

Rank 2 Totally Arithmetically Cohen-Macaulay Vector Bundles on an Abelian Surface with $\text{Num}(X) \cong \mathbb{Z}$

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ABSTRACT. *Here we classify all totally arithmetically Cohen-Macaulay rank 2 vector bundles on an abelian surface X such that $\text{Num}(X) \cong \mathbb{Z}$. They are the extensions of two numerically trivial, but not trivial, line bundles.*

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1. Introduction

Let X be an integral n -dimensional projective variety, $n \geq 2$. Let η_+ or $\eta_+(X)$ denote the ample cone of $\text{Pic}(X)$ and η_- its opposite. Let η_0 (resp. $\tilde{\eta}_0$) denote the set of all line bundles on X algebraically equivalent to \mathcal{O}_X (resp. numerically trivial). Set $\eta := \eta_+ \cup \eta_-$, $\gamma' := \eta \cup \{\mathcal{O}_X\}$, $\gamma := \eta \cup \eta_0$ and $\tilde{\gamma} := \eta \cup \tilde{\eta}_0$. Let E be a vector bundle on X . We will say that E is totally ACM or *totally arithmetically Cohen-Macaulay* (resp. totally WACM or *totally weakly arithmetically Cohen-Macaulay*, resp. totally AACM or *totally algebraically arithmetically Cohen-Macaulay*, resp. totally SACM or *totally strongly arithmetically Cohen-Macaulay*) if $H^i(X, E \otimes L) = 0$ for all $1 \leq i \leq n - 1$ and all $L \in \gamma'$ (resp. $L \in \eta$, resp. $L \in \gamma$, resp. $L \in \tilde{\gamma}$).

If $\text{Pic}(X) \cong \mathbb{Z}$, then the definition of totally ACM vector bundle is very classical, going back at least to Horrocks's splitting criterion

for vector bundles on \mathbf{P}^n . Arithmetically Cohen-Macaulay vector bundles are also classified on smooth quadric surfaces. See [2, 3, 4, 6, 7, 8, 9, 11, 12, 1] and references therein for the classical case of ACM vector bundles with respect to a unique polarization. However, as far as we know, the notion of totally ACM (or variations of it) were not studied when $\text{Pic}^0(X)$ has positive dimension. To avoid any conflict with the well-established definition of ACM vector bundle with respect to a single polarization we were forced by several friends to add the word “totally”. We think that the study of totally WACM or ACM or AACM or SACM vector bundles is very natural. The ample cone $\eta_+(X)$ is a key datum of any variety X . Hence it is very natural to study vector bundles which are ACM in the classical sense with respect to all polarizations. These vector bundles are exactly the totally ACM vector bundles. Among the other notions we think that the more interesting one is WACM. Let E be a totally WACM vector bundle on X . First of all we imposed that for every $H \in \eta_+(X)$ and every $i \in \{1, \dots, n-1\}$ there is an integer $t_{i,H}$ such that $H^i(X, E \otimes H^{\otimes t}) = 0$ for all $t \neq t_{i,H}$. Then we imposed that we may take the same $t_{i,H}$ for all i and all H . When $\omega_X \cong \mathcal{O}_X$ the choice $t_{i,H} = 0$ for all i, H fits well with Serre duality.

Our elementary tools only work for rank 2 vector bundles E on a smooth surface, essentially playing with the exact sequence obtained from a section of E with zero-dimensional zero-locus. As in the classical case our elementary tools work far better if ω_X is negative or trivial outside at most countably many “negative” curves. Here we show that it works if X is an abelian surface with Picard scheme as small as possible for an abelian variety. Let $\text{Num}(X)$ denote the quotient of $\text{Pic}(X)$ by the numerical equivalence. Here we prove the following result.

THEOREM 1.1. *Let X be an abelian surface such that $\text{Num}(X) \cong \mathbb{Z}$.*

- (i) *There is no totally SACM vector bundle on X with rank at most 2.*
- (ii) *A line bundle on X is totally WACM if and only if it is numerically trivial. A rank 2 vector bundle on X is totally WACM if and only if it is an extension of 2 numerically trivial line bundles.*

- (iii) *A rank 2 vector bundle E on X is indecomposable and totally WACM if and only if there is a numerically trivial line bundle N such that E is a non-trivial extension of N by N . The line bundle N is uniquely determined by E . E is totally ACM if and only if $N \neq \mathcal{O}_X$.*
- (iv) *A line bundle on X is totally ACM if and only if it is numerically trivial, but not trivial. A rank 2 vector bundle on X is totally ACM if and only if it is an extension of two numerically trivial, but not trivial, line bundles.*

S. Lefschetz proved that $\text{Num}(X) \cong \mathbb{Z}$ for a general Jacobian X and that the Theta-divisor is the positive generator of $\text{Num}(X)$. In the genus 2 case this is equivalent to say that $\text{Num}(X) \cong \mathbb{Z}$ for the generic complex principally polarized abelian surface. See [13], Prop. 3.4, for more.

Theorem 1.1 is easy to prove, because X is the simplest surface from the point of view of the cohomology of line bundles among all smooth projective surfaces with $h^1(X, \mathcal{O}_X) > 0$ (see Remark 2.3).

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2. The Proof

REMARK 2.1. *Let Y be any integral projective variety. Any extension of two totally ACM (resp. totally WACM, resp. totally AACM, resp. totally SACM) vector bundles on Y is totally ACM (resp. totally WACM, resp. totally AACM, resp. totally SACM). Any direct factor of a totally ACM (resp. totally WACM, resp. totally AACM, resp. totally SACM) vector bundle on Y is totally ACM (resp. totally WACM, resp. totally AACM, resp. totally SACM).*

REMARK 2.2. *Let X be an integral projective surface, $L \in \text{Pic}(X)$ and $Z \subset X$ a zero-dimensional subscheme. Since Z is zero-dimensional, the exact sequence of coherent sheaves on X :*

$$0 \rightarrow \mathcal{I}_Z \otimes L \rightarrow L \rightarrow L|_Z \rightarrow 0 \quad (1)$$

gives $h^1(X, L) \leq h^1(X, \mathcal{I}_Z \otimes L)$ and $h^2(X, \mathcal{I}_Z \otimes L) = h^2(X, L)$. Notice that $h^0(X, L) = h^1(X, \mathcal{I}_Z \otimes L) = 0$ if and only if $h^0(X, L) = h^1(X, L) = 0$ and $Z = \emptyset$ and that if $h^1(X, \mathcal{I}_Z \otimes L) = 0$, then $h^0(X, L) \geq \text{length}(Z)$.

Here we summarize everything we need concerning line bundles on our very easy abelian surface.

REMARK 2.3. Let X be an abelian surface such that $\text{Num}(X) \cong \mathbb{Z}$. For any $M \in \text{Pic}(X)$ let $\beta(M)$ denote the numerical class of M , with the convention that $\beta(M) > 0$ if and only if M is ample and that $\beta : \text{Pic}(X) \rightarrow \mathbb{Z}$ is surjective. Hence a line bundle L on X is ample (resp. numerically trivial) if and only if $\beta(L) > 0$ (resp. $\beta(L) = 0$). Obviously, $\beta(L^*) = -\beta(L)$ and $\beta(M \otimes L) = \beta(M) + \beta(L)$ for all M, L . Hence $\tilde{\gamma} = \text{Pic}(X)$. Since $h^1(X, \mathcal{O}_X) = 2$, no line bundle on X is totally SACM. Fix $R, L \in \text{Pic}(X)$ with L ample. Since $\omega_X \cong \mathcal{O}_X$, Serre duality and Kodaira's vanishing (see [10], p. 150, for the case of an ample line bundle on any abelian variety in arbitrary characteristic) give $h^1(X, L) = h^1(X, L^*) = 0$. Hence R is totally WACM if it is numerically trivial. Conversely, assume that R is not numerically trivial, i.e. assume $\beta(R) \neq 0$. Set $M := R^*$. Since $\beta(R) \neq 0$, either M is ample (case $\beta(R) < 0$) or M^* is ample (case $\beta(R) > 0$). Since $R \otimes M^* \cong \mathcal{O}_X$ and $h^1(X, \mathcal{O}_X) = 2$, R is not totally WACM. In summary, we proved that no line bundle on X is totally SACM and that a line bundle on X is totally WACM if and only if it is numerically trivial. Set $\alpha := R^2$, where R is any line bundle on X such that $\beta(R) = 1$. Fix any integer $t > 0$ and any $L \in \text{Pic}(X)$ such that $\beta(L) = t$. Since $\omega_X \cong \mathcal{O}_X$, Riemann-Roch and Serre duality gives $h^0(X, L) = \alpha t^2 / 2$ and $h^i(X, L) = 0$ for $i = 1, 2$. Hence α is a positive even integer. Since $\tilde{\gamma} = \text{Pic}(X)$, a vector bundle E is SACM if and only if $h^1(X, E \otimes L) = 0$ for all $L \in \text{Pic}(X)$. Hence if E is totally SACM, then $E \otimes M$ is totally SACM for any $M \in \text{Pic}(X)$. Serre duality gives that a vector bundle A is totally WACM (resp. totally ACM, resp. totally SACM) if and only if A^* is totally WACM (resp. totally ACM, resp. totally AACM, resp. totally SACM). Let L be a numerically trivial vector bundle on X . Riemann-Roch gives $\chi(L) = 0$. Hence $h^1(X, L) = 0$ if and only if $h^0(X, L) = h^0(X, L^*) = 0$, i.e. if and only if $L \neq \mathcal{O}_X$. Hence

a line bundle on X is totally ACM if and only if it is numerically trivial, but not trivial.

Proof of Theorem 1.1. The rank 1 case of Theorem 1.1 was proved in Remark 2.3. Hence it is sufficient to check the rank 2 case. Take $\beta : \text{Pic}(X) \rightarrow \mathbb{Z}$ and $\alpha \in 2\mathbb{Z}$, $\alpha > 0$, as in Remark 2.3. Recall that $\tilde{\gamma} = \text{Pic}(X)$ and that L is ample (resp. numerically trivial) if and only if $\beta(L) > 0$ (resp. $\beta(L) = 0$). We also saw that E is totally SACM if and only if $h^1(X, E \otimes L) = 0$ for all $L \in \text{Pic}(X)$ (Remark 2.3). Let E be a totally WACM vector bundle on X . Set $x := \beta(\det(E))$. Since X is smooth, every rank 1 reflexive sheaf on X is locally free ([5], Proposition 1.9). Hence a rank 1 subsheaf A of E is a line bundle if it is saturated in E , i.e. if E/A has no torsion. Hence there is $N \in \text{Pic}(X)$ with an inclusion $N \hookrightarrow E$ such that E/N has no torsion. We take N with the additional property that $z := \beta(N)$ is maximal among all rank 1 saturated subsheaves of E . Since E/N has no torsion and $(E/N)^{**}$ is a line bundle ([5], Proposition 1.9), we get an exact sequence

$$0 \rightarrow N \rightarrow E \rightarrow \mathcal{I}_Z \otimes \det(E) \otimes N^* \rightarrow 0 \quad (2)$$

in which Z is a zero-dimensional locally complete intersection subscheme of X . The maximality of the integer z gives $h^0(X, E \otimes L^*) = 0$ for all $L \in \text{Pic}(X)$ such that $\beta(L) \geq z + 1$. Since Z is zero-dimensional, $h^1(X, \mathcal{I}_Z \otimes L) \geq h^1(X, L)$ and $h^2(X, \mathcal{I}_Z \otimes L) = h^2(X, L)$ for all $L \in \text{Pic}(X)$ (Remark 2.2). We fix E and the exact sequence (2). Hence N and Z are fixed. Set $w := \text{length}(Z)$. Then we fix a general $A \in \text{Pic}^0(X) \cong \widehat{X}$. Since $A \neq \mathcal{O}_X$ and $\mathcal{O}_X \cong \omega_X$, Serre duality gives $h^i(X, A) = 0$ for $i = 0, 2$. Since $A^2 = 0$, Riemann-Roch gives $h^1(X, A) = 0$. Since $\dim(\widehat{A}) = 2 > 0$, for a general $A \in \widehat{X}$ we may (and will) assume $h^1(X, M \otimes A) = 0$ for $i = 0, 1, 2$, and any finite tensor product M of $\det(E)$ and /or copies of N and N^* such that $\beta(M) = 0$.

- (a) Here we assume that either $z \neq 0$ or that E is totally SACM. We tensor (2) with $N^* \otimes A$. Since $h^2(X, A) = 0$, we get $h^1(X, \mathcal{I}_Z \otimes \det(E) \otimes N^* \otimes N^* \otimes A) = 0$. Hence $h^0(X, \det(E) \otimes N^* \otimes N^* \otimes A) \geq w$. Hence either $Z = \emptyset$ or $x - 2z > 0$ and $w \leq \alpha(x - 2z)^2/2$ (Remark 2.2).

(b) Here we assume $z \neq 1$. Fix any $L \in \text{Pic}(X)$ such that $\beta(L) = 1 - z$. Hence $L \in (\eta_+ \cup \eta_-)$. Tensor (2) with L . Since E is totally WACM, $h^1(X, E \otimes L^*) = 0$. Since $\beta(N \otimes L^*) = 1 > 0$, $h^1(X, N \otimes L) = h^2(X, N \otimes L) = 0$. Since $h^2(X, N \otimes L) = h^1(X, E \otimes L) = 0$, (2) gives $h^1(X, \mathcal{I}_Z \otimes \det(E) \otimes N^* \otimes L) = 0$. Since this is true for all L such that $\beta(L) = 1 - z$ and there is $B \in \eta_0$ with $h^0(X, B) = 0$ (just because $h^1(X, \mathcal{O}_X) > 0$), the vanishing of all cohomology groups $H^1(X, \mathcal{I}_Z \otimes \det(E) \otimes N^* \otimes L)$ may occur only if one of the following cases occurs:

(b1) $Z = \emptyset$;

(b2) $x - 2z + 1 > 0$ and $w \leq \alpha(x - 2z + 1)^2/2$.

Notice that if E is assumed to be totally SACM, then taking $L := N^* \otimes A$ we may prove in the same way that either (b1) or (b2) occurs even if $z = 1$. If either $z \notin \{0, 1\}$ or E is totally SACM, then subcase (b2) cannot occur by part (a). Hence $Z = \emptyset$ if either $z \notin \{0, 1\}$ or E is totally SACM. Now assume $z = 0$. Take any $M \in \text{Pic}(X)$ such that $\beta(M) = -1$ and twist (2) with M . Since $\beta(N \otimes M) = -1$, $h^1(X, N \otimes M) = 0$. The maximality of the integer z gives $h^0(X, E \otimes M) = 0$. Hence from (2) we get $h^0(X, \mathcal{I}_Z \otimes \det(E) \otimes N^* \otimes M) = 0$. If $x = 1$, then with a general choice of M (i.e. twisting with a general $A \in \widehat{X}$) we get $h^0(X, \det(E) \otimes N^* \otimes M) = 0$ and hence $Z = \emptyset$. If $x \geq 2$, then we get $w \geq \alpha(x - 1)^2/2$. Since $z \neq 1$, part (a) shows that subcase (b2) cannot occur. In summary, if either $z \neq 1$ or E is totally SACM, then $Z = \emptyset$.

(c) Here we assume $Z = \emptyset$ and either $x \neq 2z$ or $\det(E) \otimes N^* \neq N$. Our assumptions are equivalent to the splitting of any extension of $\det(E) \otimes N^*$ by N . Hence $E \cong N \oplus \det(E) \otimes N^*$. Since no line bundle is totally SACM, E cannot be totally SACM. If E is totally WACM, then its factors must be totally WACM (Remark 2.1). Hence $z = x - z = 0$ and E is a trivial extension of two totally WACM line bundles.

(d) Here we assume that E is totally SACM. Part (b) shows that $Z = \emptyset$. Part (c) gives $x = 2z$ and $\det(E) \otimes N^* \cong N$. Thus $E \otimes N^*$ is an extension of \mathcal{O}_X by \mathcal{O}_X . Since E is totally SACM,

$h^1(X, E \otimes N^*) = 0$. Since $h^2(X, \mathcal{O}_X) = 1$, $h^1(X, \mathcal{O}_X) = 2$ and $h^1(X, E \otimes N^*) = 0$, (2) gives a contradiction, concluding the proof of part (i) of the statement of Theorem 1.1.

- (e) Here we assume $z \notin \{-1, 1\}$. Part (a) gives $Z = \emptyset$. Part (c) gives $x = 2z$ and $\det(E) \otimes N^* \cong N$. If $z = 0$, then E is an extension of two totally WACM line bundles. Assume $z \neq 0$. Since E is totally WACM, $h^1(X, E \otimes N^*) = 0$. Since $h^1(X, \mathcal{O}_X) = 2$, $h^0(X, \mathcal{O}_X) = 1$ and $Z = \emptyset$, (2) gives a contradiction. Hence we checked part (ii) of the statement of Theorem 1.1 if $z \notin \{-1, 1\}$.
- (f) Here we assume $z = -1$. Since $z \notin \{0, 1\}$, part (b) gives $Z = \emptyset$. We may also assume that the extension (2) with $Z = \emptyset$ does not split. Hence $x = -2$ and $\det(E) \otimes N^* \cong N$. Since $z \neq 0$ and E is totally WACM, $h^1(X, E \otimes N^*) = 0$. Hence twisting (2) with N^* and taking the long cohomology exact sequence we get $1 - h^0(X, E \otimes N^*) + 1 - 2 = 0$. Since $h^0(X, E \otimes N^*) > 0$, we get a contradiction.
- (g) Here we assume $z = 1$. We saw in Remark 2.3 that E is totally WACM if and only if E^* is totally WACM. Since $\text{rank}(E) = 2$, $E^* \cong E \otimes \det(E)^*$. Notice that $\beta(N \otimes \det(E)^*) = z - x$ and that $N \otimes \det(E)^*$ is a saturated rank 1 subsheaf of $E \otimes \det(E)^*$ with maximal β . If E^* is an extension of numerically trivial line bundles, then the dual of the exact sequence associated to the extension gives that E is an extension of numerically trivial line bundles. Parts (a), (b), (c), and (e) proved that E^* is an extension of numerically trivial line bundles, unless $z - x = 1$. Hence we may assume $x = 0$. Twist (2) with $B \in \text{Pic}(X)$ such that $\beta(B) = -1$, $B \neq N^* \otimes \omega_X$ and $B \neq N \otimes \det(E)^*$. Since $h^1(X, E \otimes B) = h^2(X, N \otimes B) = 0$, we get $h^1(X, \mathcal{I}_Z \otimes \det(E) \otimes N^* \otimes B) = 0$. Since $\beta(\det(E) \otimes N^* \otimes B) = -2$, we get $Z = \emptyset$ (Remark 2.2). In part (c) we obtained $x = 2z = 2$ and $\det(E) \otimes N^* \cong N^*$. Since $x \neq 0$, we get a contradiction.
- (h) Here we prove part (iii) of the statement of Theorem 1.1, except the assertion concerning a totally ACM vector bundle. Fix numerically trivial line bundles N, M, R on X . Notice

that $N \neq M$ if and only if $\text{Hom}(N, M) \neq \mathcal{O}_X$. We have $h^1(X, R) > 0$ if and only if $R \neq \mathcal{O}_X$ (Remark 2.3). Hence $h^1(X, \text{Hom}(N, M)) = 0$ if and only if $N \neq M$. Hence if $M \neq N$, then any extension of M by N splits and hence it has a decomposable vector bundle as its middle term. We have $h^1(X, \text{Hom}(N, N)) = 2$ (Remark 2.3). Fix any $\epsilon \in H^1(X, \text{Hom}(N, N)) \setminus \{0\}$ and let

$$0 \rightarrow N \rightarrow E \xrightarrow{u} N \rightarrow 0 \quad (3)$$

be the extension determined by ϵ . E is totally WACM (Remark 2.2). Assume that E is decomposable, say $E \cong L_1 \oplus L_2$ with $L_1, L_2 \in \text{Pic}(X)$. Since $u \neq 0$, there is $i \in \{1, 2\}$ such that $u|_{L_i} \neq 0$. Since L_i and N are numerically equivalent and $u|_{L_i} \neq 0$, $u|_{L_i} : L_i \rightarrow N$ is an isomorphism. The inverse of this isomorphism gives a splitting of (3), contradicting the assumption $\epsilon \neq 0$. In a similar way we prove the uniqueness of N .

- (i) Here we prove part (iv) of the statement of Theorem 1.1 and the last sentence of part (iii). The last sentence of Remark 2.3 says that a line bundle on X is totally ACM if and only if it is numerically trivial, but not trivial. Hence Remark 2.1 proves part (iv) for decomposable rank 2 vector bundles. Let E be a rank 2 indecomposable totally WACM vector bundle on X . Write E as an extension of N by N with N numerically trivial. By applying χ to (3) we get $\chi(E) = 2 \cdot \chi(N) = 0$. Hence $h^1(X, E) = 0$ if and only if $h^0(X, E) = h^2(X, E) = 0$. Assume $h^0(X, E) = 0$. The cohomology exact sequence of (3) gives $h^0(X, N) = 0$. Hence N is totally ACM. Hence if E is totally ACM, then it is an extension of two totally ACM line bundles. The converse is obvious (Remark 2.1). Take the set-up of the last sentence of part (iii). We saw using Riemann-Roch that N (resp. E) is totally ACM if and only if $h^0(X, N) = 0$ (resp. $h^0(X, E) = 0$). Since E is an extension of N by N , the cohomology exact sequence of that extension gives $h^0(X, N) \leq h^0(X, E) \leq 2 \cdot h^0(X, N)$, concluding the proof of part (iii).

□

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