A Connection between Viscous Profiles and Singular ODEs

Stefano Bianchini and Laura V. Spinolo

Abstract. *We deal with the viscous profiles for a class of mixed hyperbolic-parabolic systems in one space dimension. We focus, in particular, on the case of the compressible Navier Stokes equation in one space variable written in Eulerian coordinates. We describe the link between these profiles and a singular ordinary differential equation in the form*

$$\frac{dV}{dt} = \frac{1}{\zeta(V)} F(V). \tag{1}$$

Here $V \in \mathbb{R}^d$ and the function F takes values into $\mathbb{R}^d$ and is smooth. The real valued function ζ is as well regular: the equation is singular in the sense that $\zeta(V)$ can attain the value 0.

We focus on mixed hyperbolic-parabolic systems in one space variable in the form

$$E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx}. \tag{2}$$

Here the function u takes values in $\mathbb{R}^N$ and depends on the two scalar variables t and x. We focus on the case the matrix B is singular, namely its rank is strictly smaller than N. In particular, a conservative system $u_t + f(u)_x = \big(B(u)u_x\big)_x$ can be written in the form (2). Indeed, one can set $A(u,\, u_x) = D\,f(u) - B(u)_x$, where $D\,f$ denotes the Jacobian matrix of f. In the following we will consider explicitly the case of the compressible Navier Stokes equation in one space variable:

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ (\rho v)_t + \big(\rho v^2 + p\big)_x = \big(\nu v_x\big)_x \\ \Big(\rho e + \rho \dfrac{v^2}{2}\Big)_t + \Big(v\Big[\dfrac{1}{2}\rho v^2 + \rho e + p\Big]\Big)_x = \Big(k\theta_x + \nu v v_x\Big)_x. \end{cases} \tag{3}$$

Here the unknowns $\rho(t, x)$, $v(t, x)$ and $\theta(t, x)$ are the density of the fluid, the velocity of the particles in the fluid and the absolute temperature respectively. The function $p = p(\rho, \theta) > 0$ is the pressure and satisfies $p_\rho > 0$, while e is the internal energy. In the following we will focus on the case of a polytropic gas, so that e satisfies $e = R\theta/(\gamma - 1)$, where R is the universal gas constant and $\gamma > 1$ is a constant specific of the gas. Finally, by $\nu(\rho) > 0$ and $k(\rho) > 0$ we denote the viscosity and the heat conduction coefficients respectively.

In the following, we assume that system (2) satisfies a set of hypotheses introduced by Kawashima and Shizuta in [6] and satisfied, up to a change in the dependent variables, by the equations of the hydrodynamics. First, we assume that the rank of the matrix $B(u)$ is constant and we denote it by r. Also, $B(u)$ admits the block decomposition

$$B(u) = \begin{pmatrix} 0 & 0 \\ 0 & b(u) \end{pmatrix}. \tag{4}$$

The block $b(u)$ belongs to $\mathbb{M}^{r \times r}$ and there exists a constant $c_b > 0$ such that, for every $\vec{\xi} \in \mathbb{R}^r$, one has $\langle b(u)\vec{\xi}, \vec{\xi} \rangle \geq c_b |\vec{\xi}|^2$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^r$.

Also, we assume that for every u the matrix $A(u, 0)$ is symmetric. We denote by

$$\begin{aligned} A(u, u_x) &= \begin{pmatrix} A_{11}(u) & A_{21}^T(u) \\ A_{21}(u) & A_{22}(u, u_x) \end{pmatrix} \\ E(u) &= \begin{pmatrix} E_{11}(u) & E_{21}^T(u) \\ E_{21}(u) & E_{22}(u) \end{pmatrix} \end{aligned} \tag{5}$$

the block decomposition of A and E corresponding to (4). Note that only the block A_{22} can depend on u_x. Finally, we assume that for every u, the matrix $E(u)$ is symmetric and positive definite.

In the following, we focus on two classes of solutions of (2): traveling waves and steady solutions. A traveling wave is a one variable function $U(y)$ satisfying $\big[A(U, U') - \sigma E(U)\big]U' = B(U)U''$, where σ is a real constant. From a traveling wave U one obtains a solution of (2) by setting $u(t, x) = U(x - \sigma t)$. The steady solutions of (2) are one variable functions $U(x)$ satisfying

$$A(U, U')U' = B(U)U''. \tag{6}$$

We also require that U is bounded on $x \in [0, +\infty[$ and admits a limit as $x \to +\infty$. Steady solutions in this class are sometimes called *boundary layers*. In the applications, it is often interesting to focus on the case the speed σ of the traveling wave is close to an eigenvalue of $E^{-1}A$. Since in general 0 is not an eigenvalue of $E^{-1}A$, it is often useful to distinguish between traveling waves and boundary layers.

Travelling waves and steady solutions are powerful tools to study the viscous approximation of hyperbolic problems: being the literature concerning this topic extremely big, we just refer to the books by Dafermos [3] and by Serre [9] and to the references in there.

In a previous work [2], the authors studied a problem related to (2) by imposing a new condition of block linear degeneracy on the structure of the matrices E, A and B: let the block $A_{11}(u)$ and $E_{11}(u)$ be as in (5). The block linear degeneracy precribes that for every real number σ the dimension of the kernel of $[A_{11}(u) - \sigma E_{11}(u)]$ is constant with respect to u. In other words, such a dimension can in general vary when σ varies, but it cannot vary when u varies.

If the condition of block linear degeneracy is violated, then one might face "pathological" behaviors, in the following sense. There are systems satisfying all the Kawashima Shizuta hypotheses and violating the block linear degeneracy which admit non continuously differentiable steady solutions (note that in this case it is not completely clear, a priori, what we mean by solution of (6), because we are dealing with non regular functions: see [2, Section 2] for the details). On the other side, imposing the block linear degeneracy is restrictive in view of some applications. In particular, as pointed out by Frédéric Rousset [7], such a condition is satisfied by the compressible Navier Stokes equation written in Lagrangian coordinates, but is violated by the same equation written in Eulerian coordinates. The problem is the following.

As pointed out for example in [8], the Navier Stokes equation written in Lagrangian coordinates can be reduced to the form (2). In particular, the rank of the matrix $B(u)$ is constantly equal to two and the block A_{11} defined as in (5) is a real valued function and it is actually identically equal to 0, which implies that the condition of block linear degeneracy is satisfied. By direct computations one can then verify that the viscous profiles satisfy an ordinary differential equation which does not have the singularity exhibited by (1).

Let us consider now (3), the Navier Stokes equation written in Eulerian coordinates. We want to write it in the form (2) requiring that the matrix $A(u, 0)$ is *symmetric*. In the following, we assume that ρ is bounded away from 0, say $\rho \geq c_\rho > 0$ for a suitable constant c_ρ. This implies that the system does not reach the vacuum. We proceed as in Kawashima and Shizuta [6] and by multiplying (3) on the left by a suitable nonsingular matrix we eventually obtain that (3) can be written in the form (2) for

$$E(\rho, v, \theta) = \frac{1}{\theta} \begin{pmatrix} p_\rho/\rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho e_\theta/\theta \end{pmatrix}$$

$$B(\rho, v, \theta) = \frac{1}{\theta} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & k/\theta \end{pmatrix} \tag{7}$$

and

$$A(\rho,\, v,\, \theta,\, \rho_x,\, v_x,\, \theta_x) = \frac{1}{\theta} \begin{pmatrix} p_\rho v/\rho & p_\rho & 0 \\ p_\rho & \rho v - \nu' \rho_x & p_\theta \\ 0 & p_\theta - \nu v_x/\theta & \rho v e_\theta/\theta - k'\rho_x/\theta \end{pmatrix} \quad (8)$$

In the previous expression, we denote by p_ρ and p_θ the partial derivative of p with respect to ρ and θ respectively, and, by recalling that $e = R\theta/(\gamma - 1)$, we get that $e_\theta = R/(\gamma - 1)$. Also, we set

$$A_{21} = \frac{1}{\theta} \begin{pmatrix} p_\rho \\ 0 \end{pmatrix} \qquad A_{22} = \frac{1}{\theta} \begin{pmatrix} \rho v - \nu' \rho_x & p_\theta \\ p_\theta - \nu v_x/\theta & \rho v e_\theta/\theta - k'\rho_x/\theta \end{pmatrix} \quad (9)$$

and

$$b = \frac{1}{\theta} \begin{pmatrix} \nu & 0 \\ 0 & k/\theta \end{pmatrix} \qquad a_{11} = p_\rho/(\theta\rho).$$

The condition of block linear degeneracy is violated here. To see this, let us focus on the case $\sigma = 0$: the dimension of the kernel of $A_{11} = a_{11}v$ is 0 if $v \neq 0$, but it is 1 when $v = 0$ (we recall that $p_\rho > 0$).

To see what in principle can go wrong, we focus on steady solutions, the situation for traveling waves being analogous. We set $w = \rho_x$ and $\vec{z} = (v_x,\, \theta_x)^T$. Then (6) becomes

$$\begin{cases} a_{11}vw + A_{21}^T \vec{z} = 0 \\ A_{21}w + A_{22}\vec{z} = b\vec{z}_x \end{cases}$$

Assume $v \neq 0$, then (6) can be written as

$$\begin{cases} w = -\dfrac{A_{21}^T \vec{z}}{a_{11}v} \\ \vec{z}_x = b^{-1} \Big[A_{22} - \dfrac{A_{21}A_{21}^T}{a_{11}v} \Big] \vec{z} \end{cases} \quad (10)$$

Note that the matrix b is invertible and hence the previous expression is well defined.

By recalling that $w = \rho_x$ and $\vec{z} = (v_x,\, \theta_x)^T$ and by relying on (10), one obtains that the steady solutions of the Navier Stokes written in Eulerian coordinates satisfy a singular ordinary differential equation in the form

$$\frac{dU}{dx} = \frac{1}{v} F(U) \quad (11)$$

provided that $U = (\rho,\, v,\, \theta,\, \vec{z})^T$ and

$$F(U) = \begin{pmatrix} A_{21}^T \vec{z}/a_{11} \\ v\,\vec{z} \\ b^{-1}\Big[A_{22}v - A_{21}A_{21}^T/a_{11} \Big]\vec{z} \end{pmatrix}$$

We say that equation (11) is *singular* since in general v, the velocity of the fluid, can attain the value 0.

Let us now go back to the example in [2] concerning a system with non continuously differentiable steady solutions. It turns out that, as a consequence of the fact that the system violates the block linear degeneracy, the steady solutions V satisfy a singular ODE in the form (1), where ζ is a real-valued, smooth function that can attain the value 0. The loss of regularity experienced by V is actually due to the fact that the equation is singular. More precisely, the problem is that $\zeta(V) \neq 0$ at $t = 0$, but ζ reaches the singular value 0 at a finite time t_0, which is exactly the time when the first derivative blows up.

Summing up, we have the following: the condition of block linear degeneracy is satisfied by the compressible Navier Stokes equation written in Lagrangian coordinates, but it is violated by the same equation written in Eulerian coordinates. As a consequence, the viscous profiles of the equation written in Eulerian coordinates satisfy a singular ODE and hence might in principle experience a loss of regularity. This suggests that we should look for more general conditions than the block linear degeneracy, namely conditions weak enough to apply to the Navier Stokes equation written in Eulerian coordinates and by other systems violating the block linear degeneracy. On the other side, these conditions should be sufficiently strong to rule out any loss of regularity.

A set of conditions satisfying the previous requirements is defined in [1, Section 2] by relying on the study of the singular equation (1) in a small enough neighborhood of a point $\bar{V}$ satisfying $\zeta(\bar{V}) = 0$ and $G(\bar{V}) = \vec{0}$. The analysis in [1] is indeed local, and hence the results only apply to the study of the viscous profiles with small enough total variation.

In [1] we rely on the notions of *fast* and *slow* dynamics of the singular ODE (1): the formal definition is technical, but we can get an heuristic idea by considering the toy model

$$\begin{cases} \dfrac{dv_1}{dt} = -v_1 \\ \dfrac{dv_2}{dt} = -\dfrac{v_2}{\varepsilon}. \end{cases}$$

Here the singularity ε is just a parameter, $\varepsilon \to 0^+$ and the solution can be explicitly computed. Both v_1 and v_2 are exponentially decaying to 0, but the speed of exponential decay of v_2 gets faster and faster as $\varepsilon \to 0^+$ and hence v_2 is regarded as a *fast dynamic*. Conversely, the speed of exponential decay of v_1 is bounded with respect to ε and hence v_1 is regarded as a *slow dynamic*.

These notions can be extended to the general nonlinear case (1) and are reminiscent of the notions of *slow* and *fast time scale* discussed by Fenichel [4] (see also the lecture notes by Jones [5]).

In [1] we impose on (1) several hypotheses, the key one being that the singular set

$$\{U : \zeta(U) = 0\} \subseteq \mathbb{R}^d$$

is invariant for both the slow and the fast dynamics. Namely, if $U(t)$ is a solution of (1) which is either a slow or a fast dynamic and $\zeta\big(U(0)\big) = 0$ then $\zeta\big(U(t)\big) = 0$ for every $t \in \mathbb{R}$. As a matter of fact, this condition and the other hypotheses introduced in [1] are satisfied by the equation (11) describing the viscous profiles of the Navier Stokes equation written in Eulerian coordinates.

In [1] the analysis focuses on the solutions of (1) lying on suitable invariant manifolds (a manifold $\mathcal{M}$ is invariant for (1) if any solution $U(t)$ such that $U(0) \in \mathcal{M}$ satisfies $U(t) \in \mathcal{M}$ for every t) satisfying the following property: if $U(t)$ is a solution lying on the manifold and $\zeta\big(U(0)\big) \neq 0$ then $\zeta\big(U(t)\big) \neq 0$ for every t. Also, the viscous profiles with sufficiently small total variation lie on these manifolds, hence by applying our results to (11) we rule out the loss of regularity and we obtain that the traveling waves and the boundary layers with small total variation are continuously differentiable.

For a different approach to the analysis of the viscous profiles of the Navier Stokes equation in Eulerian coordinates see Wagner [10] and the references therein.

References

[1] S. Bianchini and L.V. Spinolo, *Invariant manifolds for a singular ordinary differential equation*, submitted, arXiv:0801.4425.

[2] S. Bianchini and L.V. Spinolo, *The boundary Riemann solver coming from the real vanishing viscosity approximation*, Arch. Rational Mech. Anal. **191** (2009), 1–96.

[3] C.M. Dafermos, *Hyperbolic conservation laws in continuum physics*, second ed., Springer, Berlin (2005).

[4] N. Fenichel, *Geometric singular perturbation theory for ordinary differential equations*, J. Differential Equations **31** (1979), 53–98.

[5] C.K.R.T. Jones, *Geometric singular perturbation theory*, Dynamical systems (Montecatini Terme, 1994), Lecture Notes in Math. vol. 1609, Springer, Berlin (1995), pp. 44–118.

[6] S. Kawashima and Y. Shizuta, *On the normal form of the symmetric hyperbolic-parabolic systems associated with the conservation laws*, Tôhoku Math. J. **40** (1988), 449–464.

[7] F. Rousset, *Navier Stokes equation and block linear degeneracy*, personal communication.

[8] F. Rousset, *Characteristic boundary layers in real vanishing viscosity limits*, J. Differential Equations **210** (2005), 25–64.

[9] D. SERRE, *Systems of conservation laws, I, II*, Cambridge University Press, Cambridge (2000).

[10] D.H. WAGNER, *The existence and behavior of viscous structure for plane detonation waves*, SIAM J. Math. Anal. **20** (1989), 1035–1054.

Authors' addresses:

Stefano Bianchini
SISSA
Via Bonomea 265
34136 Trieste, Italy
E-mail: `bianchin@sissa.it`

Laura V. Spinolo
Centro De Giorgi
Scuola Normale Superiore
Piazza dei Cavalieri 3, 56126 Pisa, Italy
E-mail: `laura.spinolo@sns.it`