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Invariants of Moduli Spaces and Modular Forms

Lothar Göttsche

Abstract. *Generating functions for invariants of moduli spaces in algebraic geometry of are often related to modular forms. In this paper we give an overview of many instances of this phenomenon and in some cases relate it to predictions from theoretical physics. In this paper we only consider moduli spaces of objects on surfaces. The examples include Euler numbers of moduli spaces of sheaves on surfaces, Donaldson invariants, and enumerative invariants of curves on surfaces.*

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1. Introduction

Moduli spaces are an important subject of study in algebraic geometry with connections to a number of other fields like physics (in particular string theory), differential geometry and number theory. A large number of mathematicians study invariants of moduli spaces such as Betti numbers and Euler numbers of moduli spaces, Donaldson invariants, Donaldson-Thomas invariants and Gromov-Witten invariants. It has been noted more and more that the best way to study invariants of moduli spaces is in terms of generating functions. In this paper I want to review work by myself and many others which relate generating functions for invariants of moduli spaces to modular forms. In this whole paper we will work over the complex numbers, and a variety is a quasiprojective variety over $\mathbb{C}$.

I start by briefly explaining some of the concepts that I used above.

1.1. Generating Functions

Assume $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ are some interesting numbers. The generating function for these numbers is the formal power series

$$f(t)=\sum_{n\geq 0}a_nt^n.$$

When we study this generating function, our aim is to find a nice closed expression for f. This will not only explicitly give us all the numbers a_n, but it will also give us an intimate relation which ties together all the different numbers a_n.

EXAMPLE 1.1 (Partitions). *Let $p(n)$ be the number of partitions of n, i.e. the number of ways to write n as a sum of smaller numbers (up to reordering). Thus $p(0) = 1$, $p(1) = 1$, $p(2) = 2$ and $p(3) = 3$ ((3), (2,1), (1,1,1)). The generating function for the numbers of partitions is*

$$\sum_{n\geq 0} p(n)t^n = \prod_{k\geq 1} \frac{1}{1-t^k}. \tag{1}$$

(This is very elementary: expand each factor as a geometric series and multiply everything out. The coefficient of t^n will be a sum of $p(n)$ times the summand 1).

1.2. Moduli Spaces

A moduli space in algebraic geometry is an algebraic variety M which parametrizes (usually up to isomorphism) in a natural way some objects we are interested in.

As an example we consider the moduli space of elliptic curves. By definition an elliptic curve over $\mathbb{C}$ is a pair $(E, 0)$ of E a nonsingular projective curve of genus 1, and a point $0 \in E$. Then $(E, 0)$ will automatically be a commutative group with neutral element 0. Topologically E is a torus. Every elliptic curve is isomorphic to the quotient $E_\tau = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ of the complex numbers divided by the lattice $\mathbb{Z}\tau + \mathbb{Z}$, where τ is an element of the complex upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ of complex numbers with positive imaginary part. The group $Sl(2, \mathbb{Z})$ acts on $\mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = \frac{a\tau + b}{c\tau + d},$$

and it is not difficult to see that $E_\tau \simeq E_{\tau'}$, if and only if there exists an $A \in Sl(2, \mathbb{Z})$ with $\tau' = A\tau$. This implies that the moduli spaces $M_{1,1}$ of elliptic curves is the quotient $M_{1,1} = \mathbb{H}/Sl(2, \mathbb{Z})$. (Strictly speaking, also for the relation with modular forms below, it is more natural to take $M_{1,1}$ as the stack quotient $[\mathbb{H}/Sl(2, \mathbb{Z})]$ and not as the quotient variety, but we will not go into this here). There is a natural compactification $\overline{M}_{1,1} = M_{1,1} \cup \infty$. The extra point ∞ corresponds to a nodal curve.

If $\Gamma \subset Sl(2, \mathbb{Z})$ is a subgroup of finite index, then $\mathbb{H}/\Gamma$ will be a moduli space of elliptic curves with some extra structure and it will allow a compactification $\overline{\mathbb{H}/\Gamma}$ which parametrizes also some degenerations of these elliptic curves with extra structure.

1.3. Modular Forms

I briefly introduce modular forms. More details and many applications can be found in [41, Chapter 2],[47],[4]. Modular forms are related to moduli spaces (or rather moduli stacks) of elliptic curves. They can be viewed as functions (or more precisely sections of line bundles) on these moduli spaces. For $\tau \in \mathbb{H}$ we write $q := e^{2\pi i\tau}$. For a in $\mathbb{Q}$ we also write $q^a := e^{2\pi i a\tau}$.

Definition 1.2. *Let $k \in \mathbb{Z}_{\geq 0}$. A modular form of weight k on $Sl(2,\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, such that*

1.
$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbb{Z}), \tag{2}$$

2. *f is "holomorphic at ∞":*
$$f(\tau) = \sum_{n\geq 0} a_n q^n \quad q = e^{2\pi i\tau},\ a_n \in \mathbb{C}\,.$$

f called is a cusp form, if also $a_0 = 0$.

Remark 1.3. 1. *$Sl(2,\mathbb{Z})/\pm 1$ is generated by the elements*
$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
Condition 1. for T implies that $f(\tau) = f(\tau+1)$. Thus f has a Fourier development (or q-development) $\sum_{n\in\mathbb{Z}} a_n q^n$. Condition 2. then means that no negative power of q occurs in this Fourier development.

2. *One can view q as local parameter of $\overline{M}_{1,1}$ at ∞, thus condition 2. of the definition really means that f is holomorphic at ∞.*

Example 1.4. 1. *Eisenstein series:*
$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{n\geq 1}\left(\sum_{d|n} d^{k-1}\right) q^n, \quad k > 2 \text{ even.}$$
Here B_k is the Bernoulli number. G_k is a modular form of weight k on $Sl(2,\mathbb{Z})$. This is proven by writing $G_k(\tau)$ in a different way: up to multiplying by a nonzero constant, $G_k(\tau)$ is equal to
$$\sum_{(n,m)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m\tau+n)^k}.$$

One sees that dividing by $(c\tau+d)^k$ and replacing τ by $\frac{a\tau+b}{c\tau+d}$ corresponds to a reordering of the summands, and thus gives the same result because the sum is absolutely convergent (see Zagiers lectures in [4] for more details). This argument fails for $k = 2$, because the sum is no longer absolutely convergent, which leads to an extra term in the transformation behaviour. In fact G_2 is only a quasimodular form (see below).

2. *Discriminant:*
$$\Delta(\tau) = q \prod_{n\geq 1}(1-q^n)^{24}$$
$\Delta(\tau)$ is a cusp form of weight 12 on $Sl(2,\mathbb{Z})$. One can prove this by noticing that $24G_2$ is the logarithmic derivative of Δ, and using the transformation behavior of G_2.
One also often considers the Dirichlet eta function
$$\eta(\tau) := \sqrt[24]{\Delta(\tau)} = q^{1/24}\prod_{n\geq 1}(1-q^n).$$
Note that $\frac{1}{\eta}$ is by (1) the generating function for partitions (up to the trivial factor $q^{1/24}$).

The sum of two modular forms of weight k is obviously a modular form of weight k, and the product fg of two modular forms of weight k and l is obviously a modular form of weight $k+l$. Thus the modular forms on $Sl(2,\mathbb{Z})$ form a graded ring $M_* = \sum_k M_k$. One shows that $M_* = \mathbb{C}[G_4, G_6]$, in particular the modular forms of a given weight form a finite dimensional vector space.

One can also consider meromorphic modular forms, which are meromorphic functions on $\mathbb{H}$ satisfying the transformation properties of modular forms and having only poles of finite order at ∞ (i.e. a Fourier development $\sum_{n\geq -m} a_n q^n$). These are obtained as quotients of modular forms. Particularly important are weakly holomorphic modular forms, which are required to be holomorphic on $\mathbb{H}$, but are allowed to have poles of finite order at ∞ (an example is $\frac{1}{\Delta}$).

In a similar way one can also define modular forms f on subgroups Γ of finite index of $Sl(2,\mathbb{Z})$, possibly with a character. In this case the transformation property (2) is required only for elements $A \in \Gamma$, and f is required to be holomorphic at all the cusps (i.e. all elements of $(\mathbb{Q}\cup\infty)/\Gamma$). One can find more details for instance in [47]. Examples of subgroups of finite index are
$$\Gamma(N) = \big\{A \in Sl(2,\mathbb{Z}) \bigm| A \simeq id \bmod N\big\},$$
$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in Sl(2,\mathbb{Z}) \middle| c \simeq 0 \bmod N\right\}.$$
A very important class of examples of modular forms are the theta functions. Let Λ be a positive definite lattice, i.e. as a group $\Lambda \simeq \mathbb{Z}^r$, but it is equipped

with a positive definite quadratic form Q with values in $\mathbb{Z}$. The theta function of Λ is then

$$\theta_\Lambda = \sum_{v\in\Lambda} q^{Q(v)}.$$

It is a modular form of weight $r/2$ on some subgroup of $Sl(2,\mathbb{Z})$.

r does not always have to be even, so we sometimes get modular forms of half integral weight, which are explained in more detail in [47]. In particular for $\Lambda = \mathbb{Z}$ with the quadratic form $Q(n) = n^2$ we get $\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$, which is a modular form of weight $1/2$ on $\Gamma_0(4)$. This is the simplest example of a modular form of half integral weight, i.e. we have the transformation formulas

$$\theta_3\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon_{c,d}(c\tau+d)^{\frac{1}{2}}\theta_3(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for $\epsilon_{c,d}$ a 4^{th} root of unity depending on (c,d). A modular form f of weight $r+1/2$ on $\Gamma_0(4)$ is then required to satisfy that f/θ_3^{2r+1} is invariant under the action of $\Gamma_0(4)$.

There are some generalizations of modular forms that one can consider: The ring $QM_* = \mathbb{C}[G_2, G_4, G_6]$ is the ring of *quasimodular forms* on $Sl(2,\mathbb{Z})$ (the formula for G_k above also makes sense for $k = 2$, only it is not a modular form). Unlike the ring of modular forms, QM_* is closed under differentiation $D = q\frac{d}{dq}$.

A further generalization are Mock modular forms, which were first introduced (as Mock theta functions) by Ramanujan in his last letter to Hardy. Recently a theory of Mock modular forms was developed by Zwegers [49]. Mock modular forms can be viewed as holomorphic parts of real analytic modular forms, i.e. real analytic functions on $\mathbb{H}$, which satisfy the transformation properties of modular forms. The interested reader can consult [48] for an overview and background.

Modular forms are important for two reasons:

1. They come up in many different parts of mathematics and physics. Their Fourier developments are the generating functions for interesting numbers from many different areas.

2. As we have seen there are very few modular forms.

These two facts together mean that there will be many interesting relations between generating functions for objects from different fields of mathematics.

In this paper we want to consider generating functions for invariants of moduli spaces. That is we will want to look at moduli spaces M_n depending on $n \in \mathbb{Z}_{\geq 0}$, and we want to study generating functions for their invariants. For instance if the invariant is the topological Euler number $e(M_n)$, then we want to consider the generating function $\sum_{n\geq 0} e(M_n)q^n$ and try to express it in terms of modular forms. At a first glance this looks like a completely impossible task: moduli spaces are usually very complicated objects, and the invariants of even one moduli space M_n are usually very difficult to determine. So how can one even dream of computing the generating function, i.e. the invariants of all M_n in one stroke?

The surprising thing is that often it is easier to determine the generating functions than it would be to determine the invariants of the moduli spaces one by one. The point is that there often is a geometric relation between the moduli spaces M_n for different n. This, if one is lucky, will lead to relations between the invariants for different n, which often translate into differential equations for the generating function, which hopefully determine it.

In this article S will be a smooth projective algebraic surface. For a line bundle L on S, we will usually just write L for $c_1(L) \in H^2(S,\mathbb{Z})$. For classes $\alpha, \beta \in H^2(S,\mathbb{Q})$ we write $\langle \alpha\beta \rangle$ (or just $\alpha\beta$ if there is no risk of confusion) for their intersection number $\int_S \alpha \cup \beta$.

2. Hilbert Schemes of Points

Now we want to consider examples of generating functions of invariants of moduli spaces which are given by modular forms. We start by looking at moduli spaces of sets of points on an algebraic surface. First we will look at a simple such moduli space, the symmetric power of the surface. One gets very nice generating functions, but they are too simple to lead to modular forms. Then we study Hilbert schemes of points, which can be viewed as a refinement. Here indeed we will find modular forms, and a very rich structure reveals itself.

2.1. Symmetric Powers

Let S be a smooth projective surface. The symmetric group $\mathfrak{S}_n$ acts on S^n by permuting the factors. The symmetric power is the quotient $S^{(n)} = S^n/\mathfrak{S}_n$. It is known that $S^{(n)}$ is a projective variety of dimension $2n$. However $S^{(n)}$ is singular. The symmetric power $S^{(n)}$ can be viewed as a moduli space of n points on S counted with multiplicities: points of $S^{(n)}$ are in bijective correspondence with sets $\big\{(p_1,n_1),\dots,(p_r,n_r)\big\}$, where $1 \le r \le n$, the p_i are distinct points on S and the $n_i \in \mathbb{Z}_{>0}$ are their multiplicities, satisfying $\sum_i n_i = n$. The singular locus of $S^{(n)}$ consists of all $\big\{(p_1,n_1),\dots,(p_r,n_r)\big\}$ for which at least one multiplicity n_i is bigger than one. In other words the smooth locus of $S^{(n)}$ consists of n distinct points of S counted with multiplicity 1. We now recall the classical MacDonald formula for the Betti numbers of the $S^{(n)}$.

We introduce the following notation: For Y a quasiprojective variety denote $\dim Y$ its complex dimension , we denote by $H^i(Y,\mathbb{Q})$ its cohomology, and by $b_i(Y) = \dim(H^i(Y,\mathbb{Q}))$ its i-th Betti number. The Poincaré polynomial of Y is $p(Y,z) := \sum_{i=0}^{2\dim Y} b_i(Y)z^i$ and the Euler number is $e(Y) = \sum_{i=0}^{2\dim Y}(-1)^i b_i(Y) = p(Y,-1)$.

THEOREM 2.1 (MacDonald formula).

$$\sum_{n\geq 0} p(S^{(n)},z)t^n = \frac{(1+zt)^{b_1(S)}(1+z^3t)^{b_3(S)}}{(1-t)^{b_0(S)}(1-z^2t)^{b_2(S)}(1-z^4t)^{b_4(S)}}.$$

COROLLARY 2.2.

$$\sum_{n\geq 0} e(S^{(n)})t^n = \frac{1}{(1-t)^{e(S)}}.$$

These are very beautiful formulas, determining all the Betti numbers (resp. all the Euler numbers) of the symmetric powers in terms of just the Betti (resp. Euler) numbers of S. It is the model for the kind of formulas that we will be looking for, except that it is a bit too simple to involve modular forms.

2.2. Hilbert Scheme of Points

The Hilbert scheme $S^{[n]}$ of points on S is very closely related to the symmetric power $S^{(n)}$. Like the symmetric power it parametrizes sets of n points on S possibly with multiplicities. For a point of multiplicity $n_i > 1$, the symmetric power only remembers the multiplicity, but the Hilbert scheme records finer information: a nonreduced scheme structure of length n_i at the point. The points of $S^{[n]}$ are in bijective correspondence to the zero-dimensional subschemes of length n on S. These can be thought of as the sets $\{(p_1,\mathcal{O}_1),\dots,(p_r,\mathcal{O}_r)\}$, where the p_i are distinct points in S and $\mathcal{O}_i$ is a quotient ring of vector space dimension n_i of the ring $\mathcal{O}_{S,p_i}$ of germs of holomorphic functions near p_i. For instance when $n_i = 2$, then giving $\mathcal{O}_i$ is equivalent to giving a complex tangent direction at p_i. It follows that $S^{[2]}$ is obtained from $S^{(2)}$ by blowing up the “diagonal” $S^{(2)}_{(2)} \simeq S$ of all $\{(p,2) \mid p \in S\}$. In general there is a natural morphism

$$\omega_n : S^{[n]} \to S^{(n)}; \{(p_1,\mathcal{O}_1),\dots,(p_r,\mathcal{O}_r)\} \mapsto \{(p_1,n_1),\dots,(p_r,n_r)\}.$$

It turns out that ω_n is a birational morphism, and that $S^{[n]}$ is nonsingular. Thus it is a natural desingularization of $S^{(n)}$. To study $S^{[n]}$ and its invariants, it is natural to study the fibres of this map. The symmetric power $S^{(n)}$ has a natural stratification into nonsingular locally closed subsets

$$S^{(n)} = \sum_{\nu\in p(n)} S^{(n)}_\nu$$

parametrized by the partitions of n. Here, if $\nu = (n_1,\dots,n_r)$ is a partition of n, we put

$$S^{(n)}_\nu := \Big\{\{(p_1,n_1),\dots,(p_r,n_r)\} \;\Big|\; p_i \text{ distinct}\Big\}.$$

Let $S_\nu^{[n]} = \omega_n^{-1}(S_\nu^{(n)})$; these form a stratification of $S^{[n]}$ into locally closed subsets. Let R_n be the set of ideals of colength n in the power series ring $\mathbb{C}[[x,y]]$. Then one proves that $S_\nu^{[n]}$ is a locally trivial fibre bundle (in the complex topology) over $S_\nu^{[n]}$ with fibre $\prod_{i=1}^r R_{n_i}$. One can use this fact to determine the invariants of $S^{[n]}$: In [13] it is used to count the points of the $S^{[n]}$ over finite fields. Then the Weil conjectures (proved by Deligne) are applied to compute their Betti numbers. Later in [21] perverse sheaves and the decomposition theorem are applied to ω_n and this stratification to give a shorter proof, which also detemines the Hodge numbers.

Theorem 2.3.

$$\sum_{n\ge 0} p(S^{[n]},z)t^n = \prod_{k\ge 1} \frac{(1+z^{2k-1}t^k)^{b_1(S)}(1+z^{2k+1}t^k)^{b_3(S)}}{(1-z^{2k-2}t^k)^{b_0(S)}(1-z^{2k}t^k)^{b_2(S)}(1-z^{2k+2}t^k)^{b_4(S)}}.$$

Corollary 2.4.

$$\sum_{n\ge 0} e(S^{[n]})q^n = \prod_{k\ge 1}\frac{1}{(1-q^k)^{e(S)}} = \left(\frac{q^{1/24}}{\eta(\tau)}\right)^{e(S)}.$$

Again we have very beautiful formulas expressing the Betti numbers and the Euler numbers of the $S^{[n]}$ in terms of those of the original surface S. Note that in both cases the first factor (for $k=1$) gives the corresponding generating functions for the symmetric powers $S^{(n)}$.

These results led to quite a lot of activity and there have been a number of generalizations and refinements.

1. The ideal sheaf of a zero-dimensional scheme $Z \in S^{[n]}$ is a coherent sheaf of rank 1 with Chern classes $c_1 = 0$ and $c_2 = n$ (see later for more details). Thus $S^{[n]}$ is also a moduli space of rank 1 coherent sheaves on S. In [43] Vafa and Witten conjecture, based on physics arguments, that the generating functions of the Euler numbers of moduli spaces of sheaves on surfaces should be given by modular forms. The above example of the Hilbert scheme of points is one of the main motivating examples.

2. The formula suggests that that one should look at all cohomology groups $H^i(S^{[n]},\mathbb{Q})$ for all n and i at the same time. There should be a structure which ties them all together. Thus we put

$$\mathbf{H} := \bigoplus_n H^*(S^{[n]},\mathbb{Q}).$$

Also in [43] Vafa and Witten suggested that $\mathbf{H}$ should be an irreducible representation of the Heisenberg algebra modelled on $H^*(S,\mathbb{Q})$. This result was then proven by Nakajima [35] and Grojnowski [23]: For any $\alpha \in H^*(S,\mathbb{Q})$, there are "creation" operators $p_k(\alpha) : H^*(S^{[n]},\mathbb{Q}) \to H^*(S^{[n+k]},\mathbb{Q})$ and "annihilation" operators $p_{-k}(\alpha) : H^*(S^{[n]},\mathbb{Q}) \to H^*(S^{[n-k]},\mathbb{Q})$ for $k \in \mathbb{Z}_{>0}$ satisfying the commutation relations of the Heisenberg algebra, such that every element in $\mathbf{H}$ can be obtained from the element $1 \in H^0(S^{[0]},\mathbb{Q})$ (note that $S^{[0]}$ is just a point) by repeated application of creation operators. In other words there is a natural way how all the cohomology of all the Hilbert schemes of points $S^{[n]}$ is created out of the cohomology of S. For a nice readable account of this see [36]

Later starting with work of Lehn [25], very rich additional algebraic structures (Virasoro algebra, vertex algebras) were discovered on $\mathbf{H}$ and related to the cohomology ring structure and natural cohomology classes of the $S^{[n]}$ (see e.g. [26],[31],[6]).

3. The formula of Corollary 2.4 has a generalization to dimension 3. For X a smooth 3-fold it is proven in [7] that

$$\sum_{n\geq 0} e(X^{[n]})q^n = \prod_{k\geq 1}\left(\frac{1}{(1-q^k)^k}\right)^{e(X)}.$$

Recently this formula has been related to Donaldson-Thomas invariants [28],[3],[27], and there is a conjectural analogue of Theorem 2.3 for a motivic version of the Donaldson-Thomas invariants corresponding to $X^{[n]}$ in [2].

3. Moduli Spaces of Sheaves

In this section we consider moduli spaces of semistable torsion-free sheaves on algebraic surfaces. Torsion-free sheaves are generalizations of vector bundles, essentially they can be viewed as vector bundles which are allowed to have some singularities: the dimensions of the fibres do not all have to be equal. The reason for considering also torsion-free sheaves and not just vector bundles is that degenerations of vector bundles do not always have to be vector bundles, thus moduli spaces of vector bundles normally would not be compact. There are too many torsion-free sheaves for a moduli space to exist. In order to obtain moduli spaces of torsion-free sheaves one has to put the restriction that the sheaves $\mathcal{E}$ are semistable. This essentially means that the subsheaves of $\mathcal{E}$ should not be too large. This is measured in terms of an ample line bundle H on S.

3.1. Stable Sheaves

Let S be a projective algebraic surface and let H be an ample line bundle on S. For a sheaf $\mathcal{E}$ on S, let $\mathcal{E}(n) := \mathcal{E} \otimes H^{\otimes n}$. Denote $\chi(S,\mathcal{E}) = \sum_{i=0}^{2}(-1)^i \dim H^i(S,\mathcal{E})$ the holomorphic Euler characteristic. Let $\mathrm{rk}(\mathcal{E})$ be the rank of $\mathcal{E}$.

Definition 3.1. *A torsion-free sheaf $\mathcal{E}$ on S is called H-semistable, if for all nonzero subsheaves $\mathcal{F} \subset \mathcal{E}$, we have*

$$\frac{\chi(S,\mathcal{F}(n))}{rk(\mathcal{F})} \leq \frac{\chi(S,\mathcal{E}(n))}{rk(\mathcal{E})}, \quad \text{for all } n \gg 0.$$

It is called H-stable if the inequality is strict.

We denote $M_S^H(r,c_1,c_2)$ the moduli space of H-semistable sheaves $\mathcal{E}$ on S of rank r and with Chern classes c_1 and c_2. We will often restrict our attention to the case of rank 2. In this case we write $M_S^H(c_1,d)$ for $M_S^H(2,c_1,c_2)$, with $c_2 - c_1^2/4 = d$.

We denote by $N_S^H(r,c_1,c_2)$, $N_S^H(c_1,d)$ the open subsets of vector bundles.

3.2. S-Duality Conjectures

In [43] Vafa and Witten consider the *partition function*

$$Z_{c_1}^{S,H}(\tau) := \sum_d e(M_S^H(c_1,d))q^d,$$

and they also consider

$$Y_{c_1}^{S,H}(\tau) := \sum_d e(N_S^H(c_1,d))q^d.$$

In other words these are just the generating functions for the Euler numbers of the moduli spaces. Their *S-duality* conjecture predicts that these should be (at least almost) meromorphic modular forms. Roughly they say the following: $Z_{c_1}^{S,H}$ is the partition function of some physical theory. The fact that we can write it in terms of q means that it is invariant under $\tau \mapsto \tau+1$. On the other hand the theory should be invariant (or rather transform nicely) if one replaces strong coupling by weak coupling, this corresponds to $\tau \mapsto -\frac{1}{\tau}$. As the operation of $Sl(2,\mathbb{Z})$ on $\mathbb{H}$ is generated by these two, it follows that $Z_{c_1}^{S,H}$, $Y_{c_1}^{S,H}$ are modular forms. A similar statement should hold for arbitrary rank; the Hilbert schemes $S^{[n]}$ are an example of rank 1, because the ideal sheaf of $W \in S^{[n]}$ has rank 1 and Chern classes $c_1 = 0$, $c_2 = n$.

We will briefly explain some instances where this conjecture applies.

3.3. Compatibility Results

We start with some compatibility results:

1. if $Z_{c_1}^{S,H}$ and $Y_{c_1}^{S,H}$ are meromorphic modular forms, obviously also their quotient must be,
2. if $Z_{c_1}^{S,H}$ is a meromorphic modular form for all S, then the blowup in a point change the generating function by a modular form.

This indeed turns out to be the case. For 2. there is a small complication: Let $\widehat{S}$ be the blowup of S in a point, and let E be the exceptional divisor. Let H be ample on S, and denote by the same letter also its pullback to $\widehat{S}$. On would like to compare $Z_{c_1}^{\widehat{S},H}$ and $Z_{c_1}^{S,H}$, but H is not ample on $\widehat{S}$. For $n \in \mathbb{Z}_{>0}$ sufficiently large $H_n := nH - E$ is ample on $\widehat{S}$. Assume that c_1H is odd. Fix $d > 0$. Then $M_{\widehat{S}}^{H_n}(c_1, d)$ and $N_{\widehat{S}}^{H_n}(c_1, d)$ are independent of n as long as n is large enough. By abuse of notation we write $M_{\widehat{S}}^{H}(c_1, d) := M_{\widehat{S}}^{H_n}(c_1, d)$, $N_{\widehat{S}}^{H}(c_1, d) := N_{\widehat{S}}^{H_n}(c_1, d)$, and put

$$Z_{c_1}^{\widehat{S},H}(\tau) := \sum_d e(M_{\widehat{S}}^{H}(c_1, d))q^d, \quad Y_{c_1}^{\widehat{S},H}(\tau) := \sum_d e(N_{\widehat{S}}^{H}(c_1, d))q^d.$$

Theorem 3.2 ([45],[17],[30],[29]). *Assume that Hc_1 is odd.*

1.

$$Z_{c_1}^{S,H} = \left(\frac{q^{1/24}}{\eta(\tau)}\right)^{2e(S)} Y_{c_1}^{S,H}.$$

2. (blowup formula) Then

$$Z_{c_1}^{\widehat{S},H} = \theta_3(\tau)\left(\frac{q^{1/24}}{\eta(\tau)}\right)^2 Z_{c_1}^{S,H}, \qquad Y_{c_1}^{\widehat{S},H} = \theta_3(\tau)Y_{c_1}^{S,H}$$

$$Z_{c_1+E}^{\widehat{S},H} = \theta_2(\tau)\left(\frac{q^{1/24}}{\eta(\tau)}\right)^2 Z_{c_1}^{S,H}, \qquad Y_{c_1+E}^{\widehat{S},H} = \theta_2(\tau)Y_{c_1}^{S,H}.$$

Here

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}, \quad \theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2}.$$

We have denoted the pullbacks of H and c_1 to $\widehat{S}$ by the same letter.

One can also verify the S-duality conjecture for special surfaces.

3.4. K3 Surfaces

A particularly nice case is that of K3 surfaces. For S a K3 surface it is easy to see that the dimension of the moduli spaces $M_S^H(r,c_1,c_2)$ is always even (in case $r=2$, the dimension is $e=4c_2-c_1^2+6$, and c_1^2 is always even).

THEOREM 3.3. *Let S be a K3 surface. Assume all H-semistable rank r sheaves are H-stable (if $c_1 \in H^2(S,\mathbb{Z})$ is a primitive class, this is true for general H). Then $M_S^H(r,c_1,c_2)$ has the same Betti numbers as $S^{[e/2]}$, where e is the dimension of $M_S^H(r,c_1,c_2)$.*

As the Euler number of a K3 surface is 24, we get from Corollary 2.4 that the generating function for the Euler numbers of the $S^{[n]}$ is $\frac{q}{\Delta(\tau)}$. It follows that the generating function $Z_{c_1}^{S,H}$ can be expressed in terms of $\frac{q}{\Delta(\tau)}$.

The theorem was first proven under some extra assumptions in the case $r=2$ in [18]. Here an explicit birational correspondence between the Hilbert scheme $S^{[e/2]}$ and the moduli space was found and analized to obtain the result. The result was subsequently generalized in a number of papers mostly by Yoshioka. Finally in [46] is proven that there is a sequence of birational correspondences and deformations which relate the Hilbert scheme $S^{[e/2]}$ to $M_S^H(r,c_1,c_2)$ and preserve the Betti numbers.

3.5. The Projective Plane

Next we look at the projective plane $\mathbb{P}^2$. In this case we actually do not get modular forms but Mock modular forms. For $m \in \mathbb{Z}_{>0}$ the Hurwitz-Kronecker class number $\mathcal{H}(m)$ is the number of quadratic forms $ax^2+bxy+cy^2$ with $a,b,c \in \mathbb{Z}$ and discriminant $b^2-ac=-m$ counted with multiplicity 1 divided by the order of their stabilizer in $PSl(2,\mathbb{Z})$; one also puts $\mathcal{H}(0)=-\frac{1}{12}$. One finds $\mathcal{H}(3)=\frac{1}{3}$, $\mathcal{H}(4)=\frac{1}{2}$, $\mathcal{H}(7)=1$. Klyachko has computed the Euler number of the moduli spaces in terms of these class numbers in [24]. Let $H=c_1(\mathcal{O}_{\mathbb{P}^2}(1))$ be the Poincaré dual to the class of a line in $\mathbb{P}^2$.

THEOREM 3.4. $e(N_{\mathbb{P}^2}(2,H,n))=3\mathcal{H}(4n-1)$.

Klyachko proves his result by using the natural $(\mathbb{C}^*)^2$ action on $\mathbb{P}^2$. It lifts to a $(\mathbb{C}^*)^2$ action on $N_{\mathbb{P}^2}(H,n)$ with finitely many fixpoints. Then Klyachko determines the Euler number $e(N_{\mathbb{P}^2}(2,H,n))$ as the number of fixpoints.

The generating function $G_{3/2} := \sum_{n\ge 0}\mathcal{H}(n)q^n$ is in many ways the simplest example of a Mock modular form (of weight $\frac{3}{2}$). In fact the real analytic function

$$G_{3/2}^* = \sum_{n\ge 0}\mathcal{H}(n)q^n + \frac{1}{16\pi\sqrt{\Im(\tau)}}\sum_{n\in\mathbb{Z}}\beta\big(4\pi n^2\Im(\tau)\big)q^{-n^2},$$

with $\beta(t) = \int_1^\infty x^{-3/2}e^{-xt}dx$, is a real analytic modular form of weight $\frac{3}{2}$ whose holomorphic part is $G_{3/2}$. Klyachko's theorem says that

$$Y_H^{\mathbb{P}^2} = \frac{3}{2}\Big(G_{3/2}(\tau/4) - G_{3/2}\big((\tau+2)/4\big)\Big)$$

Yoshioka [45] obtains a nice closed generating function for the Betti numbers of the $M_{\mathbb{P}^2}(2, H, c_2)$.

3.6. Wallcrossing and Rational Surfaces

Much of my own work on moduli spaces of sheaves has concentrated on wallcrossing: By definition the moduli spaces $M_S^H(c_1, d)$ depend on the ample line bundle H, and the question is to determine the precise dependence.

For this we view H (or rather its first Chern class) as an element in $H^2(S, \mathbb{R})$. This has the advantage that now H can be varied in a continuous way. It turns out that, when one varies H a little bit, usually the moduli spaces stay unchanged, but there are certain "walls" i.e. hyperplanes in $H^2(S, \mathbb{R})$ such that, when H crosses them, the moduli spaces will change. The change of the invariants of the moduli spaces, when H crosses a wall, will be called the wallcrossing term. It turns out to be a very fruitful approach to study the invariants of the $M_S^H(c_1, d)$ via their wallcrossing terms, because these are much easier to understand and follow a much simpler pattern than the invariants of the moduli spaces themselves. Furthermore, at least for rational surfaces, it turns out that one can always reduce to the case that the invariants of $M_S^{H_0}(c_1, d)$ vanish for some fixed H_0, thus everything is determined by wallcrossing.

Definition 3.5. *Let $c_1 \in H^2(S, \mathbb{Z})$, and $d \in \mathbb{Z} - c_1^2/4$. A class $\xi \in H^2(S, \mathbb{Z}) + c_1/2$ defines a wall of type (c_1, d) if*

1. *$d + \xi^2 \geq 0$.*
2. *$\langle \xi H \rangle = 0$ for some ample H.*

In this case we call $W_\xi := \{H \in H^2(S, \mathbb{R}) \mid \langle \xi H \rangle = 0\}$ the wall defined by ξ. Note that, as ξ is orthogonal to an ample class, its self-intersection ξ^2 is negative; condition 1. puts a bound on how negative it can be. Let H_+ and H_- be two ample classes on S. Let ξ run through the classes defining walls of type (c_1, d) with $\langle \xi H_- \rangle < 0 < \langle \xi H_+ \rangle$. In [14], is is proven that $M_S^{H_+}(c_1, d)$ is obtained from $M_S^{H_-}(c_1, d)$, by sucessively for all ξ deleting certain subsets isomorphic to $\mathbb{P}^k$-bundles over products $S^{[n_1]} \times S^{[n_2]}$ with $n_1 + n_2 = d + \xi^2$ and replacing them by $\mathbb{P}^l$-bundles for suitable k and $l = k + 2\langle \xi K_S \rangle$. In particular it follows that

$$e(M_S^{H_+}(c_1,d)) = e(M_S^{H_-}(c_1,d)) + \sum_{\xi}\sum_{n_1+n_2=d+\xi^2} 2\langle \xi K_S\rangle e(S^{[n_1]})e(S^{[n_2]}),$$

where ξ runs through the classes defining walls of type (c_1,d) with $\langle \xi H_-\rangle < 0 < \langle \xi H_+\rangle$. More precisely in [8] one obtains $M_S^{H_+}(c_1,d)$ from $M_S^{H_-}(c_1,d)$ by an explicit sequence of blowups and blowdowns with centers projective bundles over products of Hilbert schemes of points on S.

If H and L are ample line bundles on S, the walls of type $(0,c_2)$ between L and H are given by the classes $\xi\in H^2(S,\mathbb{Z})$ with $\langle L\xi\rangle < 0 < \langle H\xi\rangle$ and $-\xi^2\le c_2$. Thus it seems natural to introduce and study the "theta function"

$$\Theta^{L,H}(\tau) := \sum_{\langle H\xi\rangle<0<\langle L\xi\rangle} q^{-\xi^2}$$

for the lattice $H^2(S,\mathbb{Z})$ with the negative of the intersection form as quadratic form. Note that for a rational surface this is an indefinite lattice of type $(b_2(S)-1,1)$, i.e. with one negative eigenvalue of the intersection form, thus the standard theory of theta functions does not apply. Then one can express the difference of the invariants of $M_S^H(0,c_2)$ and $M_S^L(0,c_2)$ in terms of $\Theta^{L,H}(\tau)$, and the difference of the invariants of $M_S^H(c_1,d)$ and $M_S^L(c_1,d)$ in terms of a modification $\Theta_{c_1}^{L,H}(\tau)$. As mentioned above, for rational surfaces one can reduce to the case where all the invariants of $M_S^L(c_1,d)$ vanish, thus the invariants of $M_S^H(c_1,d)$ themselves are expressed in terms of $\Theta_{c_1}^{L,H}(\tau)$. The Donaldson invariants that we will introduce below are also invariants of the moduli spaces $M_S^H(c_1,d)$, thus this statement also applies to them. In [22] we show that under certain restrictions these theta functions $\Theta^{L,H}(\tau)$ are indeed modular forms, and this is applied to prove some structural results for the Donaldson invariants of rational surfaces. In [17] this result is used to show in some cases that the generating functions $Z_{c_1}^{S,H}$ are meromorphic modular forms.

Motivated in part by [22], Zwegers developed in [49] a general theory of theta functions $\Theta^{L,H}(\tau)$ for lattices of type $(r-1,1)$. He showed that these are always Mock modular forms. In [12] this result is used to show that for a rational surface S the generating functions $Z_{c_1}^{S,H}$ are always Mock modular forms, and to determine them explicitely in many cases.

4. Donaldson Invariants

Donaldson invariants are $\mathbb{C}^\infty$ invariants of differentiable 4-manifolds X, which are defined using gauge theory. They are defined as intersection numbers of cohomology classes on moduli spaces of anti self-dual connections (which are solutions to certain partial differential equations) on a principal fibre bundle. They depend on the choice of a Riemannian metric g on X.

If X is a projective algebraic surface S, then the Donaldson invariants can be computed as intersection numbers on moduli spaces $M = M_S^H(c_1, d)$ of H-semistable sheaves on S, and the choice of H corresponds to the choice of g.

We sketch the definition of the Donaldson invariants. Assume for simplicity that there exists a universal sheaf $\mathcal{E}$ over $S \times M$. This means in particular that the restriction of $\mathcal{E}$ to $S \times [E]$ (where $[E]$ is the point of M corresponding to the sheaf E) is just E itself. We also assume that the moduli space $M_S^H(c_1, d)$ has as dimension the expected dimension $s(d) = 4d - 3\chi(\mathcal{O}_S)$. Let $L \in H_2(S, \mathbb{Q})$. Put

$$\mu(L) := \bigl(4c_2(\mathcal{E}) - c_1(\mathcal{E})^2\bigr)/L \in H^2(M, \mathbb{Q}).$$

Here / denotes the slant product as defined in [42, page 287]. The corresponding Donaldson invariant is given by

$$\Phi_{S,c_1}^H(L^{s(d)}) = \int_{M_S^H(c_1,d)} \mu(L)^{s(d)},$$

and we put $\Phi_{S,c_1}^H(L^s) = 0$ if s is not congruent to $-c_1^2 - 3\chi(\mathcal{O}_S)$ modulo 4. We consider the generating function

$$\Phi_{S,c_1}^H(e^{Lz}) = \sum_{s \geq 0} \Phi_{S,c_1}^H(L^s) \frac{z^s}{s!}.$$

4.1. Wallcrossing and Invariants of $\mathbb{P}^2$

As the Donaldson invariants are computed using the moduli spaces $M_S^H(c_1, d)$ under variation of H they are subject to wallcrossing, with the same walls as above for the Euler numbers $e(M_S^H(c_1, d))$. For every class ξ of type (c_1, d) one can define wallcrossing terms $\delta_\xi^S(L^s)$ such that for all $s \geq 0$

$$\Phi_{S,c_1}^{H_+}(L^s) - \Phi_{S,c_1}^{H_-}(L^s) = \sum_\xi \delta_\xi^S(L^s),$$

where again ξ runs through all classes defining a wall of type (c, d) with $\langle \xi H_- \rangle < 0 < \langle \xi H_+ \rangle$. Furthermore $\delta_\xi^S(L^s) = 0$ for $s < 4d - 3\chi(\mathcal{O}_S)$. In [15] I determined a generating function for all the wallcrossing terms $\delta_\xi^S(L^s)$. I give a relatively complicated expression in meromorphic modular forms depending on the numerical invariants of S, L, ξ, and the wallcrossing term is the coefficient of q^0 in the q-development. The proof makes use of the Kotschick-Morgan conjecture, which says that the wallcrossing term indeed depends only on the numerical invariants of S, L, ξ. No proof of the Kotschick-Morgan conjecture has been published.

However there is an analogue of the Donaldson invariants for the affine plane $\mathbb{A}^2 = \mathbb{C}^2$ defined using equivariant cohomology. Their generating function is called the Nekrasov partition function. The Nekrasov conjecture relates the Nekrasov partition function to modular forms. It has been proven in [39], [37]. Using results of [8] and [9] we relate in [19] the wallcrossing terms $\delta_\xi^S(L^s)$ to the Nekrasov partition function and give a new proof of the wallcrossing formula of [15], independent of the Kotschick-Morgan conjecture.

THEOREM 4.1. *[15],[19] Let S be a simply connected algebraic surface with $b_+ = 1$. Let $L \in H^2(X, \mathbb{Q})$ and let ξ be a wall of class (c_1, d) for some d. Then*

$$\delta_\xi^X(\exp(Lz)) = -\sqrt{-1}^{\langle \xi K_S \rangle} \operatorname{Coeff}_{q^0}\Big[q^{-\xi^2} \exp\big(\langle \xi L \rangle hz + \langle L^2 \rangle T z^2\big)\theta_3^{\sigma(S)+8} h^3\Big].$$

Here $\sigma(S)$ is the signature of S, we put $u := -\frac{\theta_3^4+\theta_2^4}{\theta_3^2\theta_2^2}$, $h := \frac{2\sqrt{-1}}{\theta_3\theta_2}$, $T := -h^2 G_2(2\tau) - \frac{u}{6}$, $\theta_3 := \sum_{n\in\mathbb{Z}} q^{n^2}$, $\theta_2 = \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2}$.

As mentioned above, for rational surfaces S, by possibly blowing up one can reduce to the case where the Donaldson invariants Φ_{S,c_1}^F are zero for some F and thus all Φ_{S,c_1}^H can be termined by wallcrossing.

As an example I give a formula for the Donaldson invariants of $\mathbb{P}^2$. Here H is the class of a hyperplane.

COROLLARY 4.2. *[15]*

$$\Phi_H^{\mathbb{P}^2}(\exp(Hz)) = \sum_{0<n\leq m} \operatorname{Coeff}_{q^0}\left[\frac{q^{\frac{4m^2-(2n-1)^2}{4}}}{\sqrt{-1}^{6n-2m+5}} \exp\Big((n-1/2)hz + Tz^2\Big)\theta_3^9 h^3\right].$$

A different (and in some ways more attractive, although not simpler) formula for the Donaldson invariants of $\mathbb{P}^2$ was later proposed in [34] by Moore and Witten, based on physics arguments. The formula also contains the Eisenstein series $G_{3/2}$, that we already encountered in the context of the Euler numbers of the corresponding moduli spaces on $\mathbb{P}^2$. Recently using the theory of Mock modular forms in [33], it was shown that both formulas give the same Donaldson invariants.

A refinement of the Donaldson invariants are the K-theoretic Donaldson invariants. Instead of computing the intersection numbers

$$\Phi_{S,c_1}^H(L^s) = \int_{M_S^H(c_1,d)} \mu(L)^s,$$

one introduces a line bundle $\widetilde{L}$ on $M_S^H(c_1,d)$ with $c_1(\widetilde{L}) = \mu(L)$. The K-theoretic Donaldson invariant is then the holomorphic Euler characteristic $\chi(M_S^H(c_1,d),\widetilde{L})$, and one can again consider the generating function

$$\Psi^H_{S,c_1}(L,t) := \sum_d \chi(M_S^H(c_1,d),\widetilde{L})t^d.$$

Again these invariants are subject to wallcrossing and, based on work in [38] on a K-theoretic version of the Nekrasov partition function, we prove in [20] a generating function for the wallcrossing of the K-theoretic Donaldson invariants. It is very similar in structure to the wallcrossing formula for the usual Donaldson invariants, if somewhat more complicated. Surprisingly, using this wallcrossing formula one can prove in [11] that the generating functions of the K-theoretic Donaldson invariants of rational surfaces are always rational functions. For instance, putting $\psi_k := 1 + \Psi^H_{\mathbb{P}^2,0}(kH,t)$, we obtain:

$$\psi_1 = \frac{1}{(1-t)^3},\quad \psi_2 = \frac{1}{(1-t)^6},\quad \psi_3 = \frac{1+t^2}{(1-t)^{10}},\quad \psi_4 = \frac{1+6t^2+t^3}{(1-t)^{15}},$$
$$\psi_5 = \frac{1+21t^2+20t^3+21t^4+t^6}{(1-t)^{21}},$$
$$\psi_6 = \frac{1+56t^2+147t^3+378t^4+266t^5+148t^6+27t^7+t^8}{(1-t)^{28}}.$$

Let $g_k = \binom{k-1}{2}$ be the genus of a smooth curve of degree k in $\mathbb{P}^2$. The reader can check that

$$\psi_k = \frac{p_k(t)}{(1-t)^{\binom{k+2}{2}}},$$

with $p_k \in \mathbb{Z}[t]$ satisfying $p_k(1) = 2^{g_k}$. This is related to the strange duality conjecture of Le Potier.

5. Curve Counting

Let S be a smooth projective surface, and L a holomorphic line bundle on S. The zero set of a nonzero section of L will be a (possibly singular) curve on S. Denote by $|L| = \mathbb{P}(H^0(S,L))$ the corresponding linear system of curves. The arithmetic genus $a(C) = \chi(C,\mathcal{O}_C)$ of a curve $C \in |L|$ is the genus of a nonsingular curve in $|L|$. If C is singular its geometric genus $g(C)$ is the genus of a desingularization of C. One always has $g(C) \le a(C)$, with equality only if C is nonsingular. An irreducible curve C in $|L|$ is rational if it is the image of a nonconstant map $\mathbb{P}^1 \to S$, or equivalently if $g(C) = 0$. We want to consider generating functions for curves of given geometric genus in linear systems $|L|$ on surfaces S. The problem is related, but not identical, to the study of Gromov-Witten invariants, which are a virtual count of such curves.

5.1. Rational Curves on K3 Surfaces

Using partially physics arguments Yau and Zaslov [44] have given a generating function for the numbers of rational curves on K3-surfaces. The argument was then made precise in [1], [10].

THEOREM 5.1. *Let S be a K3 surfaces. Let L be a line bundle on S, such that all curves $C \in |L|$ are reduced and irreducible. Then:*

1. *The number of rational curves (counted with multiplicities) $C \in |L|$ depends only on $L^2 \in 2\mathbb{Z}$. Denote the number of these curves by $n_{L^2/2}$.*
2. $$\sum_{k \in \mathbb{Z}} n_k q^k = \frac{1}{\Delta(\tau)}.$$

The multiplicity with which a curve is counted depends only on the singularities of the curve. If all the singularities are just nodes, the multiplicity is 1.

Note that the generating function is the same as that for the Euler numbers of the Hilbert schemes of points $S^{[n]}$. The proof of the theorem consists in relating the problem to Hilbert schemes of points: Let $\overline{J(\mathcal{C})} \to |L|$ be the relative compactified Jacobian. The fibre of $\overline{J(\mathcal{C})}$ over a point corresponding to a curve $C \in |L|$ is the compactified Jacobian $\overline{J(C)}$ of torsion-free rank 1 sheaves. One proves that the Euler number $e(\overline{J(C)})$ vanishes unless C is a rational curve. Because of the additivity of the Euler number it follows that $e(\overline{J(\mathcal{C})})$ is equal to number of the rational curves in $|L|$ counted with $e(\overline{J(C)})$ as multiplicity. On the other hand one proves that there is a birational correspondence between $\overline{J(\mathcal{C})}$ and $S^{[n]}$ for $n = L^2/2 + 1$ which preserves the Euler number. In [10] it is shown that $e(\overline{J(C)})$ is a reasonable notion of multiplicity related to (modified) Gromov-Witten invariants. For K3 surfaces the usual Gromov-Witten invariants are trivially zero, therefore one has to consider modified Gromov-Witten invariants, which are defined by modifying the obstruction theory that is used in their definition.

A little bit later a generalization of this result to arbitrary genus was proven in [5] (not for the counting of curves, but for modified Gromov-Witten invariants) for L a primitive line bundle on a K3 surface. This has been recently vastly generalized by Maulik and Pandharipande (see [40]): in a suitable sense the generating functions of "all" reasonable modified Gromov-Witten invariants of K3 surfaces with respect to primitive line bundles L are given by quasimodular forms, and conjecturally the line bundles do not need to be primitive.

5.2. General Conjecture

Parallely to the above mentioned work of [5] a general conjecture was formulated in [16] about counting curves with a given number of nodes in linear systems $|L|$ on surfaces S. The claim is that there is a universal formula, which applies whenever the line bundle L is sufficiently ample, for instance when L is a sufficiently high power of a very ample line bundle.

Let S be a smooth projective surface and L a line bundle on S. If L is sufficiently ample we have that

$$\dim(|L|) = \chi(S, L) - 1 = L(L - K_S)/2 + \chi(\mathcal{O}_S),$$

and a general curve in $|L|$ is nonsingular of genus $L(L + K_S)/2 + 1$; the genus of a singular curve with δ nodes will be smaller by δ. One expects that a node imposes 1 linear condition on elements of L. Thus we expect the locus of curves with δ nodes to ha

ve codimension δ in $|L|$, and this will be the case provided L is sufficiently ample with respect to δ.

We denote by $a_\delta(S, L)$ the number of curves with precisely δ nodes as only singularities in a general $\chi(S, L) - 1 - \delta$ dimensional sub-linear system of $|L|$.

Conjecture 5.2. *1. There exist universal polynomials $T_\delta \in \mathbb{Q}[x, y, z, w]$ such that for all sufficiently ample L*

$$a_\delta(S, L) = T_\delta(\chi(S, L), \chi(\mathcal{O}_S), c_1(L)K_S, K_S^2).$$

2. More precisely there exist power series $B_1, B_2 \in \mathbb{Z}[[q]]$ such that

$$\sum_{\delta \geq 0} T_\delta(x, y, z, w)(DG_2)^\delta = \frac{(DG_2/q)^x B_1^z B_2^w}{(\Delta(\tau) D^2 G_2/q^2)^{y/2}}.$$

In [32] a proof of this conjecture was published. The proof introduces new gauge-theoretic invariants. It is very long (200 pages), complicated and difficult to understand. It is not accessible to algebraic geometers.

Part of the original conjecture, which is not adressed in this proof is that $T_\delta(x, y, z, w)$ is an explicit intersection number on $S^{[3\delta]}$. In this form it has just been proven by Yu-jong Tzeng, using degeneration and cobordism arguments that were originally developed by Levine-Pandharipande [27] to study Donaldson-Thomas invariants.

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Author's address:

Lothar Göttsche
International Centre for Theoretical Physics
Strada Costiera 11, 34151 Trieste, Italy
E-mail: `gottsche@ictp.it`

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