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On the Infinite Dual Goldie Dimension

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Abstract. *We analyze how the properties of Goldie dimension continue to hold or not in the infinite case, with particular interest for the dual Goldie dimension of the lattice of right ideals of a ring R. In this setting we underline the important role played by maximal ideals and we compute the dual Goldie dimension of any Boolean ring and of any endomorphism ring of an infinite dimensional vector space over a division ring.*

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1. Introduction

The invariant of modules that now we call Goldie dimension was first used by A. Goldie in [7] in the study of the right R-module R_R. Subsequently, the Goldie dimension was dualized in different ways by Fleury [5] and Varadarajan [13]. In [9], Grzeszczuk and Puczyłowski extended the Goldie dimension to the case of modular lattices, proving that the Goldie dimension of the opposite lattice L^{op} of the lattice L of all submodules of a module M is equal to the dual dimension of M studied by Varadarajan. Grzeszczuk and Puczyłowski also extended in [10] the Goldie dimension and the dual Goldie dimension to the infinite case.

The aim of this paper is to continue this line of research in the infinite case, focusing mainly on the dual Goldie dimension of the right R-module R_R. We start from the most general setting were the Goldie dimension makes sense, that is, the setting of bounded modular lattices. In Section 2, we analyze which properties of the finite Goldie dimension still hold in the infinite case and which do not. In Section 3, we restrict our attention to the case of the dual Goldie dimension of the right R-module R_R. For this study, in which we are particularly interested, we can consider only the maximal right ideals of R instead of the whole lattice of right ideals of R. In Section 4, we study in detail two relevant examples, computing their dual Goldie dimension. These examples show the difficulties that arise in passing from the finite case to the infinite one.

2. Goldie Dimension on Lattices

Let L be a bounded modular lattice, that is, a lattice L that satisfies the modular law $x \leq b \Rightarrow x \vee (a \wedge b) = (x \vee a) \wedge b$ and has a greatest element 1 and a smallest element 0. A finite subset $\{ a_i \mid i \in I \}$ of $L \setminus \{ 0 \}$ is said to be *join-independent*, or simply *independent*, if $a_i \wedge (\bigvee_{j \neq i} a_j) = 0$ for every $i \in I$. The empty subset of $L \setminus \{ 0 \}$ is join-independent. An arbitrary subset A of $L \setminus \{ 0 \}$ is *join-independent* if all its finite subsets are join-independent.

We recall that the *Goldie dimension* of L, denoted by $dim(L)$, is defined as the supremum of all cardinals κ such that L contains a join-independent subset of cardinality κ (see for example [4]).

Considering these concepts in the opposite lattice L^{op}, we define a subset $A = \{ a_i \mid i \in I \}$ of L to be *coindependent* if for every finite subset $F \subseteq I$ and $i \in F$ we have $a_i \vee (\bigwedge_{j \neq i \in F} a_j) = 1$.

The *dual Goldie dimension* of L, denoted by $codim(L)$, is the Goldie dimension of L^{op}, i.e. the supremum of all cardinals κ such that L contains a coindependent subset of cardinality κ.

Given a cardinal number κ, say that κ is *attained* in L if L contains a join-independent subset of cardinality κ. We recall that an infinite cardinal κ is called *regular* if $\kappa_i < \kappa$ for $i \in I$ with $|I| < \kappa$ implies $\sum \kappa_i < \kappa$. Otherwise it is called *singular*. An uncountable, regular, limit cardinal is said to be *inaccessible*. We remind that the existence of inaccessible cardinals can not be proved in ZFC (Zermelo-Fraenkel with the axiom of choice) and that there are no such cardinals in the constructible universe (see for example [3]). In [12], Santa-Clara and Silva proved, generalizing results in [2] and [3], that the Goldie dimension of L can be not attained only if it is an inaccessible cardinal.

Definition 2.1. *Let L be a bounded modular lattice and let $A = \{ a_i \mid i \in I \}$ be a subset of L. A is called an essential subset if for every non-zero element $b \in L$, there exists a finite subset F of I such that $(\bigvee_{i \in F} a_i) \wedge b \neq 0$.*

Similarly, A is a superfluous subset if for every $1 \neq b \in L$, there exists a finite subset F of I such that $(\bigwedge_{i \in F} a_i) \vee b \neq 1$. Obviously A is a superfluous subset in L if and only if A is an essential subset in L^{op}.

Let a be an element of L and $A = \{ a_i \mid i \in I \}$ a subset of L such that $a_i \leq a$. We say that A is essential in a if it is an essential subset of the lattice $[0, a]$.

A finite subset $A = \{ a_i \mid i \in I \} \subseteq L$ is essential if and only if $\bigvee_{i \in I} a_i$ is essential in L. Similarly, A is superfluous if and only if $\bigwedge_{i \in I} a_i$ is superfluous in L.

Theorem 2.2. *Let $L \neq 0$ be a bounded modular lattice such that every non-zero element of L contains a uniform element. Let κ be a cardinal number. Then the following are equivalent:*

(a) L does not contain join-independent subsets of cardinality $\geq \kappa$;

(b) L contains an essential join-independent subset $\{ a_i \mid i \in I \}$ of cardinality strictly less than κ, with a_i uniform for every $i \in I$;

(c) there exists a cardinal $\lambda < \kappa$ such that every join-independent subset of L has cardinality $\leq \lambda$.

Moreover, if these equivalent conditions hold, every essential join-independent subset $\{ a_i \mid i \in I \}$, with a_i uniform for every $i \in I$, attains the Goldie dimension of L.

Proof. (a)⇒(b) Let F be the set of all join-independent subsets of L consisting only of uniform elements. Since every element of L contains a uniform element, F is non-empty. By Zorn's lemma, F has a maximal element X with respect to inclusion. By (a), $card(X) < \kappa$. At this point, we claim that X is an essential subset of L; otherwise there would exist a non-zero element $x \in L$ such that $(\bigvee_{y \in F} y) \wedge x = 0$ for every finite subset $F \subseteq X$ and, by hypothesis, there would be a uniform element $b \in L$ such that $b \leq x$. Then $X \cup \{ b \}$ would be a join-independent set of uniform elements strictly containing X, a contradiction to the maximality of X.

(b)⇒(c) Suppose that there exists an essential join-independent subset $A = \{ a_i \mid i \in I \}$ with every a_i uniform, $card(I) = \lambda < \kappa$.

We claim that A is maximal between the join-independent subsets of L; otherwise there exists a non-zero element $b \in L$ such that $A \cup \{ b \}$ is join-independent. This means that for every finite subset F of I we have that $(\bigvee_{i \in F} a_i) \wedge b = 0$, but this clearly contradicts the hypothesis that A is an essential subset.

Then, by Theorem 1 of [10], we have $card(J) \leq card(I)$ for every join-independent subset $\{ b_j \mid j \in J \}$ of L.

(c)⇒(a) Obvious.

The final remark is clear from the proof $(b) \Rightarrow (c)$. □

I'd like to recall that Grzeszczuc and Puczyłowski ([10], Prop. 2) say that our hypothesis that every non-zero element contains a uniform element is not only necessary, but also sufficient to claim that the lattice has a basis, i.e. a maximal independent subset of uniform elements.

In the finite case one has that also the following statement is equivalent to the ones in the theorem:

- if $a_0 \leq a_1 \leq a_2 \leq \dots$ is an ascending chain of elements of L, then there exists $i \geq 0$ such that a_i is essential in a_j for every $j \geq i$.

One can try to generalize this to the infinite case and ask if the following condition is equivalent to the ones in the theorem:

(d) *there does not exist an ascending chain X of elements of L of cardinality κ such that $\{ b \in X \mid b < a \}$ is not an essential set in a, for every a in the chain.*

What happens is that just one implication continues to hold. We have that (c)⇒(d): if (d) does not hold, there exists an ascending chain X of elements of L of cardinality κ such that $\{ b \in X \mid b < a \}$ is not an essential set in a, for every a in the chain. Then, for every a in the chain, there exist a non-zero element $c_a \leq a$ in L such that $(\bigvee_F b) \wedge c_a = 0$, for any finite subset F of $\{ b \in X \mid b < a \}$. This implies that these elements c_a form a join-independent subset of L of cardinality κ. Thus (c) does not hold.

The problem is that the other implication is no longer true when we pass to the infinite case. Let us show that the implication (d)⇒(a) is false in general. Let X be a set of cardinality κ; consider L to be the sublattice of $(\wp(X), \subseteq)$ consisting of $\emptyset$, X itself and all the finite subsets of X. It is clear that L is modular, since it is a sublattice of a distributive lattice. Every singleton is an uniform element in L, then every non-zero element of L contains a uniform element.

Since $x \wedge (\bigvee_{i=1}^{n} x_i) = 0$ for x, x_i singletons of X with $x \neq x_i$, $i = 1, \ldots, n$, we have that the set of the singletons is a join-independent subset of L.

On the other hand it is also obvious that every chain in L can have at most countable cardinality.

Before stating the next proposition, if L and L' are two bounded modular lattices, we denote by $L \oplus L'$ the product of L and L', which, as a set, consists of the elements (l, l'), with $l \in L$ and $l' \in L'$ and have the operations defined componentwise.

Proposition 2.3. *Let L be a bounded modular lattice.*

(a) *$dim(L) = 0$ if and only if $L = 0$;*

(b) *$dim(L) = 1$ if and only if L is uniform;*

(c) *$dim([0, a]) \leq dim(L)$ for every $a \in L$;*

(d) *$dim([0, a]) = dim(L)$ if a is essential in L;*

(e) *if L' is another modular lattice bounded, then $dim(L \oplus L') = dim(L) + dim(L')$.*

Proof. The proof of $(a), (b), (c)$ and (d) are elementary (the original article of Alfred Goldie where these things were observed is [8]). To prove (e) it is enough to observe that if $\{ a_i \mid i \in I \}$ is an essential join-independent subset of uniform elements of L_1 and $\{ b_j \mid j \in J \}$ is an essential join-independent subset of uniform elements of L_2, then $\{ (a_i, 0) \mid i \in I \} \cup \{ (0, b_j) \mid j \in J \}$ is an essential join-independent subset of uniform elements of $L_1 \oplus L_2$. □

We notice that the converse implication of (d) holds only in the finite case.

Remark 2.4. *The hypothesis that every element of the lattice contains a uniform element is always satisfied by the dual lattice of the right (left) ideals of a ring, since every right (left) ideal is contained in a maximal one. Therefore the dual Goldie dimension (left or right) of a ring is always attained.*

3. Dual Goldie Dimension on Rings

In view of the previous remark, now we restrict to the case of the right dual Goldie dimension of a ring R. We can easily observe here a certain number of facts:

- the Jacobson radical is a superfluous ideal ([1], Prop. 9.18). This means that $codim(R_R) = codim(R_R/J(R_R))$ and so we can restrict our attention to semiprimitive rings;
- when we look for coindependent sets we can restrict to maximal right ideals. If $I_1, \dots, I_n$ are coindependent right ideals, i.e. $I_i + (\bigcap_{j \neq i} I_j) = R$, choosing maximal ideals $M_i \supseteq I_i$, we have that $M_i + (\bigcap_{j \neq i} M_j) = R$, which means that $M_1, \dots, M_n$ are coindependent maximal right ideals. Moreover $M_1, \dots, M_n$ are all distinct; in fact, if $M_i = M_j$, we have that $M_i = M_i + M_j \supseteq I_i + I_j$, and this contradicts the fact that I_i and I_j are coindependent.

Let us see now how the concepts that we introduced above translate in this particular case. Let $\{ M_i \mid i \in I \}$ be a set of maximal right ideals.

The set $\{ M_i \mid i \in I \}$ is coindependent if for every finite subset $F \subseteq I$ and $i \in F$, we have $M_i + (\bigcap_{i \neq j \in F} M_j) = R_R$, that is equivalent to saying that $\bigcap_{i \neq j \in F} M_j \nsubseteq M_i$; this, thanks to the chinese remainder theorem, is also equivalent to

$$\frac{R}{\bigcap_{i \in F} M_i} \cong \bigoplus_{i \in F} \frac{R}{M_i}.$$

We have that $\bigcap_{i \in I} M_i$ is superfluous in R if and only if it is equal to the Jacobson radical $J(R)$. It is clear that $\bigcap_{i \in I} M_i \supseteq J(R)$, since $J(R)$ is the intersection of all maximal right ideals; on the other side the Jacobson radical of a ring R is the biggest superfluous right ideal ([1], Prop. 9.18), and therefore $\bigcap_{i \in I} M_i$ must be contained in it.

The set $\{ M_i \mid i \in I \}$ is superfluous, by definition, if for any proper right ideal $J \subseteq R_R$ there exists a finite subset $F \subseteq I$ such that $J + \bigcap_{i \in F} M_i \neq R_R$. Clearly, without loss of generality, we can take J to be a maximal ideal; therefore the condition we get is that for any maximal right ideal $M \subseteq R_R$ there exists a finite subset $F \subseteq I$ such that $M + \bigcap_{i \in F} M_i \neq R_R$; this is equivalent to saying that for every maximal right ideal there exists a finite subset $F \subseteq I$ such that $\bigcap_{i \in F} M_i \subseteq M$.

It is clear that if the set is superfluous, then $\bigcap_{i\in I} M_i = J(R)$ is superfluous. The converse implication is not true in general.

Example 3.1. *Consider the polynomial ring* $\mathbb{Q}[x]$ *in one variable over the rational numbers. It is a principal ideal domain and* $(p(x))$ *is a maximal ideal if and only if* $p(x)$ *is an irreducible polynomial. Let* $T = \{ (x-a) \mid a \in \mathbb{Z} \}$*; we can notice that, since* $(x-a)+\prod_{i=1}^{n}(x-b_i) = \mathbb{Q}[x]$ *for any distinct* $a, b_1, \dots, b_n$*,* T *is a coindependent set. It is clear also that* $\bigcap_T (x-a) = (0) = J(\mathbb{Q}[x])$*. Anyway, if we take* $p(x)$ *to be an irreducible polynomial of degree bigger than 1, there does not exist a finite number of elements in* T *such that* $(p(x)) \subseteq \bigcap_{i=1}^{n}(x-a_i)$*.*

Example 3.2. *In this example we will show that the cardinalities of a superfluous coindependent set of maximal right ideals and of a coindependent set of maximal right ideals with superfluous intersection can be different. Let us consider the ring of continuous functions of the real numbers* $C(\mathbb{R})$*; for every* $a \in \mathbb{R}$*, the subset* $M_a = \{ f \in C(\mathbb{R}) \mid f(a) = 0 \} \subseteq C(\mathbb{R})$ *is a maximal ideal. The set* $\{ M_a \mid a \in \mathbb{Q} \}$ *is a coindependent set of cardinality* $\aleph_0$*, but it is not coindependent since none of the maximal ideals* M_a*, with* $a \in \mathbb{R} \setminus \mathbb{Q}$*, contains a finite intersection of maximal ideals in this set. On the other hand it is clear that to have a coindependent superfluous set it is necessary that it contains all the maximal ideals* M_a*, with* $a \in \mathbb{R}$*, and therefore it has cardinality at least* $\mathfrak{c}$*.*

We saw that if $\{ M_i \mid i \in I \}$ is a coindependent set we have, for any finite subset $F \subseteq I$, an isomorphism $\frac{R}{\bigcap_{i\in F} M_i} \cong \bigoplus_{i\in F} \frac{R}{M_i}$. Therefore, in this situation, $\{ M_i \mid i \in I \}$ is a superfluous set if, for any maximal right ideal M of R_R, we have that there exists a finite subset $F \subseteq I$ such that $\frac{M}{\bigcap_{i\in F} M_i}$ is a maximal ideal of the semisimple ring $\bigoplus_{i\in F} \frac{R}{M_i}$.

Proposition 3.3. *Let* $\mathcal{M}_R$ *be the set of all maximal right ideals of a ring* R *and* $\mathcal{F} \subseteq \mathcal{P}(\mathcal{M}_R)$ *the set of all subsets* T *of* $\mathcal{M}_R$ *such that every maximal ideal of* R *contains the intersection of a finite number of elements of* T*. If* $\mathcal{F}$ *has a minimal element* T^**, then* $|T^*| = codim(R_R)$*.*

Proof. Thanks to Theorem 2.2, it is enough to prove that T^* is a superfluous coindependent set of couniform elements. Our hypothesis clearly implies that T is a superfluous set and it is obvious that any maximal ideal is couniform. It remains to prove that T^* is a coindependent set. To do this we have to prove that for any $M_1, \dots, M_n \in T$, we have that $M_i + (\bigcap_{j\neq i} M_j) = R$ for any $i = 1, \dots, n$. Suppose not, i.e. for an i we have $M_i + (\bigcap_{j\neq i} M_j) \subsetneq R$; this means that $M_i + (\bigcap_{j\neq i} M_j) \supseteq M_i$ is contained in a maximal ideal, that clearly must be M_i itself. From this it follows that $\bigcap_{j\neq i} M_j \subseteq M_i$ contradicting the fact that T^* is minimal. □

Corollary 3.4. *All minimal sets* T *of maximal right ideals of a ring* R *such that every maximal right ideal of* R *contains the intersection of a finite number of elements of* T*, have the same cardinality.*

If we want these minimal sets to exist, the easiest and more obvious way to proceed would be to assume the ring R to be right artinian, but, if we do so, since we assumed our ring to be semiprimitive, we fall in the case where R is a semisimple artinian ring, and therefore it is semilocal, case that is already completely understood (see for example [4], Prop. 2.43).

4. Examples and Computations

When one defines new concepts or tries to generalize existing ones, it is natural to try them on the easy examples. In the case of the infinite dual Goldie dimension, one of these examples could be the power set $(\mathcal{P}(X), \Delta, \cap)$ of an infinite set X. We saw above that to investigate the dual Goldie dimension of a ring we have to know its maximal ideals; in the case of $\mathcal{P}(X)$ they correspond bijectively to the ultrafilters of X, where the bijection is just the complement in $\mathcal{P}(X)$. The set of all ultrafilters is the underlying set of the so-called Stone-Čech compactification of X, which is indicated by βX, where X is considered as a topological space with the discrete topology (for a precise definition and more details look in [6]).

The easiest example would be to consider X a countable set, but even in this case the things are complicated. In fact, if one looks at $\beta\mathbb{N}$, the only elements that one can understand and exhibit explicitly are the principal ultrafilters, that correspond to the maximal ideals of the form $\mathcal{P}(X \setminus \{ n \})$, for $n \in \mathbb{N}$. These are only countably many, but the whole $\beta\mathbb{N}$ has cardinality $2^{\mathfrak{c}}$. Since the maximal ideals we mentioned do not form a superfluous set, it does not seem possible to find the dual Goldie dimension of an infinite power set in an elementary way.

In general, in the setting of Boolean rings, we have a bijective correspondence between maximal ideals and ultrafilters given by the complement. Therefore we can try to translate our conditions to compute the dual Goldie dimension from the language of maximal ideals to the one of ultrafilters. The two conditions that we have to translate are the following:

- the set of maximal ideals $\{ M_i \mid i \in I \}$ is coindependent, i.e. for every finite subset $F \subseteq I$ and $i \in F$, we have $M_i + (\bigcap_{i \neq j \in F} M_j) = R$. In other terms, we can express this condition saying that for every finite subset $F \subseteq I$ and $i \in F$, we have $\bigcap_{i \neq j \in F} M_j \nsubseteq M_i$. Taking the complement of this last formula we get $R \setminus \bigcap_{j \neq i} M_j = \bigcup_{j \neq i} R \setminus M_j \nsupseteq R \setminus M_i$. Therefore, according to this, we call a set of ultrafilters $\{ U_i \mid i \in I \}$ *coindependent* if for every finite subset $F \subseteq I$ and $i \in F$, we have $U_i \nsubseteq \bigcup_{j \neq i} U_j$;
- the set of maximal ideals $\{ M_i \mid i \in I \}$ is superfluous, i.e. for any maximal ideal $M \subseteq R$ there exists a finite subset $F \subseteq I$ such that $M + \bigcap_{i \in F} M_i \neq R$. Similarly to what we did above, we translate this condition defining

a set of ultrafilters $\{ U_i \mid i \in I \}$ to be *superfluous* if for any ultrafilter U, there exists a finite subset $F \subseteq I$ such that $U \subseteq \bigcup_{i \in F} U_i$.

According to these definitions it is clear that we get the following:

PROPOSITION 4.1. *Given a Boolean ring R, its dual Goldie dimension is equal to the cardinality of a coindependent superfluous set of ultrafilters of R.*

What is nice in the setting of Boolean rings is that the set of ultrafilters is not only a set, but it becomes a topological space. In fact, let B be a Boolean algebra and $S(B)$ its set of ultrafilters; the collection of all the sets of the form $\{ x \in S(B) \mid b \in x \}$, for $b \in B$, forms a basis for the so-called Stone topology on $S(B)$. In this topology a generic open set is of the form $\bigcup_{b \in B_0 \subseteq B} \{ x \in S(B) \mid b \in x \}$ and a generic closed set of the form $\bigcap_{b \in B_0 \subseteq B} \{ x \in S(B) \mid b \notin x \}$. It is not difficult to see that $S(B)$ is a $T1$ topological space; this is equivalent to saying that every point is closed, and in fact we have

$$\overline{\{ y \}} = \bigcap_{b \notin y} \{ x \in S(B) \mid b \notin x \} = \{ x \in S(B) \mid x \subseteq y \} = \{ y \}.$$

Therefore every finite subset of $S(B)$ is closed in the Stone topology.

Now that we have this topological structure, we can try to interpret our conditions of being superfluous and coindependent in these terms. All we have to do is to translate the condition $U \subseteq \bigcup_{i=1}^{n} U_i$. This is equivalent to saying that

$$b \in U \Rightarrow \exists i \in \{ 1, \dots, n \} \mid b \in U_i.$$

Reversing the implication we get

$$b \notin U_i \forall i \in \{ 1, \dots, n \} \Rightarrow b \notin U;$$

we can translate this to

$$U_i \in \{ x \in S(B) \mid b \notin x \} \forall i \in \{ 1, \dots, n \} \Rightarrow U \in \{ x \in S(B) \mid b \notin x \}.$$

At this point, this is equivalent to saying

$$U \in \overline{\{ U_1, \dots, U_n \}} = \{ U_1, \dots, U_n \}.$$

Now it becomes clear that for every finite subset $F \subseteq I$ and $i \in F$, we have $U_i \nsubseteq \bigcup_{j \neq i} U_j$, and therefore every set of ultrafilters is coindependent; if we require also that the set is superfluous we get the following.

THEOREM 4.2. *Given a Boolean ring B, its dual Goldie dimension is given by the cardinality of its Stone space $S(B)$.*

At this point, following ([6],9.2), we get

COROLLARY 4.3. *Let X be a set. The dual Goldie dimension of the ring* $(\mathcal{P}(X), \Delta, \cap)$ *is equal to* $2^{2^{|X|}}$.

It's not difficult to generalize the previous situation to the case K^X, where X is a set and K is any field. To do this it is enough that we prove that there is a bijective correspondence between maximal ideals of K^X and ultrafilters of X. To do this we explicit an order-preserving bijection between proper ideals of K^X and filters of X.

Let I be a proper ideal of the ring K^X, we want to show that the collection of zero sets of elements of I form a filter of X. To do this we observe the following:

- the empty set is not a zero set, since the identity function does not belong to I;
- if $A \subseteq X$ is a zero set of an element $f \in I$ and $A \subseteq B \subseteq X$, then B is the zero set of the element $f \cdot \chi_{X \setminus B} \in I$, where with χ_Y we denote the characteristic function of the set $Y \subseteq X$;
- let A and B be the zero sets of two elements in I. Multiplying by the appropriate elements in the ring we find that also $\chi_{X\setminus A}$ and $\chi_{X\setminus B}$ are in the ideal I; therefore we have that $\chi_{X\setminus A} + \chi_{X\setminus B} - \chi_{X\setminus A} \cdot \chi_{X\setminus B}$ (the last term is needed only when the characteristic of K is 2) is an element of the ideal I, having as zero set $A \cap B$.

On the other hand, let F be a filter on X. We want to prove that the set $I = \{\, f : X \to K \mid Z(f) \in F \,\}$, where $Z(f)$ indicates the zero set of f, is a proper ideal of the ring K^X. To do this we notice the following:

- the identity of K^X is not in I since the empty set is not in F;
- if $f \in I$ and $g \in K^X$, we have that $fg \in I$ since $Z(f) \subseteq Z(fg) \subseteq X$;
- if f and g are elements of I, then $f + g \in I$ since $Z(f+g) \supseteq Z(f) \cap Z(g)$.

It is easy to see that these two functions are mutually inverse. Therefore we can conclude that

PROPOSITION 4.4. *Let X be a set and K any field. Then the dual Goldie dimension of the ring K^X is equal to* $2^{2^{|X|}}$.

A similar situation happens when one considers the endomorphism ring of an infinite dimensional vector space. If D is a division ring and V_D is an infinite dimensional right vector space over D, we have that $R = End(V_D)$ is a non-commutative ring. If we want to compute its right or left dual Goldie dimension, we need to know its maximal right or left ideals. Following [11] we have that maximal left ideals of R correspond to the ultrafilters of the lattice

$L(V)$ of subspaces of V while maximal right ideals of R correspond to the maximal ideals of $L(V)$.

It is clear that $L(V)$ is a Boolean lattice and therefore we can repeat what we did above and we find that

Proposition 4.5. *Let V be an infinite dimensional vector space over a division ring D and let $R = End(V_D)$ be its endomorphism ring. Then the left dual Goldie dimension of R is equal to the cardinality of the set of ultrafilters of $L(V)$.*

Following [11] we find that

Corollary 4.6. *Let V be a vector space over a division ring D of infinite dimension d. Then the left dual Goldie dimension of $R = End(V)$ is at least 2^{2^d}. Moreover, if $|D| = \alpha \leq d$, the left dual Goldie dimension of R is exactly 2^{2^d}.*

Proof. The first part is clear since it is analogous to what we did above. For the second part, we have to verify that 2^{2^d} is also an upper bound on the cardinality of the set of maximal ideals of R. Just counting the subsets of R, it is easy to deduce that

$$|Max(R)| \leq 2^{(max\{\, d, \alpha\,\}^d)}.$$

If we suppose $d \geq \alpha$, we have $d \geq max\{\, d, \alpha\,\} \Rightarrow 2^d > max\{\, d, \alpha\,\} \Rightarrow 2^d = (2^d)^d \geq (max\{\, d, \alpha\,\})^d$; at this point we can conclude that

$$|Max(R)| \leq 2^{(max\{\, d, \alpha\,\}^d)} \leq 2^{(2^d)}. \qquad \square$$

To see what happens on the right we have to do a further step. We said that maximal right ideals of R correspond to the maximal ideals of $L(V)$, but these are exactly the ultrafilters of the opposite lattice $L(V)^{op}$. Since V is a vector space, there is a lattice isomorphism between $L(V)$ and $L(V)^{op}$ induced by the complement. Therefore we obtain the right version of the previous proposition and the following corollary.

Proposition 4.7. *Let V be an infinite dimensional vector space over a division ring D and let $R = End(V_D)$ be its endomorphism ring. Then the right dual Goldie dimension of R is equal to the cardinality of the set of ultrafilters of $L(V)^{op} \cong L(V)$. Moreover we have $codim(R_R) \leq 2^{2^d}$ and, if $|D| = \alpha \leq d$, equality holds.*

Comparing right and left, we have the following:

Corollary 4.8. *Let V be an infinite dimensional vector space over a division ring D and let $R = End(V_D)$ be its endomorphism ring. Then $codim({}_RR) = codim(R_R)$.*

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