Rend. Istit. Mat. Univ. Trieste
Vol. XL, 117–137 (2009)

Lifts of Structures on Product Manifolds

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Abstract. *This article presents the further steps of the previously done studies taking into consideration the k-th order extensions of a complex manifold. In the previous studies higher order vertical and complete lifts of structures on the complex manifold were introduced. Presently, k-th extended spaces of a product manifold have been set and the higher order vertical, complete, complete- vertical and horizontal lifts of geometric structures on the product manifold have been presented.*

Keywords: Complex, Kaehlerian and Product Manifolds, Extensions, Lifting Theory
MS Classification 2000: 35C15; 20K35; 28A51

1. Introduction

Lifting theory that permits to extend the differentiable structures has also an important role in differential geometry (see [1, 2, 3, 4, 5] and there in). In [2], the structure of extended vector bundles has been obtained, especially the canonical k-th order extended vector bundle π^k of a vector bundle $\pi = (E, \pi, M)$. Extensions of a complex manifold were defined and also higher order vertical, complete and horizontal lifts of complex functions, vector fields, 1-forms and tensor fields of type (1,1) and type (0,2) on any complex manifold to its extension spaces were studied in [3, 4, 5].

The paper is structured as follows. In section 2, we recall extended complex manifolds and higher order vertical and complete lifts of differential elements on any complex manifold to its extension spaces and also extended Kaehlerian manifolds. In section 3, using structures obtained in [2, 3, 4, 5] we define the k-th order extension ${}^kN$ of any product manifold N of dimension $2m+1$. Also we obtain the higher order vertical, complete, complete-vertical and horizontal lifts of functions, vector fields and 1-forms on N to ${}^kN$. Then we find the higher order vertical and complete lifts of tensor field of type (1,1) on N to ${}^kN$.

Throughout the paper, all objects are assumed to be differentiable of class C^∞ and the sum is taken over repeated indices. Unless otherwise stated it will be accepted $0 \leq r \leq k$, $1 \leq i \leq m$. Also, v and c will denote the vertical and complete lifts of geometric structures either from ${}^{k-1}M$ to ${}^kM$ or from ${}^{k-1}N$ to ${}^kN$. The symbol C_j^r called combination is the binomial coefficient $\binom{r}{j}$.

2. Preliminaries

In this section, we recall k-th order extension of a complex manifold and higher order vertical and complete lifts of geometric elements on complex manifold to its extension spaces, and also extended Kaehlerian manifolds given in [3, 4].

2.1. Extended Complex Manifolds

Let M be 2m-real dimensional manifold and ${}^kM$ its k-th order extended manifold. A tensor field J_k on ${}^kM$ is called an *extended almost complex structure* on ${}^kM$ if J_k is endomorphism of the tangent space $T_p({}^kM)$ such that $(J_k)^2 = -I$ at every point p of ${}^kM$. An extended manifold ${}^kM$ with extended almost complex structure J_k is called *extended almost complex manifold*. If $k = 0$, J_0 is called *almost complex structure* and a manifold ${}^0M = M$ with almost complex structure J_0 is said to be *almost complex manifold*.

Let (x^{ri}, y^{ri}) be a real coordinate system on a neighborhood ${}^kU$ of any point p of ${}^kM$. In this situation, we respectively define by $\{\frac{\partial}{\partial x^{ri}}, \frac{\partial}{\partial y^{ri}}\}$ and $\{dx^{ri}, dy^{ri}\}$ the natural bases over $\mathbf{R}$ of tangent space $T_p({}^kM)$ and cotangent space $T_p^*({}^kM)$ of ${}^kM$. The manifold ${}^kM$ is called *extended complex manifold* if there exists an open covering $\left\{{}^kU\right\}$ of ${}^kM$ satisfying the following condition:

There is a local coordinate system (x^{ri}, y^{ri}) on each ${}^kU$ such that for each point of ${}^kU$,

$$J_k\left(\frac{\partial}{\partial x^{ri}}\right) = \frac{\partial}{\partial y^{ri}}, J_k\left(\frac{\partial}{\partial y^{ri}}\right) = -\frac{\partial}{\partial x^{ri}}.$$

If $k = 0$, then the manifold ${}^0M = M$ with almost complex structure J_0 is said to be *complex manifold.* Let $z^{ri} = x^{ri} + \mathbf{i}\, y^{ri}, \mathbf{i} = \sqrt{-1}$ be an extended complex local coordinate system on a neighborhood ${}^kU$ of any point p of ${}^kM$.

Then, one puts:

$$\begin{aligned} \frac{\partial}{\partial z^{ri}} &= \frac{1}{2}\left\{\frac{\partial}{\partial x^{ri}} - \mathbf{i}\frac{\partial}{\partial y^{ri}}\right\}, \\ \frac{\partial}{\partial \overline{z}^{ri}} &= \frac{1}{2}\left\{\frac{\partial}{\partial x^{ri}} + \mathbf{i}\frac{\partial}{\partial y^{ri}}\right\}, \\ dz^{ri} &= dx^{ri} + \mathbf{i} dy^{ri}, \\ d\overline{z}^{ri} &= dx^{ri} - \mathbf{i} dy^{ri}. \end{aligned}$$

Note that $\{\frac{\partial}{\partial z^{ri}}, \frac{\partial}{\partial \overline{z}^{ri}}\}$ and $\{dz^{ri}, d\overline{z}^{ri}\}$ are bases of the tangent space $T_p({}^kM)$ and of the cotangent space $T_p^*({}^kM)$ of ${}^kM$, respectively. The endomorphism J_k is given by

$$J_k\left(\frac{\partial}{\partial z^{ri}}\right) = \mathbf{i}\frac{\partial}{\partial z^{ri}},\ J_k\left(\frac{\partial}{\partial \overline{z}^{ri}}\right) = -\mathbf{i}\frac{\partial}{\partial \overline{z}^{ri}}.$$

If J_k^* is an endomorphism of the cotangent space $T_p^*({}^kM)$ such that $J_k^{*2} = -I$, then it holds

$$J_k^*(dz^{ri}) = \mathbf{i} dz^{ri},\ J_k^*(d\overline{z}^{ri}) = -\mathbf{i} d\overline{z}^{ri}.$$

2.2. Higher Order Lifts of Complex Functions

In this subsection, we give definitions about higher order vertical and complete lifts of complex functions defined on any complex manifold M to k-th order extension ${}^kM$. The *vertical lift* of a function f defined on M to ${}^kM$ is the function f^{v^k} on ${}^kM$ given by the equality:

$$f^{v^k} = f \circ \tau_M \circ \tau_{2M} \circ ... \circ \tau_{k-1M},$$

where $\tau_{k-1_M} :^k M \to^{k-1} M$ is a canonical projection. The *complete lift* of function f to ${}^kM$ is the function f^{c^k} denoted by

$$f^{c^k} = \dot{z}^{ri}\left(\frac{\partial f^{c^{k-1}}}{\partial z^{ri}}\right)^v + \dot{\overline{z}}^{ri}\left(\frac{\partial f^{c^{k-1}}}{\partial \overline{z}^{ri}}\right)^v.$$

Using the induction method, the properties about vertical and complete lifts of complex functions have been extended as follows:

i) $(f+g)^{v^r} = f^{v^r} + g^{v^r}, \qquad (f.g)^{v^r} = f^{v^r}.g^{v^r}$

ii) $(f+g)^{c^r} = f^{c^r} + g^{c^r}, \qquad (f.g)^{c^r} = \sum_{j=0}^{r} \mathrm{C}_j^r f^{c^{r-j}v^j}.g^{c^j v^{r-j}},$

iii) $\left(\frac{\partial f}{\partial z^{0i}}\right)^{v^r} = \frac{\partial f^{c^r}}{\partial z^{ri}}, \qquad \left(\frac{\partial f}{\partial \overline{z}^{0i}}\right)^{v^r} = \frac{\partial f^{c^r}}{\partial \overline{z}^{ri}},$

vi) $\left(\frac{\partial f}{\partial z^{0i}}\right)^{c^r} = \frac{\partial f^{c^r}}{\partial z^{0i}}, \qquad \left(\frac{\partial f}{\partial \overline{z}^{0i}}\right)^{c^r} = \frac{\partial f^{c^r}}{\partial \overline{z}^{0i}},$

where f and g are complex functions, and C_j^r is the combination.

2.3. Higher Order Lifts of Complex Vector Fields

Here, the definitions and propositions about higher order vertical and complete lifts of complex vector fields defined on any complex manifold M to k-th order extension ${}^kM$ are presented. The *vertical lift* of X to ${}^kM$ is the complex vector field X^{v^k} on ${}^kM$ formulated as below:

$$X^{v^k}(f^{c^k}) = (Xf)^{v^k}.$$

Now, we give the local expression of the vertical lift of X to ${}^kM$.

PROPOSITION 2.1. *Let M be any complex manifold and ${}^kM$ its k-th order extension. Consider $X = Z^{0i}\frac{\partial}{\partial z^{0i}} + \overline{Z}^{0i}\frac{\partial}{\partial \overline{z}^{0i}}$. Then the vertical lift of X to ${}^kM$ is*

$$X^{v^k} = (Z^{0i})^{v^k}\frac{\partial}{\partial z^{ki}} + (\overline{Z}^{0i})^{v^k}\frac{\partial}{\partial \overline{z}^{ki}}.$$

The complete lift of X to ${}^kM$ is the complex vector field X^{c^k} such that

$$X^{c^k}(f^{c^k}) = (Xf)^{c^k}.$$

The local expression of the complete lift of X to ${}^kM$ is obtained as follows.

PROPOSITION 2.2. *Let M be any complex manifold and ${}^kM$ its k-th order extension. Let $X = Z^{0i}\frac{\partial}{\partial z^{0i}} + \overline{Z}^{0i}\frac{\partial}{\partial \overline{z}^{0i}}$. Then the complete lift of X to ${}^kM$ is*

$$X^{c^k} = \mathrm{C}_j^r (Z^{0i})^{v^{k-r}c^r}\frac{\partial}{\partial z^{ri}} + \mathrm{C}_j^r (\overline{Z}^{0i})^{v^{k-r}c^r}\frac{\partial}{\partial \overline{z}^{ri}}.$$

The extended properties about vertical and complete lifts of complex vector fields by using the induction method are formulated as follows:

i) $(X+Y)^{v^r} = X^{v^r} + Y^{v^r}, \qquad (X+Y)^{c^r} = X^{c^r} + Y^{c^r},$

ii) $(fX)^{v^r} = f^{v^r}X^{v^r}, \qquad (fX)^{c^r} = \sum_{j=0}^{r} \mathrm{C}_j^r \, f^{c^{r-j}v^j} X^{c^j v^{r-j}},$

iii) $X^{v^k}(f^{v^k}) = 0, \qquad X^{c^k}(f^{c^k}) = (Xf)^{c^k},$
$X^{c^k}(f^{v^k}) = X^{v^k}(f^{c^k}) = (Xf)^{v^k}$

iv) $\left[X^{v^k}, Y^{v^k}\right] = 0, \qquad \left[X^{c^k}, Y^{c^k}\right] = [X,Y]^{c^k},$
$\left[X^{v^k}, Y^{c^k}\right] = \left[X^{c^k}, Y^{v^k}\right] = [X,Y]^{v^k}$

$$\Im_0^1(M) = Sp\left\{\frac{\partial}{\partial z^{0i}}, \frac{\partial}{\partial \overline{z}^{0i}}\right\} \qquad \Im_0^1({}^kM) = Sp\left\{\frac{\partial}{\partial z^{ri}}, \frac{\partial}{\partial \overline{z}^{ri}}\right\},$$

v) $\left(\frac{\partial}{\partial z^{0i}}\right)^{c^r} = \frac{\partial}{\partial z^{0i}}, \qquad \left(\frac{\partial}{\partial \overline{z}^{0i}}\right)^{c^r} = \frac{\partial}{\partial \overline{z}^{0i}},$
$\left(\frac{\partial}{\partial z^{0i}}\right)^{v^r} = \frac{\partial}{\partial z^{ri}}, \qquad \left(\frac{\partial}{\partial \overline{z}^{0i}}\right)^{v^r} = \frac{\partial}{\partial \overline{z}^{ri}}$

2.4. Higher Order Lifts of Complex 1-Forms

This subsection covers definitions and propositions about higher order vertical and complete lifts of complex 1-forms defined on any complex manifold M to k-th order extension ${}^kM$. The *vertical lift* of α to ${}^kM$ is the complex 1-form α^{v^k} on ${}^kM$ defined by

$$\alpha^{v^k}(X^{c^k}) = (\alpha X)^{v^k}.$$

Now, we state a proposition on the vertical lift of α to ${}^kM$.

PROPOSITION 2.3. *Let M be any complex manifold and ${}^kM$ its k-th order extension. Set $\alpha = \alpha_{0i}dz^{0i} + \overline{\alpha}_{0i}d\overline{z}^{0i}$. Then the vertical lift of α to ${}^kM$ is*

$$\alpha^{v^k} = (\alpha_{0i})^{v^k}dz^{0i} + (\overline{\alpha}_{0i})^{v^k}d\overline{z}^{0i}.$$

The complete lift of α to ${}^kM$ is the complex 1-form α^{c^k} on ${}^kM$ defined by

$$\alpha^{c^k}(X^{c^k}) = (\alpha X)^{c^k}.$$

Now, we state a proposition on the complete lift of α to ${}^kM$.

PROPOSITION 2.4. *Let M be any complex manifold and ${}^kM$ its k-th order extended complex manifold. Put $\alpha = \alpha_{0i}dz^{0i} + \overline{\alpha}_{0i}d\overline{z}^{0i}$. Then the complete lift of α to ${}^kM$ is*

$$\alpha^{c^k} = (\alpha_{0i})^{c^{k-r}v^r}dz^{ri} + (\overline{\alpha}_{0i})^{c^{k-r}v^r}d\overline{z}^{ri}.$$

Using the induction method, the extended properties about vertical and complete lifts of complex 1-forms are given as follows:

i) $(\alpha + \lambda)^{v^r} = \alpha^{v^r} + \lambda^{v^r}, \qquad (\alpha + \lambda)^{c^r} = \alpha^{c^r} + \lambda^{c^r},$

ii) $(f\alpha)^{v^r} = f^{v^r}\alpha^{v^r}, \qquad (f\alpha)^{c^r} = \sum_{j=0}^{r} \mathrm{C}_j^r f^{c^{r-j}v^j}\alpha^{c^j v^{r-j}},$

$$\Im_1^0(M) = Sp\left\{dz^{0i}, d\overline{z}^{0i}\right\}, \quad \Im_1^0({}^kM) = Sp\left\{dz^{ri}, d\overline{z}^{ri}\right\},$$

iii) $(dz^{0i})^{c^r} = dz^{ri}, \qquad (d\overline{z}^{0i})^{c^r} = d\overline{z}^{ri},$

$(dz^{0i})^{v^r} = dz^{0i}, \qquad (d\overline{z}^{0i})^{v^r} = d\overline{z}^{0i}$

2.5. Higher Order Lifts of Almost Complex Structure

This subsection presents the definitions about higher order vertical and complete lifts of an almost complex structure defined on any complex manifold M to the extended complex manifold ${}^kM$. The *vertical lifts* of J_0 and J_0^* to ${}^kM$ are respectively the structures $J_0^{v^k}$ and $J_0^{*v^k}$ on ${}^kM$ defined by

$$J_0^{v^k}(X^{c^k}) = (J_0X)^{v^k}, J_0^{*v^k}(\alpha^{c^k}) = (J_0^*\alpha)^{v^k}.$$

The *complete lifts* of J_0 and J_0^* to ${}^kM$ are respectively the structures $J_0^{c^k}$ and $J_0^{*c^k}$ on ${}^kM$ denoted by

$$J_0^{c^k}(X^{c^k}) = (J_0X)^{c^k}, J_0^{*c^k}(\alpha^{c^k}) = (J_0^*\alpha)^{c^k}.$$

2.6. Higher Order Lifts of Hermitian Metric

This subsection gives the definitions about higher order vertical and complete lifts of a Hermitian metric on any complex manifold M to the extension space ${}^kM$. In order to define higher order vertical and complete lifts of a Hermitian metric on M, firstly the definition of higher order vertical and complete lifts of complex tensor fields of type (0,2) is given. The *vertical lift* of G to ${}^kM$ is the tensor field of type (0,2) G^{v^k} on ${}^kM$ formulated as

$$G^{v^k}(X^{c^k}, Y^{c^k}) = (G(X, Y))^{v^k}.$$

Denote by g a Hermitian metric and by J_0 an almost complex structure on any complex manifold M. Since g is a tensor field of type (0,2), we have the equality

$$g^{v^k}(X^{c^k}, Y^{c^k}) = (g(X, Y))^{v^k} = g^{v^k}(J_0^{c^k}X^{c^k}, J_0^{c^k}Y^{c^k}),$$

for any complex vector fields X, Y on M. Hence the *vertical lift* of g to ${}^kM$ is a Hermitian metric g^{v^k} on ${}^kM$. The extended complex manifold ${}^kM$ with Hermitian metric g^{v^k} is called the *vertical lift* of order k of Hermitian manifold (M, J_0, g). The *complete lift* of G to ${}^kM$ is the tensor field of type (0,2) G^{c^k} on ${}^kM$ given by equality

$$G^{c^k}(X^{c^k}, Y^{c^k}) = (G(X, Y))^{c^k}.$$

Also we find the equality

$$g^{c^k}(X^{c^k}, Y^{c^k}) = (g(X,Y))^{c^k} = g^{c^k}(J_0^{c^k} X^{c^k}, J_0^{c^k} Y^{c^k}),$$

for any complex vector fields X, Y on M. Hence the *complete lift* of g to ${}^kM$ is g^{c^k}on ${}^kM$. The extended complex manifold ${}^kM$ with Hermitian metric g^{c^k} is called *complete lift* of order k of Hermitian manifold.

2.7. Higher Order Lifts of Kaehlerian Form

In this part, the higher order vertical and complete lifts of a Kaehlerian form on any complex manifold M to its extension ${}^kM$ are defined. Given a Hermitian manifold (M, J_0, g) since the Kaehlerian form Φ is tensor field of type (0,2), we obtain the equality:

$$\Phi^{v^k}(X^{c^k}, Y^{c^k}) = (\Phi(X,Y))^{v^k} = g^{v^k}(X^{c^k}, J_0^{c^k} Y^{c^k}),$$

for any complex vector fields X, Y on M. Hence the *vertical lift* of Φ to ${}^kM$ is a Kaehlerian form Φ^{v^k}on ${}^kM$. Also we have equality

$$\Phi^{c^k}(X^{c^k}, Y^{c^k}) = (\Phi(X,Y))^{c^k} = g^{c^k}(X^{c^k}, J_0^{c^k} Y^{c^k}),$$

for any complex vector fields X, Y on M. Thus the *complete lift* of Φ to extended complex manifold ${}^kM$ is a Kaehlerian form Φ^{c^k}on ${}^kM$.

2.8. Higher Order Lifts of Kaehlerian Metric

Higher order vertical and complete lifts of a Kaehlerian form associated with any Hermitian manifold M to its extension ${}^kM$ are introduced in this subsection. Also we give definitions about higher order vertical and complete lifts of a Kaehlerian metric defined on M to ${}^kM$.

One needs to specify in the statement that (M, J_0, g) is a Kaehlerian manifold, since only in this case $d\Phi = 0$. In fact, some authors call Kaehlerian form the 2-form Φ associated with any Hermitian manifold.

Let Φ^{v^k} (resp. Φ^{c^k}) its k-th order vertical lift (resp. complete lift). Then we have

$$d\Phi^{v^k} = 0, d\Phi^{c^k} = 0.$$

Since the Kaehlerian form Φ^{v^k} (resp. Φ^{c^k}) on ${}^kM$ is closed, the Hermitian metric g^{v^k} (resp. g^{c^k}) on ${}^kM$ is said to be the *vertical lift* (resp. *complete lift*) of Kaehlerian metric g to ${}^kM$. The extended Hermitian manifold ${}^kM$ with Kaehlerian metric g^{v^k} (resp. g^{c^k}) is called the *vertical lift* (resp. *complete lift*) of order k of Kaehlerian manifold.

3. Product Manifolds and Lifted Structures

From this point onwards the definitions and structures given in [1, 2, 3] can be extended as follows.

Definition 3.1. *Let* ${}^0N = \mathbf{R}\times{}^0M$, ${}^1N = \mathbf{R}\times{}^1M$ *and* ${}^2N = \mathbf{R}\times{}^2M$ *be manifolds. Consider a sequence given by*

$$ {}^0\mathbf{N}\overleftarrow{\tau_{0N}}\ {}^1\mathbf{N}\overleftarrow{\tau_{1N}}\ {}^2\mathbf{N} \qquad (1)$$

where τ_{0N} *and* τ_{1N} *are smooth maps. If the kernel of the map* τ_{0N} *is equal to the image set of the map* τ_{1N}, *then the sequence (1) is said to be a short exact sequence.*

Definition 3.2. *If* N *is a manifold, then a sequence of* N *is a sequence* S *of manifolds and maps determined by sequence*

$$ {}^0\mathbf{N}\overleftarrow{\tau_{0N}}\ {}^1\mathbf{N}\overleftarrow{\tau_{1N}}\ {}^2\mathbf{N}\ \ \ldots \qquad (2)$$

where ${}^0N = N$ *and each short sequence of the sequence (2) is an exact sequence of* N. *If* S *has a last term* ${}^mN$ *then we say that* S *has length* m, *otherwise we say that it has infinite length or length* ∞. *Denote by* $T(\tau_{kN})$ *the differential (tangent functor) of* τ_{kN}, *by* $T({}^kN)$ *the tangent bundle of* ${}^kN$ *and by* τ_{kN} *the natural projection of* $T({}^kN)$ *to* ${}^kN$.

Definition 3.3. *If* N *is a manifold, a sequence* S *of length* $m \leq \infty$ *of* N *yields the following properties:*

i) For each integer $1 \leq k \leq m$ *there exists on imbedding* ${}^{k-1}I{:}{}^kN \rightarrow T({}^{k-1}N)$ *with* ${}^0I$ *onto such that* $\tau_{k-1N} = \theta_{kN}\circ{}^{k-1}I$;

ii) For each integer $1 \leq k \leq m$ *the diagram*

$$\begin{array}{ccc} T({}^kN) & \stackrel{T(\theta_{k-1_N})}{\longrightarrow} & T({}^{k-1}N) \\ \theta_{k_N} \downarrow & & I \downarrow \\ {}^kN & \stackrel{{}^{k-1}I}{\longrightarrow} & T({}^{k-1}N) \end{array}$$

commutes on ${}^kI({}^{k+1}M)$ *exactly.*

In this case S *is said to be an extended sequence of length* m *of* N*, and* ${}^kN = \mathbf{R} \times^k M$ *is called a k-th order extension of product manifold* $N = \mathbf{R} \times M$ *(or* ${}^0N = \mathbf{R} \times {}^0M$*) of dimension* $2m+1$*, where* ${}^kM$ *is an extended complex manifold.*

Let $(t, z^{ri}, \overline{z}^{ri})$ be a coordinate system on a neighborhood ${}^kV$ of any point p of ${}^kN$. Therefore, we respectively define by $\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial z^{ri}}, \frac{\partial}{\partial \overline{z}^{ri}}\right\}$ and $\left\{dt, dz^{ri}, d\overline{z}^{ri}\right\}$ the natural bases over coordinate system of tangent space $T_p({}^kN)$ and cotangent space $T_p^*({}^kN)$ of ${}^kN$.

Let f be a function defined on N and $(t, z^{0i}, \overline{z}^{0i})$ be coordinates of N. So, the 1- form defined by

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial z^{0i}}dz^{0i} + \frac{\partial f}{\partial \overline{z}^{0i}}d\overline{z}^{0i} \tag{3}$$

is the differential of f. Denote by $\chi(N)$ the set of vector fields and by $\chi^*(N)$ the set of dual vector fields on N. In this case, elements Z and ω of $\chi(N)$ and $\chi^*(N)$ are determined by

$$Z = \frac{\partial}{\partial t} + Z^{0i}\frac{\partial}{\partial z^{0i}} + \overline{Z}^{0i}\frac{\partial}{\partial \overline{z}^{0i}} \tag{4}$$

and

$$\omega = dt + \omega_{0i}dz^{0i} + \overline{\omega}_{0i}d\overline{z}^{0i}, \tag{5}$$

respectively, where $Z^{0i}, \omega_{0i}, \overline{Z}^{0i}, \overline{\omega}_{0i} \in \mathcal{F}(M)$.

3.1. Higher Order Lifts of Functions

In this subsection, extensions of definitions and properties about the higher order vertical, complete, complete-vertical and horizontal lifts of functions on product manifold N of dimension $2m+1$ to ${}^{k}N$ are obtained.

Let ${}^{k-1}N$ be the $(k-1)$-th order extension of N and $\tau_{{}^{k-1}N} :{}^{k} N \to{}^{k-1} N$ the natural projection. Consider the linear isomorphism as follows:

$$\begin{array}{llll} v: & \Im_0^0({}^{k-1}N) & \to & \Im_0^0({}^{k}N) \\ & \widetilde{f} & \to & v(\widetilde{f}) = \widetilde{f}^{v}. \end{array} \tag{6}$$

Thus, the *vertical lift* of function $\widetilde{f}$ to ${}^{k}N$ is the function $\widetilde{f}^{v} = \widetilde{f} \circ \tau_{{}^{k-1}N}$.

Let $f^{v^{k-1}}$ be vertical lift of a function f on N to ${}^{k-1}N$. In (6), if $\widetilde{f} = f^{v^{k-1}}$, then the *vertical lift* of function f to ${}^{k}N$ is the function $f^{v^{k}}$ defined by the equality

$$f^{v^{k}} = f \circ \tau_N \circ \tau_{{}^{2}N} \circ ... \circ \tau_{{}^{k-1}N}. \tag{7}$$

Let $\widetilde{f}$ be a function on ${}^{k-1}N$ and $(t, z^{ri}, \overline{z}^{ri}), 0 \le r \le k-1$ be extended coordinates of ${}^{k-1}N$. Therefore, the 1- form defined by the equality

$$d\widetilde{f} = \frac{\partial \widetilde{f}}{\partial t}dt + \frac{\partial \widetilde{f}}{\partial z^{ri}}dz^{ri} + \frac{\partial \widetilde{f}}{\partial \overline{z}^{ri}}d\overline{z}^{ri} \tag{8}$$

is a differential of $\widetilde{f}$.

Suppose that let ι_k be the linear isomorphism such that

$$\iota_k : \Im_1^0({}^{k-1}N) \to \Im_0^0({}^{k}N) \tag{9}$$

$$\iota_k(dt) = t, \iota_k(dz^{ri}) = \dot{z}^{ri},\ \iota_k(d\overline{z}^{ri}) = \dot{\overline{z}}^{ri},$$

where $Sp\left\{dt, dz^{ri}, d\overline{z}^{ri} : 0 \le r \le k-1\right\} = \Im_1^0({}^{k-1}N)$. Taking account of (8), we say that the *complete lift* of the function $\widetilde{f}$ to ${}^{k}N$ is the function $\widetilde{f}^{c}$ defined by

$$\widetilde{f}^{c} = \iota_k(d\widetilde{f}) = t\bigg(\frac{\partial \widetilde{f}}{\partial t}\bigg)^{v} + \dot{z}^{ri}\bigg(\frac{\partial \widetilde{f}}{\partial z^{ri}}\bigg)^{v} + \dot{\overline{z}}^{ri}\bigg(\frac{\partial \widetilde{f}}{\partial \overline{z}^{ri}}\bigg)^{v}. \tag{10}$$

Let $f^{c^{k-1}}$ be complete lift of a function f on N to ${}^{k-1}N$. In (10), if $\widetilde{f} = f^{c^{k-1}}$, the *complete lift* of function f to extended manifold ${}^{k}N$ is the function $f^{c^{k}}$ defined by equality

$$f^{c^{k}} = t\bigg(\frac{\partial f^{c^{k-1}}}{\partial t}\bigg)^{v} + \dot{z}^{ri}\bigg(\frac{\partial f^{c^{k-1}}}{\partial z^{ri}}\bigg)^{v} + \dot{\overline{z}}^{ri}\bigg(\frac{\partial f^{c^{k-1}}}{\partial \overline{z}^{ri}}\bigg)^{v}. \tag{11}$$

Let f^{c^r} be r -th order complete lift of a function f on N to ${}^rN$. Then if it is taken s-th order vertical lift of function f^{c^r} on ${}^rN$ to ${}^kN$, by *complete-vertical lift of order* (r,s) of f on N to ${}^kN$ we call the function $f^{c^r v^s}$determined by

$$(f^{c^r})^{v^s} = f^{c^r v^s} = f^{c^r} \circ \tau_{{}^rN} \circ ... \circ \tau_{{}^{r+s-1}N}, \tag{12}$$

where $0 \leq r, s \leq k$ and $r + s = k$.

There exists commutative property taking into complete-vertical lift of functions. i.e., it means that complete-vertical lifts of order (r,s) and complete-vertical lifts of order (s,r) of functions on N to its extension ${}^kN$ are equal. The *horizontal lift* of f on N to ${}^kN$ is the function f^{H^k} on ${}^kN$ given by

$$f^{H^k} = f^{c^k} - \gamma(\nabla f^{c^{k-1}}), \quad (\gamma(\nabla f^{c^{k-1}}) = \nabla_\gamma f^{c^{k-1}}), \tag{13}$$

where ∇ is an affine linear connection on ${}^{k-1}N$ with local components $\Gamma^{ri}_{rj}, 1 \leq i,j \leq m$, $\nabla f^{c^{k-1}}$ is gradient of $f^{c^{k-1}}$ and γ is an operator given by

$$\gamma : \Im^r_s({}^{k-1}N) \to \Im^r_{s-1}({}^kN).$$

Thus, one has $f^{H^k} = 0$, since

$$\nabla_\gamma f^{c^{k-1}} = \left(\frac{\partial f^{c^{k-1}}}{\partial t}\right)^v + \dot{z}^{ri}\left(\frac{\partial f^{c^{k-1}}}{\partial z^{ri}}\right)^v + \dot{\overline{z}}^{ri}\left(\frac{\partial f^{c^{k-1}}}{\partial \overline{z}^{ri}}\right)^v,$$

where dots mean derivation with respect to time. The generic properties of the higher order vertical, complete and horizontal lifts of functions on N are

i) $(f+g)^{v^r} = f^{v^r} + g^{v^r}, \qquad (f+g)^{H^k} = 0,$
$(f+g)^{c^r} = f^{c^r} + g^{c^r}$

ii) $(f.g)^{v^r} = f^{v^r}.g^{v^r}, \qquad (f.g)^{H^k} = 0,$
$(f.g)^{c^r} = \sum_{j=0}^{r} \mathrm{C}^r_j f^{c^{r-j}v^j}.g^{c^j v^{r-j}}$

iii) $\left(\frac{\partial f}{\partial z^{0i}}\right)^{v^r} = \frac{\partial f^{c^r}}{\partial z^{ri}}, \qquad \left(\frac{\partial f}{\partial \overline{z}^{0i}}\right)^{v^r} = \frac{\partial f^{c^r}}{\partial \overline{z}^{ri}},$
$\left(\frac{\partial f}{\partial t}\right)^{v^r} = \frac{\partial f^{c^r}}{\partial t}$

iv) $\left(\frac{\partial f}{\partial z^{0i}}\right)^{c^r} = \frac{\partial f^{c^r}}{\partial z^{0i}}, \qquad \left(\frac{\partial f}{\partial \overline{z}^{0i}}\right)^{c^r} = \frac{\partial f^{c^r}}{\partial \overline{z}^{0i}},$

$\left(\frac{\partial f}{\partial t}\right)^{c^r} = \frac{\partial f^{c^r}}{\partial t}$

for all $f, g \in \mathcal{F}(N)$.

3.2. Higher Order Lifts of Vector Fields

Extensions of definitions and propositions about higher order vertical, complete, complete-vertical and horizontal lifts of vector fields on product manifold N of dimension $2m+1$ to ${}^kN$ are derived in this subsection.

Let ${}^{k-1}N$ be the $(k-1)$-th order extension of N. Denote by $\widetilde{Z}$ a vector field on ${}^{k-1}N$. Then the *vertical lift* of the vector field $\widetilde{Z}$ to ${}^kN$ is the vector field $\widetilde{Z}^v$ on ${}^kN$ defined by:

$$\widetilde{Z}^v(\widetilde{f}^c) = (\widetilde{Z}\widetilde{f})^v. \tag{14}$$

We denote by $f^{c^{k-1}}$ and $Z^{v^{k-1}}$ the complete lift of a function f and the vertical lift of a vector field Z on N to ${}^{k-1}N$, respectively. In (14), if $\widetilde{f} = f^{c^{k-1}}$ and $\widetilde{Z} = Z^{v^{k-1}}$, then the *vertical lift* of vector field Z on N to ${}^kN$ is the vector field Z^{v^k} on ${}^kN$ given by

$$Z^{v^k}(f^{c^k}) = (Zf)^{v^k}. \tag{15}$$

Proposition 3.4. *Let ${}^kN$ be k-th extension of N. Assume that the vector field Z defined on N is given by (4). Then vertical lift of order k of Z to ${}^kN$ is*

$$Z^{v^k} = \frac{\partial}{\partial t} + (Z^{0i})^{v^k}\frac{\partial}{\partial z^{ki}} + (\overline{Z}^{0i})^{v^k}\frac{\partial}{\partial \overline{z}^{ki}}. \tag{16}$$

Proof. Considering a coordinate system $(t, z^{ri}, \overline{z}^{ri})$ on a neighborhood ${}^kV$ of any point p of ${}^kN$, we put $Z^{v^k} = \frac{\partial}{\partial t} + Z^{ri}\frac{\partial}{\partial z^{ri}} + \overline{Z}^{ri}\frac{\partial}{\partial \overline{z}^{ri}}$. Let f^{c^k} be k-th order complete lift of function f to ${}^kN$. Then from vertical lift properties we can obtain equations

$$Z^{v^k}(f^{c^k}) = \frac{\partial f^{c^k}}{\partial t} + Z^{ri}\frac{\partial f^{c^k}}{\partial z^{ri}} + \overline{Z}^{ri}\frac{\partial f^{c^k}}{\partial \overline{z}^{ri}} \tag{17}$$

and

$$\begin{aligned}(Zf)^{v^k} &= \left(\frac{\partial f}{\partial t} + Z^{0i}\frac{\partial f}{\partial z^{0i}} + \overline{Z}^{0i}\frac{\partial f}{\partial \overline{z}^{0i}}\right)^{v^k} \\ &= \frac{\partial f^{c^k}}{\partial t} + (Z^{0i})^{v^k}\frac{\partial f^{c^k}}{\partial z^{ki}} + (\overline{Z}^{0i})^{v^k}\frac{\partial f^{c^k}}{\partial \overline{z}^{ki}}. \end{aligned} \qquad (18)$$

By (15), (17), (18) one obtains

$$Z^{ri} = 0, \quad \overline{Z}^{ri} = 0, \quad Z^{ki} = (Z^{0i})^{v^k}, \quad \overline{Z}^{ki} = (\overline{Z}^{0i})^{v^k}, \quad 0 \le r \le k-1.$$

Hence, the proof finishes. □

Let ${}^{k-1}N$ be the $(k-1)$-th order extended manifold of N and $\widetilde{Z}$ a vector field on ${}^{k-1}N$. Then, the *complete lift* of $\widetilde{Z}$ to ${}^{k}N$ is defined by:

$$\widetilde{Z}^c(\widetilde{f}^c) = (\widetilde{Z}\widetilde{f})^c, \qquad (19)$$

for any function $\widetilde{f}$ on ${}^{k-1}N$. Let $f^{c^{k-1}}$ and $Z^{c^{k-1}}$ be respectively complete lifts of a function f and a vector field Z defined on N to ${}^{k-1}N$. In (4), if $\widetilde{f} = f^{c^{k-1}}$ and $\widetilde{Z} = Z^{c^{k-1}}$, then the *complete lift* of vector field Z on N to extended manifold ${}^{k}N$ is the vector field Z^{c^k} on ${}^{k}N$ given by

$$Z^{c^k}(f^{c^k}) = (Zf)^{c^k}. \qquad (20)$$

Similar to the proof of Proposition 3.4, one may prove the following:

Proposition 3.5. *Let ${}^{k}N$ be extension of order k of N. Assume that the vector field Z on N is given by (4). Then k-th order complete lift of Z to ${}^{k}N$ is*

$$Z^{c^k} = \frac{\partial}{\partial t} + \mathrm{C}_r^k (Z^{0i})^{v^{k-r}c^r}\frac{\partial}{\partial z^{ri}} + \mathrm{C}_r^k (\overline{Z}^{0i})^{v^{k-r}c^r}\frac{\partial}{\partial \overline{z}^{ri}}. \qquad (21)$$

Let Z be a vector field on manifold N. Then the complete-vertical lift of order (r,s) of Z to ${}^{k}N$ is the vector field $Z^{c^r v^s} \in \chi({}^{k}N)$ determined by equality

$$Z^{c^r v^s}(f^{c^k}) = (Zf)^{c^r v^s}, \qquad (22)$$

where $0 \le r, s \le k$ and $r + s = k$.

PROPOSITION 3.6. *Let N be any product manifold of dimension $2m+1$ and ${}^kN$ its k-th order extended manifold. Consider the vector field Z on N given by (4). Then the complete-vertical lift of order (r,s) of Z to ${}^kN$ is*

$$Z^{c^rv^s}:\begin{pmatrix} 1 \\ \mathrm{C}^r_{k-1}(Z^{0i})^{v^{s+k-l}c^{l-s}} \\ \mathrm{C}^r_{k-1}(\overline{Z}^{0i})^{v^{s+k-l}c^{l-s}} \end{pmatrix}, \qquad 0\le l\le k\,.$$

Proof. Considering a coordinate system $(t, z^{li}, \overline{z}^{li})$ on a neighborhood ${}^kV$ of any point p of ${}^kN$, we put $Z^{c^rv^s}=\frac{\partial}{\partial t}+Z^{li}\frac{\partial}{\partial z^{li}}+\overline{Z}^{li}\frac{\partial}{\partial \overline{z}^{li}}$. Let f^{c^k} be k-th order complete lift of function f to the extended manifold ${}^kN$. Then from complete and vertical lift properties we calculate

$$Z^{c^rv^s}(f^{c^k})=\frac{\partial f^{c^k}}{\partial t}+Z^{hi}\frac{\partial f^{c^k}}{\partial z^{li}}+\overline{Z}^{hi}\frac{\partial f^{c^k}}{\partial \overline{z}^{li}} \tag{23}$$

and

$$\begin{aligned}(Zf)^{c^rv^s} &= \left(\frac{\partial f}{\partial t}+Z^{0i}\frac{\partial f}{\partial z^{0i}}+\overline{Z}^{0i}\frac{\partial f}{\partial \overline{z}^{0i}}\right)^{c^rv^s} \qquad (24)\\ &= \frac{\partial f^{c^k}}{\partial t}+\mathrm{C}^r_h(Z^{0i})^{v^{s+h}c^{r-h}}\frac{\partial f^{c^k}}{\partial z^{k-hi}}\\ &\quad +\mathrm{C}^r_h(\overline{Z}^{0i})^{v^{s+h}c^{r-h}}\frac{\partial f^{c^k}}{\partial \overline{z}^{k-hi}}.\end{aligned}$$

According to (22), by (23) and (24) and also using $l=k-h$ from the following equalities

$$\frac{\partial f^{c^k}}{\partial z^{li}}=\frac{\partial f^{c^k}}{\partial z^{k-hi}} \text{ and } \frac{\partial f^{c^k}}{\partial \overline{z}^{li}}=\frac{\partial f^{c^k}}{\partial \overline{z}^{k-hi}}$$

we have

$$Z^{li}=\mathrm{C}^r_{k-1}(Z^{0i})^{v^{s+k-l}c^{l-s}}, \overline{Z}^{ri}=\mathrm{C}^r_{k-1}(\overline{Z}^{0i})^{v^{s+k-l}c^{l-s}}, 0\le l\le k$$

Hence, the proof is completed. □

There exists commutative property considering complete-vertical lift of vector fields. i.e., it means that complete-vertical lifts of order

(r,s) and complete-vertical lifts of order (s,r) of vector fields on N to its extension ${}^kN$ are equal. The complete-vertical lifts of order (r,s) of vector fields on a manifold N obey the following property

$$(fZ)^{c^r v^s} = \mathrm{C}^r_{k-1} f^{v^{s+h} c^{r-h}} Z^{c^h v^{k-h}},\ 0 \le r,s \le k,\ (r+s=k).$$

The *horizontal lift* of a vector field Z on N to ${}^kN$ is the vector field Z^{H^k} on ${}^kN$ given by

$$Z^{H^k} f^{v^k} = (Zf)^{v^k}. \tag{25}$$

Obviously, we have

$$Z^{H^k} = \frac{\partial}{\partial t} + Z^{ri} D_{ri} + \overline{Z}^{ri} \overline{D}_{ri}$$

such that $D_{ri} = \frac{\partial}{\partial z^{ri}} - \Gamma^{ri}_{rj} \frac{\partial}{\partial z^{r+1i}}$ and $\overline{D}_{ri} = \frac{\partial}{\partial \overline{z}^{ri}} - \overline{\Gamma}^{ri}_{rj} \frac{\partial}{\partial \overline{z}^{r+1i}}, 1 \le i,j \le m$. By an *extended frame* adapted to a connection ∇ on ${}^kN$, we mean the set of local vector fields $\left\{ \frac{\partial}{\partial t}, D_{ri}, \overline{D}_{ri}, V_{ri} = \frac{\partial}{\partial z^{r+1i}}, \overline{V}_{ri} = \frac{\partial}{\partial \overline{z}^{r+1i}} \right\}$. The higher order vertical, complete and horizontal lifts of vector fields on N obey the general properties

i) $(Z+W)^{v^r} = Z^{v^r} + W^{v^r}, \qquad (Z+W)^{c^r} = Z^{c^r} + W^{c^r},$
$(Z+W)^{H^k} = Z^{H^k} + W^{H^k}$

ii) $(fZ)^{v^r} = f^{v^r} Z^{v^r}, \qquad (fZ)^{c^r} = \sum_{j=0}^{r} \mathrm{C}^r_j f^{c^{r-j} v^j} Z^{c^j v^{r-j}},$

iii) $Z^{v^k}(f^{v^k}) = 0, \qquad Z^{c^k}(f^{v^k}) = Z^{v^k}(f^{c^k}) = (Zf)^{v^k},$
$Z^{c^k}(f^{c^k}) = (Zf)^{c^k}, \qquad Z^{H^k}(f^{v^k}) = (Zf)^{v^k}$

iv) $\chi(V) = Sp\left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial z^{0i}}, \frac{\partial}{\partial \overline{z}^{0i}} \right\},\ \chi({}^kV) = Sp\left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial z^{ri}}, \frac{\partial}{\partial \overline{z}^{ri}} \right\},$

$$\left(\frac{\partial}{\partial z^{0i}}\right)^{c^r} = \frac{\partial}{\partial z^{0i}}, \qquad \left(\frac{\partial}{\partial \overline{z}^{0i}}\right)^{c^r} = \frac{\partial}{\partial \overline{z}^{0i}}, \qquad \left(\frac{\partial}{\partial t}\right)^{c^r} = \frac{\partial}{\partial t},$$

$$\left(\frac{\partial}{\partial z^{0i}}\right)^{v^r} = \frac{\partial}{\partial z^{ri}}, \qquad \left(\frac{\partial}{\partial \overline{z}^{0i}}\right)^{v^r} = \frac{\partial}{\partial \overline{z}^{ri}},$$

$$\left(\frac{\partial}{\partial t}\right)^{v^r} = \frac{\partial}{\partial t}, \qquad \left(\frac{\partial}{\partial t}\right)^{H^k} = \frac{\partial}{\partial t},$$
$$\left(\frac{\partial}{\partial z^{0i}}\right)^{H^k} = D_{ri}, \qquad \left(\frac{\partial}{\partial \overline{z}^{0i}}\right)^{H^k} = \overline{D}_{ri},$$

for all $Z, W \in \chi(N)$ and $f \in \mathcal{F}(N)$.

3.3. Higher Order Lifts of 1-Forms

In this subsection, we introduce definitions and propositions about higher order vertical, complete, complete-vertical and horizontal lifts of a 1-form defined on a product manifold N of dimension $2m+1$ to ${}^kN$.

Let ${}^{k-1}N$ be $(k-1)$-th order extension of N. Given a 1-form by $\widetilde{\omega}$ the *vertical lift* of $\widetilde{\omega}$ to ${}^kN$ is the 1-form $\widetilde{\omega}^v$ on ${}^kN$ defined by:

$$\widetilde{\omega}^v(\widetilde{Z}^c) = (\widetilde{\omega}\widetilde{Z})^v, \tag{26}$$

for any vector field $\widetilde{Z}$ on ${}^{k-1}N$. Denote by $Z^{c^{k-1}}$ complete lift of a vector field Z and by $\omega^{v^{k-1}}$ vertical lift of a 1-form ω defined on N to ${}^{k-1}N$. In (26), if $\widetilde{Z} = Z^{c^{k-1}}$ and $\widetilde{\omega} = \omega^{v^{k-1}}$, then the *vertical lift* of 1-form ω on N to ${}^kN$ is the 1-form ω^{v^k} on ${}^kN$ determined by

$$\omega^{v^k}(Z^{c^k}) = (\omega Z)^{v^k}. \tag{27}$$

PROPOSITION 3.7. *Let ω be a 1-form on N locally expressed by (5). Then k-th order vertical lift of ω to ${}^kN$ is given by:*

$$\omega^{v^k} = dt + (\omega_{0i})^{v^k} dz^{0i} + (\overline{\omega}_{0i})^{v^k} d\overline{z}^{0i}. \tag{28}$$

Proof. Locally, we write $\omega^{v^k} = dt + \omega_{ri} dz^{ri} + \overline{\omega}_{ri} d\overline{z}^{ri}$. Let Z^{c^k} be complete lift of a vector field Z to the extended manifold ${}^kN$. Then from vertical lift properties we get

$$\begin{aligned} \omega^{v^k}(Z^{c^k}) &= (dt + \omega_{ri} dz^{ri} + \overline{\omega}_{ri} d\overline{z}^{ri})(Z^{c^k}) \\ &= 1 + \omega_{ri} \mathrm{C}^k_r (Z^{0i})^{v^{k-r}c^r} + \overline{\omega}_{ri} \mathrm{C}^k_r (Z^{0i})^{v^{k-r}c^r} \end{aligned} \tag{29}$$

and

$$\begin{aligned} (\omega Z)^{v^k} &= (dt + \omega_{0i} Z^{0i} + \overline{\omega}_{0i} \overline{Z}^{0i})^{v^k} \\ &= 1 + (\omega_{0i})^{v^k} (Z^{0i})^{v^k} + (\overline{\omega}_{0i})^{v^k} (\overline{Z}^{0i})^{v^k}. \end{aligned} \tag{30}$$

By (27), (29), (30) we have

$$\omega_{ri} = 0, \overline{\omega}_{ri} = 0, \omega_{0i} = (\omega_{0i})^{v^k}, \overline{\omega}_{0i} = (\overline{\omega}_{0i})^{v^k}, 1 \leq r \leq k.$$

Hence, the proof is complete. □

Let ${}^{k-1}N$ be $(k-1)$-th order extension of N. Given by $\widetilde{\omega}$ a 1-form and by $\widetilde{Z}$ a vector field defined on ${}^{k-1}N$. Then, the *complete lift* of 1-form $\widetilde{\omega}$ on ${}^{k-1}N$ to ${}^{k}N$ is the 1-form $\widetilde{\omega}^c$ on ${}^{k}N$ defined by

$$\widetilde{\omega}^c(\widetilde{Z}^c) = (\widetilde{\omega}\widetilde{Z})^c. \tag{31}$$

By $Z^{c^{k-1}}$ and $\omega^{c^{k-1}}$, let denote complete lifts of a vector field Z and a 1-form ω defined on N to ${}^{k-1}N$. In (31), if $\widetilde{Z} = Z^{c^{k-1}}$ and $\widetilde{\omega} = \omega^{c^{k-1}}$, then the *complete lift* of 1-form ω on N to ${}^{k}N$ is the 1-form ω^{c^k} on ${}^{k}N$ given by equality

$$\omega^{c^k}(Z^{c^k}) = (\omega Z)^{c^k}. \tag{32}$$

Similar to the proof of Proposition 3.7, one can prove the following:

PROPOSITION 3.8. *Let ω be a 1-form on N locally expressed by (5). Then k-th order complete lift of ω to ${}^{k}N$ is given by:*

$$\omega^{c^k} = dt + (\omega_{0i})^{c^{k-r}v^r} dz^{ri} + (\overline{\omega}_{0i})^{c^{k-r}v^r} d\overline{z}^{ri}. \tag{33}$$

Let Z be a vector field on manifold N. Then the complete-vertical lift of order (r, s) of $\omega \in \chi^*(N)$ to ${}^{k}N$ is the 1-form $\omega^{c^r v^s} \in \chi^*({}^{k}N)$ given by equality

$$\omega^{c^r v^s}(Z^{c^k}) = (\omega Z)^{c^r v^s}. \tag{34}$$

PROPOSITION 3.9. *Let N be a product manifold of dimension $2m+1$ and ${}^{k}N$ its k-th order extension. Suppose that the 1-form $\omega \in \chi^*(N)$ is given by (5). Then complete-vertical lift of order (r, s) of ω to ${}^{k}N$ is*

$$\omega^{c^r v^s} : \left(1, \frac{\mathrm{C}_l^r}{\mathrm{C}_l^k}(\omega_{0i})^{v^{s+l}c^{r-l}}, \frac{\mathrm{C}_l^r}{\mathrm{C}_l^k}(\overline{\omega}_{0i})^{v^{s+l}c^{r-l}}\right), 0 \leq l \leq k.$$

Proof. Since $\omega^{c^r v^s}$ a 1-form on ${}^kN$, with respect to a coordinate system $(t, z^{li}, \overline{z}^{li})$ one writes $\omega^{c^r v^s} = dt + \omega_{li} dz^{li} + \overline{\omega}_{li} d\overline{z}^{li}$. Let Z^{c^k} be complete lift of order k of vector field Z to ${}^kN$. Then, from complete and vertical lift properties we have

$$\omega^{c^r v^s}(Z^{c^k}) = \{1 + \mathrm{C}_l^r \omega_{li} (Z^{0i})^{v^{k-l} c^l} + \mathrm{C}_l^r \overline{\omega}_{li} (\overline{Z}^{0i})^{v^{k-l} c^l}\} \tag{35}$$

and

$$\begin{aligned} (\omega Z)^{c^r v^s} &= (1 + \omega_{0i} Z^{0i} + \overline{\omega}_{0i} \overline{Z}^{0i})^{c^r v^s} \\ &= 1 + \mathrm{C}_h^r (\omega_{0i})^{v^{s+h} c^{r-h}} (Z^{0i})^{v^{k-h} c^h} \\ &\quad + \mathrm{C}_h^r (\overline{\omega}_{0i})^{v^{s+h} c^{r-h}} (\overline{Z}^{0i})^{v^{k-h} c^h}. \end{aligned} \tag{36}$$

By (34), (35)and (36), using $l = h$ from the following equalities

$$(Z^{0i})^{v^{k-l} c^l} = (Z^{0i})^{v^{k-h} c^h} \quad \text{and} \quad (\overline{Z}^{0i})^{v^{k-l} c^l} = (\overline{Z}^{0i})^{v^{k-h} c^h}$$

we have

$$\omega_{li} = \left(\frac{\mathrm{C}_l^r}{\mathrm{C}_l^k} (\omega_{0i})^{v^{s+l} c^{r-l}} \right), \quad \overline{\omega}_{li} = \left(\frac{\mathrm{C}_l^r}{\mathrm{C}_l^k} (\overline{\omega}_{0i})^{v^{s+l} c^{r-l}} \right), \quad 0 \leq l \leq k$$

Hence, the proof is complete. □

There exists the commutative property for complete-vertical lift of 1-forms. Clearly, it means that complete-vertical lifts of order (r, s) and complete-vertical lifts of order (s, r) of 1-forms on N to its extended manifold ${}^kN$ are equal. The property of complete-vertical lifts of order (r, s) of 1-forms on manifold N is

$$(f\omega)^{c^r v^s} = \mathrm{C}_h^r f^{v^{s+h} c^{r-h}} \omega^{c^h v^{k-h}}, \ 0 \leq r, s \leq k, \ (r + s = k).$$

The *horizontal lift* of a 1-form ω on N to ${}^kN$ is the 1-form ω^{H^k} on ${}^kN$ given by

$$\omega^{H^k}(Z^{H^k}) = 0, \ \omega^{H^k}(Z^{v^k}) = (\omega Z)^{v^k}.$$

Considering $\omega = dt + \omega_{0i} dz^{0i} + \overline{\omega}_{0i} d\overline{z}^{0i}$, we obtain

$$\omega^{H^k} = dt + \omega_{ri} \eta^{ri} + \overline{\omega}_{ri} \overline{\eta}^{ri},$$

where $\eta^{ri} = \overline{d}z^{r+1i} + \Gamma^{ri}_{rj}\overline{d}z^{ri}$, $\overline{\eta}^{ri} = \overline{d}\overline{z}^{r+1i} + \overline{\Gamma}^{ri}_{rj}\overline{d}\overline{z}^{ri}, 1 \leq i,j \leq m$. An *extended coframe* adapted to ∇ on ${}^kN$ is the dual coframe $\left\{dt, \theta^{ri} = dz^{ri}, \overline{\theta}^{\alpha ri} = d\overline{z}^{ri}, \eta^{ri}, \overline{\eta}^{ri}\right\}$. The properties of the higher order vertical, complete and horizontal lifts of 1-forms on N are

i) $(\omega+\lambda)^{v^r} = \omega^{v^r} + \lambda^{v^r}, \qquad (\omega+\lambda)^{c^r} = \omega^{c^r} + \lambda^{c^r},$
$(\omega+\lambda)^{H^k} = \omega^{H^k} + \lambda^{H^k}$

ii) $(f\omega)^{v^r} = f^{v^r}\omega^{v^r}, \qquad (f\omega)^{c^r} = \sum_{j=0}^{r} \mathrm{C}^r_j f^{c^{r-j}v^j}\omega^{c^j v^{r-j}},$

$\omega^{H^k}(Z^{H^k}) = 0, \qquad \omega^{H^k}(Z^{v^k}) = (\omega Z)^{v^k}$

iii) $\chi^*(V) = Sp\left\{dt, dz^{0i}, d\overline{z}^{0i}\right\}, \qquad \chi^*({}^kV) = Sp\left\{dt, dz^{ri}, d\overline{z}^{ri}\right\},$
$(dz^{0i})^{v^r} = dz^{0i}, \qquad (d\overline{z}^{0i})^{v^r} = d\overline{z}^{0i}, \qquad (dt)^{v^r} = dt,$
$(dz^{0i})^{c^r} = dz^{ri}, \qquad (d\overline{z}^{0i})^{c^r} = d\overline{z}^{ri}, \qquad (dt)^{c^r} = dt$
$(dt)^{H^k} = dt, \qquad (dz^{0i})^{H^k} = \eta^{0i}, \qquad (d\overline{z}^{0i})^{H^k} = \overline{\eta}^{0i},$

for all $\omega, \lambda \in \chi^*(N)$ and $f \in \mathcal{F}(N)$.

3.4. Higher Order Lifts of Tensor Fields of Type (1,1)

This subsection studies the extended definitions and properties about higher order lifts of a tensor field of type (1,1) ϕ defined on N to ${}^kN$.

Let ${}^{k-1}N$ be $(k-1)$-th order extension of N. Denote by $\widetilde{\phi} = \phi_{k-1}$ a tensor field of type (1,1), by $\widetilde{\xi}$ a vector field and by $\widetilde{\eta}$ a 1-form defined on ${}^{k-1}N$. Then the *vertical lift* of a tensor field of type (1,1) $\widetilde{\phi}$ to ${}^kN$ is the tensor field $\widetilde{\phi}^v$ such that

$$\widetilde{\phi}^v(\widetilde{\xi}^c) = (\widetilde{\phi}\widetilde{\xi})^v, \widetilde{\eta}^v(\widetilde{\phi}^v) = (\widetilde{\eta}\widetilde{\phi})^v. \tag{37}$$

Now, let $Z^{c^{k-1}}$ and $\phi^{v^{k-1}}$ be respectively complete lift of a vector field $Z \in \chi(N)$ and vertical lift of a tensor field of type (1,1) $\phi \in \Im^1_1(N)$ to ${}^{k-1}N$. In (37), if $\widetilde{\xi} = \xi^{c^{k-1}}, \widetilde{\eta} = \eta^{v^{k-1}}$ and $\widetilde{\phi} = \phi^{v^{k-1}}$, then the *vertical lift* of ϕ to ${}^kN$ is the tensor field ϕ^{v^k} on ${}^kN$ given by

$$\phi^{v^k}(\xi^{c^k}) = (\phi\xi)^{v^k}, \eta^{v^k}(\phi^{v^k}) = (\eta\phi)^{v^k}. \tag{38}$$

Similarly, we define higher order complete lift of a tensor field of type (1,1) ϕ on N. Denote by $\widetilde{\phi}$ a tensor field of type (1,1), by $\widetilde{\xi}$ a vector field and by $\widetilde{\eta}$ a 1-form defined on ${}^{k-1}N$. Then the *complete lift* of $\widetilde{\phi}$ to ${}^{k}N$ is the structure $\widetilde{\phi}^{c} \in \Im_1^1({}^{k}N)$ determined by

$$\widetilde{\phi}^{c}(\widetilde{\xi}^{c}) = (\widetilde{\phi}\widetilde{\xi})^{c}, \widetilde{\eta}^{c}(\widetilde{\phi}^{c}) = (\widetilde{\eta}\widetilde{\phi})^{c}. \tag{39}$$

Presently, let $\xi^{c^{k-1}}, \eta^{c^{k-1}}$ and $\phi^{c^{k-1}}$be respectively complete lifts of a vector field $\xi \in \chi(N)$,a 1-form $\eta \in \chi(N)$ and a tensor field of type (1,1) $\phi \in \Im_1^1(N)$ to ${}^{k-1}N$. In (39), if $\widetilde{\xi} = \xi^{c^{k-1}}$, $\widetilde{\eta} = \eta^{c^{k-1}}$ and $\widetilde{\phi} = \phi^{c^{k-1}}$, then the *complete lift* of $\phi \in \Im_1^1(N)$ to ${}^{k}N$ is the structure ϕ^{c^k} on ${}^{k}N$ given by

$$\phi^{c^k}(\xi^{c^k}) = (\phi\xi)^{v^k}, \eta^{c^k}(\phi^{c^k}) = (\eta\phi)^{c^k}. \tag{40}$$

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Received March 13, 2007
Revised February 7, 2009