \Big) \right|^\rho dy.$$

First, we consider the case of $M > 0$. There is a formula (See formula 15.3.6 of Ch.15 [1] and [3].) that ties together points $z = 0$ and $z = 1$:

$$F\,(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F\,(a,b;a+b-c+1;1-z) \tag{13}$$
$$+(1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F\,(c-a,c-b;c-a-b+1;1-z)\,,$$

where $|\arg(1-z)| < \pi$. Each term of the last formula has a pole when $c = a+b\pm m$, $(m = 0, 1, 2, \ldots)$; this case is covered by formula 15.3.10 of Ch.15 [1]:

$$\begin{aligned} F\left(a, b; a+b; z\right) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} (1-z)^n \frac{(a)_n (b)_n}{(n!)^2} \\ &\quad \times \left[2\psi(n+1) - \psi(a+n) - \psi(b+n) - \ln(1-z)\right] , \end{aligned} \tag{14}$$

where $|\arg(1-z)| < \pi$, $|1-z| < 1$. If $\Re(c-a-b) > 0$, then $F(a, b; c; 1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$. For every given $\varepsilon \in (0, 1)$ the right hand side of (13) implies $\left|F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; z\right)\right| \leq C_{M,\varepsilon}$ for all $z \in [\varepsilon, 1)$ and, consequently, together with the formula (14), means

$$\left|F\Big(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; z\Big)\right| \leq C_M \left(1 - \ln(1-z)\right)^{1-\operatorname{sgn} M} \quad \text{for all } z \in [0, 1). \tag{15}$$

Thus,

$$\int_{-(\phi(t)-\phi(b))}^{\phi(t)-\phi(b)} |E(x, t; 0, b)|^{\rho} dx \leq C_M e^{b-b\rho} \int_0^{e^{t-b}-1} \left((e^{t-b}+1)^2 - y^2\right)^{-\frac{\rho}{2}} dy.$$

For all $z > 1$ the following equality holds:

$$\int_0^{z-1} ((z+1)^2 - r^2)^{-\frac{\rho}{2}} dr = (z-1)(z+1)^{-\rho} F\Big(\frac{1}{2}, \frac{\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\Big) \tag{16}$$

provided that $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. In particular, if $\rho < 2$, then

$$\int_0^{z-1} ((z+1)^2 - r^2)^{-\frac{\rho}{2}} dr \quad \leq \quad C_\rho (z-1)(z+1)^{-\rho} .$$

The last estimate completes the proof of the theorem in the case of $M > 0$. Next we consider the case of $M = 0$. Thus,

$$\begin{aligned} &\int_{-(\phi(t)-\phi(b))}^{\phi(t)-\phi(b)} |E(x, t; 0, b)|^{\rho} dx \\ &= 2e^{b-b\rho} \int_0^{e^{t-b}-1} \left((e^{t-b}+1)^2 - y^2\right)^{-\frac{\rho}{2}} \left|F\Big(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{t-b}-1)^2 - y^2}{(e^{t-b}+1)^2 - y^2}\Big)\right|^{\rho} dy. \end{aligned}$$

LEMMA 2.3. [*12*] *For all $z > 1$ the following estimate is fulfilled:*

$$\begin{aligned} &\int_0^{z-1} ((z+1)^2 - r^2)^{-\frac{\rho}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right)^{\rho} dr \\ &\qquad \leq C(1 + \ln z)^{\rho} (z-1)(z+1)^{-\rho} F\Big(\frac{1}{2}, \frac{\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\Big) \end{aligned}$$

provided that $1<p<\rho'$, $\frac{1}{q}=\frac{1}{p}-\frac{1}{\rho'}$, $\frac{1}{\rho}+\frac{1}{\rho'}=1$. *In particular, if* $\rho<2$, *then*

$$\int_0^{z-1}((z+1)^2-r^2)^{-\frac{\rho}{2}}F\left(\frac{1}{2},\frac{1}{2};1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\right)^{\rho}dr \\ \leq C(1+\ln z)^{\rho}(z-1)(z+1)^{-\rho}\,.$$

Thus for $\rho<2$ and $z=e^{t-b}$ we have

$$\begin{aligned}\|u(\cdot,t)\|_{L^q(\mathbb{R})} &\leq c\int_0^t e^{\frac{b}{\rho}-b}(1+t-b)(e^{t-b}-1)^{\frac{1}{\rho}}(e^{t-b}+1)^{-1}\|f(\cdot,b)\|_{L^p(\mathbb{R})}db\\ &\leq c\int_0^t e^{\frac{b}{\rho}-b}(1+t-b)e^{\frac{t}{\rho}-\frac{b}{\rho}}e^{-t+b}\|f(\cdot,b)\|_{L^p(\mathbb{R})}\,db\,.\end{aligned}$$

The last inequality implies the estimate of the statement of the theorem if $M=0$. This concludes the proof of Theorem 2.2. ☐

PROPOSITION 2.4. *The solution* $u=u(x,t)$ *of the Cauchy problem*

$$u_{tt}-e^{2t}u_{xx}+M^2u=0\,,\qquad u(x,0)=\varphi_0(x)\,,\qquad u_t(x,0)=\varphi_1(x)\,,$$

with φ_0, $\varphi_1\in C_0^\infty(\mathbb{R})$ *satisfies the following estimate:*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})}\leq C\Big(\|\varphi_0\|_{L^q(\mathbb{R})}+(1+t)^{1-\operatorname{sgn}M}(1-e^{-t})\|\varphi_1\|_{L^q(\mathbb{R})}\Big) \tag{17}$$

for all $t\in(0,\infty)$.

Proof. First we consider the problem with the second datum, that is, the case of $\varphi_0=0$. We apply the representation (10) for the solution $u=u(x,t)$ of the problem, and we obtain

$$u(x,t) \;=\; \int_0^{e^t-1}\Big[\varphi_1(x-z)+\varphi_1(x+z)\Big]K_1(z,t)dz\,,$$

where the kernel $K_1(z,t)$ is defined by (12). Hence, we arrive at the inequality

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})}\leq 2\|\varphi_1\|_{L^q(\mathbb{R})}\int_0^{e^t-1}|K_1(r,t)|dr=2\|\varphi_1\|_{L^q(\mathbb{R})} \\ \times\int_0^{e^t-1}\frac{1}{\sqrt{(e^t+1)^2-y^2}}\left|F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\right)\right|dy\,.$$

To estimate the last integral we introduce $z=e^t>1$ and denote the integral by I_1,

$$I_1(z):=\int_0^{z-1}\frac{1}{\sqrt{(z+1)^2-y^2}}\left|F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\right)\right|dy\,.$$

First we consider the case of $M > 0$. Then, according to (16) (with $\rho = 1$) we have for that integral the following estimate:

$$I_1(e^t) \le C_M(e^t - 1)(e^t + 1)^{-1}\,.$$

The last inequality implies the $L^q - L^q$ estimate (17) for the case of $M > 0$. Then we consider the case of $M = 0$. According to Lemma 2.3 (with $\rho = 1$) we have for $I_1(z)$ the following estimate:

$$I_1(e^t) \le C(1+t)(e^t - 1)(e^t + 1)^{-1}\,. \tag{18}$$

Finally, (18) implies the $L^q - L^q$ estimate (17) for the case of $M = 0$ and $\varphi_0 = 0$.

Next we consider the equation without source but with the first datum. We apply the representation (10) for the solution $u = u(x,t)$ of the Cauchy problem with $\varphi_1 = 0$, and we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{2}e^{-\frac{t}{2}}\Big[\varphi_0(x + e^t - 1) + \varphi_0(x - e^t + 1)\Big] \\ &\quad + \int_0^{e^t-1} \big[\varphi_0(x-z) + \varphi_0(x+z)\big] K_0(z,t)\,dz\,, \end{aligned}$$

where the kernel $K_0(r,t)$ is defined by (11). Then we easily obtain the following two estimates:

$$\|u(x,t) - \int_0^{e^t-1} \big[\varphi_0(x-r) + \varphi_0(x+r)\big] K_0(r,t)\,dr\|_{L^q(\mathbb{R})} \le e^{-\frac{t}{2}}\|\varphi_0\|_{L^q(\mathbb{R})}$$

and

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \le e^{-\frac{t}{2}}\|\varphi_0\|_{L^q(\mathbb{R})} + 2\|\varphi_0\|_{L^q(\mathbb{R})}\int_0^{e^t-1} |K_0(z,t)|\,dz\,.$$

Finally, the following lemma completes the proof of the proposition.

LEMMA 2.5. *The kernel $K_0(r,t)$ has an integrable singularity at $r = e^t - 1$, more precisely, one has*

$$\int_0^{e^t-1} |K_0(r,t)|\,dr \le C_M(1 - e^{-t}) \quad \textit{for all} \quad t \in [0,\infty)\,.$$

Proof. For the integral we obtain

$$\begin{aligned} \int_0^{e^t-1} |K_0(r,t)|\,dr &\le \int_0^{z-1} \frac{1}{[(1-z)^2 - r^2]\sqrt{(1+z)^2 - r^2}} \\ &\quad \times\Big|\big(z - 1 - iM(z^2 - 1 - r^2)\big)F\Big(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(1-z)^2 - r^2}{(1+z)^2 - r^2}\Big) \\ &\quad + \big(1 - z^2 + r^2\big)\Big(\frac{1}{2} - iM\Big)F\Big(-\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(1-z)^2 - r^2}{(1+z)^2 - r^2}\Big)\Big|\,dr \end{aligned}$$

for all $z := e^t > 1$. We fix $\varepsilon \in (0,1)$ and we divide the domain of integration into two zones,

$$Z_1(\varepsilon, z) \;:=\; \left\{(z,r) \,\Big|\, \frac{(z-1)^2-r^2}{(z+1)^2-r^2} \le \varepsilon,\ 0 \le r \le z-1\right\}, \tag{19}$$

$$Z_2(\varepsilon, z) \;:=\; \left\{(z,r) \,\Big|\, \varepsilon \le \frac{(z-1)^2-r^2}{(z+1)^2-r^2},\ 0 \le r \le z-1\right\}, \tag{20}$$

and we split the integral into two parts:

$$\int_0^{e^t-1} |K_0(r,t)|\, dr \;=\; \int_{(z,r)\in Z_1(\varepsilon,z)} |K_0(r,t)|\, dr + \int_{(z,r)\in Z_2(\varepsilon,z)} |K_0(r,t)|\, dr\,.$$

In the first zone we have

$$\begin{aligned} &F\Big(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big) \\ &\quad = 1 + \left(\frac{1}{2}+iM\right)^2 \frac{(z-1)^2-r^2}{(z+1)^2-r^2} + O\left(\left(\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\right)^2\right), \end{aligned} \tag{21}$$

$$\begin{aligned} &F\Big(-\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big) \\ &\quad = 1 - \left(\frac{1}{4}+M^2\right) \frac{(z-1)^2-r^2}{(z+1)^2-r^2} + O\left(\left(\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\right)^2\right). \end{aligned} \tag{22}$$

We use the last formulas to estimate the terms containing the hypergeometric functions:

$$\begin{aligned} &\Bigg|\big(z-1-iM(z^2-1-r^2)\big)F\Big(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(1-z)^2-r^2}{(1+z)^2-r^2}\Big) \\ &+\big(1-z^2+r^2\big)\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(1-z)^2-r^2}{(1+z)^2-r^2}\Big)\Bigg| \\ &\le \frac{1}{2}\big[(z-1)^2-r^2\big] + \Bigg|\frac{1}{8}(z^2+2z-3-r^2) - iM\frac{1}{2}((z-1)^2-r^2) \\ &+\frac{1}{2}M^2(3z^2-3r^2-2z-1)\Bigg|\frac{(z-1)^2-r^2}{(z+1)^2-r^2} \\ &+\Big(\big|z-1-iM(z^2-1-r^2)\big| + \big|1-z^2+r^2\big|\Big)O\bigg(\Big(\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)^2\bigg). \end{aligned} \tag{23}$$

Hence, we have to consider the following three integrals, which can be easily evaluated and estimated,

$$
\begin{aligned}
A_1 &:= \int_{(z,r)\in Z_1(\varepsilon,z)} \frac{1}{\sqrt{(z+1)^2-r^2}}dr \le \mathrm{Arcsin}\left(\frac{z-1}{z+1}\right) \le \frac{\pi}{2}\frac{z-1}{z+1}\,, \\
A_2 &:= \int_{(z,r)\in Z_1(\varepsilon,z)} \left\{ \frac{(z^2+2z-3-r^2)+M((z-1)^2-r^2)}{((z+1)^2-r^2)\sqrt{(z+1)^2-r^2}} \right. \\
&\qquad \left. + \frac{M^2(3z^2-3r^2-2z-1)}{((z+1)^2-r^2)\sqrt{(z+1)^2-r^2}} \right\} dr \\
&\le \frac{3\pi}{2}(1+M+M^2)\frac{z-1}{z+1} \qquad \text{for all} \qquad z\in[1,\infty),
\end{aligned}
$$

and

$$
\begin{aligned}
A_3 &:= \int_{Z_1(\varepsilon,z)} \frac{(z^2+z-2-r^2)+M^2(z^2-1-r^2)}{\sqrt{(z+1)^2-r^2}} \frac{(z-1)^2-r^2}{((z+1)^2-r^2)^2}dr \\
&\le \int_{Z_1(\varepsilon,z)} \frac{1+M^2}{\sqrt{(z+1)^2-r^2}}dr \le \frac{\pi}{2}(1+M^2)\,\frac{z-1}{z+1} \quad \text{for all} \quad z\in[1,\infty).
\end{aligned}
$$

Finally, for the integral over the first zone we have obtained

$$
\int_{Z_1(\varepsilon,z)} |K_0(r,t)|\; dr \quad\le\quad C_M\frac{z-1}{z+1} \qquad \text{for all} \qquad z\in[1,\infty).
$$

In the second zone we have

$$
\varepsilon \le \frac{(z-1)^2-r^2}{(z+1)^2-r^2} \le 1\,, \quad \text{hence} \quad \frac{1}{(z-1)^2-r^2} \le \frac{1}{\varepsilon[(z+1)^2-r^2]}\;. \tag{24}
$$

First consider the case of $M>0$. According to (13) the hypergeometric functions obey the estimates

$$
\left|F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;x\Big)\right| \le C\;\;,\;\; \left|F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;x\Big)\right| \le C_M \tag{25}
$$

for all $x\in[\varepsilon,1)$. This allows us to estimate the integral over the second zone:

$$
\begin{aligned}
\int_{Z_2(\varepsilon,z)} |K_0(r,t)|\,dr &\le C_M \int_{Z_2(\varepsilon,z)} \frac{|z-1|+(1+M)|z^2-1-r^2|}{[(1-z)^2-r^2]\sqrt{(1+z)^2-r^2}}\,dr \\
&\le C_{M,\varepsilon}\int_0^{z-1} \frac{1}{\sqrt{(1+z)^2-r^2}}\,dr \le C_{M,\varepsilon}\frac{z-1}{z+1}
\end{aligned}
$$

for all $z \in [1,\infty)$. In the case of $M=0$ we apply formula 15.3.10 of [3, Ch.15]. According to that formula the hypergeometric functions obey the estimates

$$\Big|F\Big(-\frac{1}{2},\frac{1}{2};1;x\Big)\Big| \le C \quad\text{and}\quad \Big|F\Big(\frac{1}{2},\frac{1}{2};1;x\Big)\Big| \le C\big(1-\ln(1-x)\big) \tag{26}$$

for all $x \in [\varepsilon,1)$. This allows to prove the estimate for the integral over the second zone

$$\int_{Z_2(\varepsilon,z)} |K_0(r,t)|dr \le C\frac{z-1}{z+1} \quad \text{for all } z\in[1,\infty). \tag{27}$$

Indeed, for the argument of the hypergeometric functions we have

$$\varepsilon \le \frac{(z-1)^2-r^2}{(z+1)^2-r^2} = 1-\frac{4z}{(z+1)^2-r^2} < 1, \qquad \frac{4z}{(z+1)^2-r^2} < 1-\varepsilon \tag{28}$$

for all $(z,r)\in Z_2(\varepsilon,z)$. Hence,

$$\left|F\Big(\frac{1}{2},\frac{1}{2};1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)\right| \le C\left(1-\ln\frac{4z}{(z+1)^2-r^2}\right) \le C\,(1+\ln z) \tag{29}$$

for all $(z,r)\in Z_2(\varepsilon,z)$. To prove (27) we have to estimate the following two integrals:

$$\begin{aligned} A_4 &:= \int_{Z_2(\varepsilon,z)} \frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}}\,\big|(1-z^2+r^2)\big|\,dr\,,\\ A_5 &:= \int_{Z_2(\varepsilon,z)} \frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}}\,|(z-1)\,(1+\ln z)|\,dr\,. \end{aligned}$$

We apply (24) to A_4 and obtain

$$A_4 \le C_\varepsilon \int_0^{z-1} \frac{1}{\sqrt{(z+1)^2-r^2}}\,dr \le C_\varepsilon\frac{z-1}{z+1}\,,$$

while

$$\begin{aligned} A_5 &\le C_\varepsilon(z-1)\,(1+\ln z)\int_0^{z-1}\frac{1}{((z+1)^2-r^2)^{3/2}}dr\\ &\le C_\varepsilon(z-1)^2\,(1+\ln z)\,\frac{1}{\sqrt{z}(z+1)^2} \le C_\varepsilon\,(1+\ln z)\,\frac{1}{\sqrt{z}}\frac{z-1}{z+1}\,. \end{aligned}$$

Thus, (27) is proven. This concludes the proof of lemma.

3. $L^p - L^q$ estimates in 1D without source term. Some estimates of the kernels K_0 and K_1

THEOREM 3.1. *Let $u = u(x,t)$ be a solution of the Cauchy problem*

$$u_{tt} - e^{2t}u_{xx} + M^2 u = 0\,, \qquad u(x,0) = \varphi_0(x)\,, \qquad u_t(x,0) = \varphi_1(x)\,,$$

with φ_0, $\varphi_1 \in C_0^\infty(\mathbb{R})$. If $\rho \in (1,2)$, then

$$\begin{aligned}\|u(\cdot,t)\|_{L^q(\mathbb{R})} \;\leq\; & e^{-\frac{t}{2}}\|\varphi_0\|_{L^q(\mathbb{R})} + C_{M,\rho}(e^t-1)^{\frac{1}{\rho}}e^{-t}\|\varphi_0\|_{L^p(\mathbb{R})} \\ & + C_{M,\rho}(1+t)^{1-\operatorname{sgn} M}(e^t-1)^{\frac{1}{\rho}}e^{-t}\|\varphi_1\|_{L^p(\mathbb{R})}\end{aligned}$$

for all $t \in (0,\infty)$. Here $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. If $\rho = 1$, then

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq C\Big(\|\varphi_0\|_{L^q(\mathbb{R})} + (1+t)^{1-\operatorname{sgn} M}(e^t-1)e^{-t}\|\varphi_1\|_{L^q(\mathbb{R})}\Big)$$

for all $t \in (0,\infty)$.

Proof. For $\rho = 1$ we just apply Proposition 2.4. To prove this theorem for $\rho > 1$ we need some auxiliary estimates for the kernels K_0 and K_1. We start with the case of $\varphi_0 = 0$, where the kernel K_1 appears. The representation (10) and Young's inequality lead to

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \;\leq\; 2\left(\int_0^{e^t-1} |K_1(r,t)|^\rho dr\right)^{1/\rho} \|\varphi_1\|_{L^p(\mathbb{R})},$$

where $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. Now we have to estimate the last integral.

PROPOSITION 3.2. *We have*

$$\left(\int_0^{e^t-1} |K_1(r,t)|^\rho dr\right)^{\frac{1}{\rho}} \leq C(1+t)^{1-\operatorname{sgn} M}(e^t-1)^{\frac{1}{\rho}}(e^t+1)^{-1} \text{ for all } t \in (0,\infty)\,.$$

Proof. For $M = 0$ one can write

$$\left(\int_0^{e^t-1} |K_1(r,t)|^\rho dr\right)^{\frac{1}{\rho}} \leq \left(\int_0^{e^t-1} \left|\frac{F\Big(\frac{1}{2},\frac{1}{2};1;\frac{(e^t-1)^2-r^2}{(e^t+1)^2-r^2}\Big)}{\sqrt{(1+e^t)^2-r^2}}\right|^\rho dr\right)^{\frac{1}{\rho}}.$$

Denote $z := e^t > 1$ and consider the integral

$$\int_0^{z-1} \left|\frac{1}{\sqrt{(1+z)^2-r^2}}F\Big(\frac{1}{2},\frac{1}{2};1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)\right|^\rho dr$$

on the right hand side. According to Lemma 2.3 we obtain that for all $z>1$ the following estimate is fulfilled:

$$\int_0^{z-1} \left| \frac{1}{\sqrt{(1+z)^2-r^2}} F\Big(\frac{1}{2},\frac{1}{2};1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)\right|^\rho dr$$
$$\leq C(1+\ln z)^\rho (z-1)(z+1)^{-\rho} F\Big(\frac{1}{2},\frac{\rho}{2};\frac{3}{2};\frac{(z-1)^2}{(z+1)^2}\Big)$$

provided that $1<p<\rho'$, $\frac{1}{q}=\frac{1}{p}-\frac{1}{\rho'}$, $\frac{1}{\rho}+\frac{1}{\rho'}=1$. In particular, if $\rho<2$, then

$$\left(\int_0^{z-1} ((z+1)^2-r^2)^{-\frac{\rho}{2}} F\Big(\frac{1}{2},\frac{1}{2};1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)^\rho dr\right)^{1/\rho}$$
$$\leq C(1+\ln z)(z-1)^{1/\rho}(z+1)^{-1}\,.$$

This concludes the proof of the proposition in the case of $M=0$. For $M>0$ we apply (15):

$$\left(\int_0^{e^t-1} |K_1(r,t)|^\rho dr\right)^{\frac{1}{\rho}} \leq C_M \left(\int_0^{e^t-1} \left|\frac{1}{\sqrt{(1+e^t)^2-r^2}}\right|^\rho dr\right)^{\frac{1}{\rho}}$$
$$\leq C_M (e^t-1)^{1/\rho}(e^t+1)^{-1}.$$

This completes the proof of proposition. ☐

Thus, the theorem in the case of $\varphi_0=0$ is proven.

Now we turn to the case of $\varphi_1=0$, where the kernel K_0 appears. From the representation (10) of the solution we have

$$\|u(x,t)\|_{L^q(\mathbb{R})}$$
$$\leq e^{-\frac{t}{2}}\|\varphi_0\|_{L^q(\mathbb{R})} + \left\|\int_0^{e^t-1} [\varphi_0(x-r)+\varphi_0(x+r)]\,K_0(r,t)\,dr\right\|_{L^q(\mathbb{R})}.$$

Similarly to the case of the second datum we apply the Young's inequality and arrive at

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq e^{-\frac{t}{2}}\|\varphi_0\|_{L^q(\mathbb{R})} + 2\|\varphi_0\|_{L^p(\mathbb{R})}\left(\int_0^{e^t-1}|K_0(r,t)|^\rho dr\right)^{1/\rho}.$$

The next proposition gives an estimate for the integral in the last inequality.

Proposition 3.3. *Let $1<p<\rho'$, $\frac{1}{q}=\frac{1}{p}-\frac{1}{\rho'}$, $\frac{1}{\rho}+\frac{1}{\rho'}=1$, and $\rho\in[1,2)$. We have*

$$\left(\int_0^{e^t-1}|K_0(r,t)|^\rho dr\right)^{1/\rho} \leq C_\rho (e^t-1)^{\frac{1}{\rho}}(e^t+1)^{-1} \quad \textit{for all} \quad t\in(0,\infty)\,.$$

Proof. The case of $\rho = 1$ is just Lemma 2.5 therefore we bring up details, which in the case of $\rho > 1$ are different from those ones used in the proof of that lemma. We turn to the integral ($z := e^t > 1$)

$$\left(\int_0^{e^t-1} |K_0(r,t)|^\rho \, dr \right)^{\frac{1}{\rho}} \leq \left(\int_0^{z-1} \frac{1}{[(1-z)^2 - r^2]^\rho (\sqrt{(1+z)^2 - r^2})^\rho} \right.$$
$$\times \left| \big(z - 1 - iM(z^2 - 1 - r^2)\big) F\Big(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(1-z)^2 - r^2}{(1+z)^2 - r^2}\Big) \right.$$
$$\left. \left. + \big(1 - z^2 + r^2\big) \Big(\frac{1}{2} - iM\Big) F\Big(-\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(1-z)^2 - r^2}{(1+z)^2 - r^2}\Big) \right|^\rho dr \right)^{\frac{1}{\rho}} .$$

The formulas (21) and (22) describe the behavior of the hypergeometric functions in the neighbourhood of zero. Consider therefore two zones, $Z_1(\varepsilon, z)$ and $Z_2(\varepsilon, z)$, defined in (19) and (20), respectively. We split the integral into two parts:

$$\int_0^{e^t-1} |K_0(r,t)|^\rho \, dr = \int_{(z,r)\in Z_1(\varepsilon,z)} |K_0(r,t)|^\rho \, dr + \int_{(z,r)\in Z_2(\varepsilon,z)} |K_0(r,t)|^\rho \, dr \, .$$

In the proof of Lemma 2.5 the relation (23) was checked in the first zone. If $1 \leq z \leq N$ with some constant N, then the argument of the hypergeometric functions is bounded,

$$\frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \leq \frac{(z-1)^2}{(z+1)^2} \leq \frac{(N-1)^2}{(N+1)^2} < 1 \quad \text{for all} \quad r \in (0, z-1), \tag{30}$$

and we obtain with $z = e^t$,

$$\int_0^{e^t-1} |K_0(r,t)|^\rho \, dr \leq C \int_0^{z-1} \left\{ \frac{1}{(\sqrt{(z+1)^2 - r^2})} \left[1 + \left| \frac{1}{8}(z^2 + 2z - 3 - r^2) \right. \right. \right.$$
$$\left. - iM\frac{1}{2}((z-1)^2 - r^2) + \frac{1}{2}M^2(3z^2 - 3r^2 - 2z - 1) \right| \frac{1}{(z+1)^2 - r^2}$$
$$\left. \left. + \Big(\big|z - 1 - iM(z^2 - 1 - r^2)\big| + \big|1 - z^2 + r^2\big| \Big) \frac{(z-1)^2 - r^2}{((z+1)^2 - r^2)^2} \right] \right\}^\rho dr \, ,$$

then

$$\left(\int_0^{z-1} |K_0(r,t)|^\rho \, dr\right)^{\frac{1}{\rho}} \le C_{M,N}\left(\int_0^{z-1}\left|\frac{1}{\sqrt{(z+1)^2-r^2}}\right|^\rho dr\right)^{\frac{1}{\rho}}$$
$$\le C_{M,N}\left((z-1)(z+1)^{-\rho}F\Big(\frac{1}{2},\frac{\rho}{2};\frac{3}{2};\frac{(z-1)^2}{(z+1)^2}\Big)\right)^{\frac{1}{\rho}}$$
$$\le C_{M,N}\,(z-1)^{1/\rho}(z+1)^{-1}\,.$$

Thus, we can restrict ourselves to the case of large $z \ge N$ in both zones. Consider therefore for $\rho \in (1,2)$ the following integrals over the first zone:

$$A_6 := \int\limits_{Z_1(\varepsilon,z)} \left|\frac{1}{\sqrt{(z+1)^2-r^2}}\right|^\rho dr \le C_{M,N,\varepsilon}(z-1)(z+1)^{-\rho}\,,$$
$$A_7 := \int\limits_{Z_1(\varepsilon,z)} \left|\frac{z^2+2z-3-r^2+M((z-1)^2-r^2)+M^2(3z^2-3r^2-2z-1)}{((z+1)^2-r^2)\sqrt{(z+1)^2-r^2}}\right|^\rho dr$$
$$\le C_{M,N,\varepsilon}(z-1)(z+1)^{-\rho} \quad \text{for all} \quad z\in[1,\infty),$$

and

$$A_8 := \int\limits_{Z_1(\varepsilon,z)} \left|\frac{(z^2+z-2-r^2)+M^2(z^2-1-r^2)}{\sqrt{(z+1)^2-r^2}}\,\frac{(z-1)^2-r^2}{((z+1)^2-r^2)^2}\right|^\rho dr$$
$$\le \int\limits_{Z_1(\varepsilon,z)} \left|\frac{1+M^2}{\sqrt{(z+1)^2-r^2}}\right|^\rho dr \le C_{M,N,\varepsilon}\frac{z-1}{(z+1)^\rho} \quad \text{for all} \quad z\in[1,\infty).$$

In the second zone for the argument of the hypergeometric functions we have the inequalities (28) and $\dfrac{1}{(z-1)^2-r^2} \le \dfrac{1}{\varepsilon[(z+1)^2-r^2]}$, $0\le r\le z-1$. Hence,

$$\left|F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)\right| \le C\left(1-\ln\frac{4z}{(z+1)^2-r^2}\right)^{1-\operatorname{sgn} M}$$
$$\le C\,(1+\ln z)^{1-\operatorname{sgn} M}$$

for all $(z,r)\in Z_2(\varepsilon,z)$. First consider the case of $M=0$. We have to estimate the following two integrals:

$$
\begin{aligned}
A_9 &:= \int\limits_{Z_2(\varepsilon,z)} \left[\frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}}(z^2-1-r^2)\right]^\rho dr\,,\\
A_{10} &:= \int\limits_{Z_2(\varepsilon,z)} \left[\frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}}(z-1)\,(1+\ln z)\right]^\rho dr\,.
\end{aligned}
$$

We apply (24) and obtain

$$
\begin{aligned}
A_9 &\leq C\int_0^{z-1} \left[\frac{1}{\sqrt{(z+1)^2-r^2}}\right]^\rho dr\\
&\leq C(z-1)(z+1)^{-\rho} F\Big(\frac{1}{2},\frac{\rho}{2};\frac{3}{2};\frac{(z-1)^2}{(z+1)^2}\Big) \leq C(z-1)(z+1)^{-\rho}\,,
\end{aligned}
$$

$$
\begin{aligned}
A_{10} &\leq C_\varepsilon (z-1)^\rho\,(1+\ln z)^\rho \int_{Z_2(\varepsilon,z)} ((z+1)^2-r^2)^{-3\rho/2} dr\\
&\leq C_\varepsilon (z-1)^\rho\,(1+\ln z)^\rho\,(z-1)(z+1)^{-3\rho} F\Big(\frac{1}{2},\frac{3\rho}{2};\frac{3}{2};\frac{(z-1)^2}{(z+1)^2}\Big)\\
&\leq C_\varepsilon (z-1)(z+1)^{-\rho}\,.
\end{aligned}
$$

This concludes the proof of the proposition in the case of $M=0$. Now consider the case of $M>0$. Inequalities (25) allow us to estimate the integral over the second zone:

$$
\begin{aligned}
\int\limits_{Z_2(\varepsilon,z)} |K_0(r,t)|^\rho\, dr &\leq C_M \int\limits_{(z,r)\in Z_2(\varepsilon,z)} \left(\frac{|z-1|+(1+M)|z^2-1-r^2|}{[(z-1)^2-r^2]\sqrt{(z+1)^2-r^2}}\right)^\rho dr\\
&\leq C_{M,\varepsilon}\int_0^{z-1} ((z+1)^2-r^2)^{-\rho/2}\, dr\\
&\leq C_{M,\varepsilon}(z-1)(z+1)^{-\rho} \qquad \text{for all} \qquad z\in[1,\infty).
\end{aligned}
$$

The proposition is proven. ☐

4. $L^p - L^q$ estimates in the higher-dimensional case with source

According to Theorems 5, 6 [14] for higher-dimensional equation with $n \geq 2$ the solution $u = u(x,t)$ to the Cauchy problem

$$
u_{tt} - e^{2t}\Delta u + M^2 u = f, \quad u(x,0) = \varphi_0, \quad u_t(x,0) = \varphi_1, \tag{31}
$$

with $f \in C^\infty(\mathbb{R}^{n+1})$, $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$, is given by

$$\begin{aligned} u(x,t) &= 2\int_0^t db \int_0^{e^t-e^b} dr\, v(x,r;b)(4e^{b+t})^{iM} \left((e^t+e^b)^2-r^2\right)^{-\frac{1}{2}-iM} \\ &\qquad \times F\left(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(e^b-e^t)^2-r^2}{(e^b+e^t)^2-r^2}\right) + e^{-\frac{t}{2}} v_{\varphi_0}(x,\phi(t)) \\ &\qquad + \sum_{i=0}^{1} 2\int_0^1 v_{\varphi_i}(x,\phi(t)s) K_i(\phi(t)s,t)\phi(t)\, ds, \quad x \in \mathbb{R}^n,\ t>0\,, \quad (32) \end{aligned}$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation

$$v_{tt} - \triangle v = 0\,, \quad v(x,0;b) = f(x,b)\,, \quad v_t(x,0;b) = 0\,.$$

The function $v_\varphi(x,t)$ is the solution of the Cauchy problem

$$v_{tt} - \triangle v = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = 0\,.$$

For the wave equation, Duhamel's principle allows us to reduce the case with a source term to the case of the Cauchy problem without source term and consequently to derive the $L^p - L^q$-decay estimates for their solutions. For (3) the Duhamel's principle is not applicable straight-forwardly and we have to appeal to the representation formula (32). In this section we consider the Cauchy problem (31) for the equation with the source term and with zero initial data.

THEOREM 4.1. *Let $u = u(x,t)$ be the solution of the Cauchy problem (31) with $\varphi_0 = \varphi_1 = 0$. Then for $n \geq 2$ one has the following estimate:*

$$\|(-\triangle)^{-s} u(\cdot,t)\|_{L^q(\mathbb{R}^n)}$$

$$\leq C\int_0^t db \|f(\cdot,b)\|_{L^p(\mathbb{R}^n)} \int_0^{e^t-e^b} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{\left(F\left(\frac{1}{2},\frac{1}{2};1;\frac{(e^t-e^b)^2-r^2}{(e^t+e^b)^2-r^2}\right)\right)^{1-\operatorname{sgn} M}}{\sqrt{(e^t+e^b)^2-r^2}}\, dr$$

provided that $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q} = 1$, $\frac{1}{2}(n+1)\left(\frac{1}{p}-\frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p}-\frac{1}{q}\right) < 2s+1$.

Proof. From the representation of the solution (32) and due to the results of [4, 8] for the wave equation, we have

$$\begin{aligned} \|(-\triangle)^{-s} u(\cdot,t)\|_{L^q(\mathbb{R}^n)} &\leq C\int_0^t db\, \|f(\cdot,b)\|_{L^p(\mathbb{R}^n)} \int_0^{e^t-e^b} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \\ &\quad \times \frac{1}{\sqrt{(e^t+e^b)^2-r^2}} \left(F\left(\frac{1}{2},\frac{1}{2};1;\frac{(e^t-e^b)^2-r^2}{(e^t+e^b)^2-r^2}\right)\right)^{1-\operatorname{sgn} M} dr. \end{aligned}$$

The theorem is proven. □

We are going to write the estimate of the last theorem in a more comfortable form. To this aim we estimate the last integral of the right hand side. If we replace $e^t/e^b > 1$ with $z > 1$, then the integral will be simplified.

LEMMA 4.2. *Assume that* $0 \geq 2s - n(\frac{1}{p} - \frac{1}{q}) > -1$. *Then*

$$\int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} \left(F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right)\right)^{1-\operatorname{sgn} M} dr$$
$$\leq C z^{-1}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}(1+\ln z)^{1-\operatorname{sgn} M} \quad \textit{for all } z > 1\,.$$

Proof. If $1 < z \leq N$ with some constant N, then the argument of the hypergeometric functions is less than one, see (30), and the inequality above follows.

Hence, we can restrict ourselves to the case of large z, that is $z \geq N$. In particular, we choose $N > 6$ and we split the integral into two parts:

$$\int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} \left(F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right)\right)^{1-\operatorname{sgn} M} dr$$
$$= \int_0^{\sqrt{(z+1)^2-8z}} \star \; dr + \int_{\sqrt{(z+1)^2-8z}}^{z-1} \star \; dr \; = J_1(z) + J_2(z).$$

For the second part $J_2(z)$ we have $r \geq \sqrt{(z+1)^2 - 8z}$, then

$$\frac{4z}{(z+1)^2 - r^2} \geq \frac{1}{2} \Longrightarrow 0 < 1 - \frac{4z}{(z+1)^2 - r^2} \leq \frac{1}{2} \tag{33}$$

for such r and z implies

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right)\right| \leq C\,. \tag{34}$$

Then (33) and (34) imply

$$J_2(z) \leq C \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}}\, dr \leq C(1+z)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}$$

for all $z \geq N > 6$. For the first integral, $r \leq \sqrt{(z+1)^2 - 8z}$ and $z \geq N > 6$ imply $8z \leq (z+1)^2 - r^2$. It follows

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right)\right| \leq C \left| \ln\left(\frac{4z}{(z+1)^2 - r^2}\right)\right| \leq C(1 + \ln z)\,.$$

Then we obtain

$$\begin{aligned} J_1(z) &\le C(1+\ln z)^{1-\operatorname{sgn} M} \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2-r^2}}\, dr \\ &\le C(1+\ln z)^{1-\operatorname{sgn} M} (1+z)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} . \end{aligned}$$

Lemma is proven. □

COROLLARY 4.3. *Let $u=u(x,t)$ be the solution of the Cauchy problem (31). Then, for $n \ge 2$, one has the following decay estimate:*

$$\|(-\triangle)^{-s}u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \le Ce^{t\left(2s-n(\frac{1}{p}-\frac{1}{q})\right)}\int_0^t \|f(\cdot,b)\|_{L^p(\mathbb{R}^n)}(1+t-b)^{1-\operatorname{sgn} M}\, db \tag{35}$$

provided that $s\ge 0$, $1<p\le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{2}(n+1)\left(\frac{1}{p}-\frac{1}{q}\right)\le 2s\le n\left(\frac{1}{p}-\frac{1}{q}\right)$, $-1+n\left(\frac{1}{p}-\frac{1}{q}\right)<2s$.

Proof. Indeed, from Theorem 4.1 we derive

$$\begin{aligned} \|(-\triangle)^{-s}u(\cdot,t)\|_{L^q(\mathbb{R}^n)} &\le C\int_0^t \|f(\cdot,b)\|_{L^p(\mathbb{R}^n)} e^{b(2s-n(\frac{1}{p}-\frac{1}{q}))}db \int_0^{e^{t-b}-1} dl \\ &\quad \times \frac{l^{2s-n(\frac{1}{p}-\frac{1}{q})}}{\sqrt{(e^{t-b}+1)^2-r^2}} \left(F\left(\frac{1}{2},\frac{1}{2};1;\frac{(e^{t-b}-1)^2-l^2}{(e^{t-b}+1)^2-l^2}\right)\right)^{1-\operatorname{sgn} M} . \end{aligned}$$

Next we apply Lemma 4.2 with $z=e^{t-b}$,

$$\begin{aligned} &\|(-\triangle)^{-s}u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \\ \le\ & Ce^{t\left(2s-n(\frac{1}{p}-\frac{1}{q})\right)} \int_0^t \|f(\cdot,b)\|_{L^p(\mathbb{R}^n)}(1-e^{b-t})(1+t-b)^{1-\operatorname{sgn} M}db, \end{aligned}$$

and we arrive at (35). This completes the proof of the corollary. □

5. $L^p - L^q$ estimates in the higher-dimensional case without source

The L^p-L^q-decay estimates for the energy of the solution of the Cauchy problem for the wave equation without source can be proved by the representation formula, L^1-L^∞ and L^2-L^2 estimates, and interpolation argument. (See, e.g., [9, Theorem 2.1].) One can prove L^p-L^q-decay estimates for the solutions applying the method used in [4, 8]. This approach is based on the microlocal consideration and dyadic decomposition of the phase space. As it was mentioned in the introduction the L^p-L^q estimate obtained using this approach in [5] contains the additional derivative loss $r_0 > 0$. To avoid such additional derivative loss and obtain more sharp estimates we appeal to the representation formula (32) and then apply the results of [4, 8].

THEOREM 5.1. *The solution $u = u(x,t)$ of the Cauchy problem (31) with $f = 0$ satisfies the following $L^p - L^q$ estimate:*

$$\begin{aligned} &\|(-\triangle)^{-s} u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \\ &\leq C_M (e^t - 1)^{2s - n(\frac{1}{p} - \frac{1}{q})} \Big\{ \|\varphi_0\|_{L^p(\mathbb{R}^n)} + (1 - e^{-t})(1+t)^{1 - \operatorname{sgn} M} \|\varphi_1\|_{L^p(\mathbb{R}^n)} \Big\} \end{aligned}$$

for all $t \in (0,\infty)$, provided that $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)\left(\frac{1}{p} - \frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p} - \frac{1}{q}\right) < 2s + 1$.

Proof. We start with the case of $\varphi_0 = 0$. Due to the representation (32) for the solution $u = u(x,t)$ of the Cauchy problem (31) with $f = 0$, $\varphi_0 = 0$ and to the results of [4, 8] we have:

$$\|(-\triangle)^{-s} u(x,t)\|_{L^q(\mathbb{R}^n)} \leq C \|\varphi_1\|_{L^p(\mathbb{R}^n)} \int_0^{e^t - 1} r^{2s - n(\frac{1}{p} - \frac{1}{q})} \, |K_1(r,t)| \, dr \,.$$

To continue we need the next lemma.

LEMMA 5.2. *The following inequality holds:*

$$\int_0^{z-1} r^{2s - n(\frac{1}{p} - \frac{1}{q})} \, |K_1(r,t)| \, dr \leq C(1 + \ln z)^{1 - \operatorname{sgn} M} z^{-1} (z-1)^{1 + 2s - n(\frac{1}{p} - \frac{1}{q})}$$

for all $z > 1$.

Proof. In fact, we have to estimate the integral:

$$\begin{aligned} &\int_0^{z-1} r^{2s - n(\frac{1}{p} - \frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} F\Big(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\Big) \, dr \\ &\leq \int_0^{z-1} r^{2s - n(\frac{1}{p} - \frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} \left(F\Big(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\Big) \right)^{1 - \operatorname{sgn} M} dr, \end{aligned}$$

where $z = e^t$. The estimate for that integral is given by Lemma 4.2. □

Thus, for the case of $\varphi_0 = 0$ the theorem is proven. Next we turn to the case of $\varphi_1 = 0$. From representation (32) with $f = 0$, $\varphi_1 = 0$, and the results of [8] we have:

$$\begin{aligned} \|(-\triangle)^{-s} u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq\; & C e^{-\frac{t}{2}} (e^t - 1)^{2s - n(\frac{1}{p} - \frac{1}{q})} \|\varphi_0\|_{L^p(\mathbb{R}^n)} \\ & + C \|\varphi_0\|_{L^p(\mathbb{R}^n)} \int_0^{e^t - 1} r^{2s - n(\frac{1}{p} - \frac{1}{q})} |K_0(r,t)| \, dr. \end{aligned}$$

The following proposition gives the estimate for the last integral and completes the proof of the theorem.

PROPOSITION 5.3. *If* $2s-n(\frac{1}{p}-\frac{1}{q})>-1$, *then*

$$\int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})}|K_0(r,t)|\,dr \le Cz^{-1}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})} \quad \textit{for all} \quad z>1.$$

Proof. We follow the arguments used in the proof of Proposition 3.3. If $1\le z\le N$ with some constant N, then due to (30) the argument of the hypergeometric functions is less than one, and we have

$$\begin{aligned}\int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\,|K_0(r,t)|\,dr &\le C\int_0^{z-1}\frac{1}{\sqrt{(z+1)^2-r^2}}r^{2s-n(\frac{1}{p}-\frac{1}{q})}dr\\ &\le C_N(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}, \qquad 1<z\le N\,.\end{aligned}$$

Thus, we can restrict ourselves to the case of large $z\ge N$ in both zones $Z_1(\varepsilon,z)$ and $Z_2(\varepsilon,z)$. In the first zone we have (21) and (22). Consider therefore the following inequalities:

$$\begin{aligned}A_{11} &:= \int_{Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{\sqrt{(z+1)^2-r^2}}\,dr \le Cz^{2s-n(\frac{1}{p}-\frac{1}{q})}\,,\\ A_{12} &:= \int_{Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{\sqrt{(z+1)^2-r^2}}\frac{\left|3-z^2-2z+r^2\right|}{(z+1)^2-r^2}dr\\ &\le C\int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{\sqrt{(z+1)^2-r^2}}dr \;\le\; Cz^{2s-n(\frac{1}{p}-\frac{1}{q})}\,,\\ A_{13} &:= \int_{Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{\sqrt{(z+1)^2-r^2}}\frac{\left|3z^2-3r^2-2z-1\right|}{(z+1)^2-r^2}dr\\ &\le C\int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{\sqrt{(z+1)^2-r^2}}dr \;\le\; Cz^{2s-n(\frac{1}{p}-\frac{1}{q})}\,,\\ A_{14} &:= \int_{Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{\sqrt{(z+1)^2-r^2}}\\ &\qquad\times\frac{(z^2+z-2-r^2)+M^2(z^2-1-r^2)}{((z-1)^2-r^2)}\left(\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\right)^2 dr\\ &\le \int_{Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{\sqrt{(z+1)^2-r^2}}\frac{1}{4z}dr \;\le\; Cz^{2s-n(\frac{1}{p}-\frac{1}{q})-1}\end{aligned}$$

for all $z\in[1,\infty)$. Finally,

$$\int_{Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})}|K_0(r,t)|dr \le C_M z^{2s-n(\frac{1}{p}-\frac{1}{q})} \qquad \text{for all} \qquad z\in[1,\infty).$$

In the second zone we use (24), (26), and (29). Thus, we have to estimate the next two integrals:

$$A_{15} := \int_{Z_2(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}} \left|1-z^2+r^2\right| dr\,,$$
$$A_{16} := \int_{Z_2(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{|(z-1)\,(1+\ln z)|}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}} dr\,.$$

We apply (24) to A_{14} and we obtain

$$\begin{aligned} A_{15} &\leq C_\varepsilon \int_{Z_2(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{\left|z^2-1-r^2\right|}{((z+1)^2-r^2)\sqrt{(z+1)^2-r^2}} dr \\ &\leq C_\varepsilon \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2-r^2}} dr \;\leq\; C_\varepsilon z^{2s-n(\frac{1}{p}-\frac{1}{q})} \end{aligned}$$

for all $z \in [1,\infty)$, while

$$A_{16} \;\leq\; C_\varepsilon (z-1)\,(1+\ln z) \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{((z+1)^2-r^2)^{3/2}} dr\,.$$

For $0 \geq a > -1$ and $z \geq N$, the following integral can be easily estimated:

$$\int_0^{z-1} r^a \frac{1}{((z+1)^2-r^2)^{3/2}} dr \;\leq\; C z^{-3+a+1} + C z^{a-3/2} \;\leq\; C z^{a-3/2}\,.$$

Then $A_{15} \leq C_\varepsilon (z-1)\,(1+\ln z)\, z^{a-3/2} \leq C_\varepsilon z^a$. The proposition is proven. □

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, Washington, 1964.

[2] D. R. Baskin, *A Strichartz estimate for de Sitter space*, arXiv:0909.2696.

[3] H. Bateman and A. Erdelyi, *Higher Transcendental Functions*, McGraw-Hill, New York, 1953.

[4] P. Brenner, *On $L^p - L^q$ estimates for the wave-equation*, Math. Z. **145** (1975), 251–254.

[5] A. Galstian, *$L_p - L_q$ decay estimates for the wave equations with exponentially growing speed of propagation*, Appl. Anal. **82** (2003), 197–214.

[6] A. Galstian and M. Reissig, *$L_p - L_q$ decay estimates for a Klein-Gordon type model equation*, Proceedings of the Second ISAAC Congress (Fukuoka, Japan,1999) (Heinrich G. W. Begehr, ed.), vol. 2, Kluwer Academic Publishers, Dordrecht, 2000, pp. 1355–1369.

[7] C. Møller, *The theory of relativity*, (The international series of monographs on physics.) Oxford: At the Clarendon Press, 1952.

[8] H. Pecher, *L^p-Abschätzungen und klassische Lösungen für nichtlineare Wellengleichungen.I*, Math. Z. **150** (1975), 159–183.

[9] R. Racke, *Lectures on Nonlinear Evolution Equations*, Aspects of Mathematics, Vieweg, Braunschweig/ Wiesbaden, 1992.

[10] M. Reissig and K. Yagdjian, *The wave equation on asymptotically Anti-de Sitter spaces*, arXiv:0911.5440.

[11] M. Reissig and K. Yagdjian, *$L_p - L_q$ estimates for the solutions of strictly hyperbolic equations of second order with increasing in time coefficients*, Math. Nachr. **214** (2000), 71–104.

[12] K. Yagdjian and A. Galstian, *Fundamental Solutions for Wave Equation in de Sitter Model of Universe*, University of Potsdam, August, Preprint 2007/06.

[13] K. Yagdjian and A. Galstian, *Fundamental Solutions for the Klein-Gordon Equation in de Sitter Spacetime*, Communications in Mathematical Physics **285** (2009), 293–344.

[14] K. Yagdjian and A. Galstian, *The Klein-Gordon Equation in anti-de Sitter Spacetime*, Rend. Sem. Math. Univ. Pol. Torino **67** (2009), 271–292.

Author's address:

Anahit Galstian
Department of Mathematics
University of Texas-Pan American,
1201 W. University Drive, Edinburg, TX 78541-2999, USA
E-mail: agalstyan@utpa.edu

Received February 3, 2010
Revised April 6, 2010