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Exponents in $\mathbb{R}$ of Elements in a Uniformly Complete Φ-Algebra

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Abstract. *In this paper we give a new and constructive proof of exponents in $\mathbb{R}$ of elements in a uniformly complete Φ-algebra. As an application we establish the Young's inequality and we give a short proof of the Hölder's inequality on uniformly complete Φ-algebras.*

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1. Introduction

The proofs as well as extensions and applications of the well-known Hölder's inequality can be found in many works about real functions, analysis, functional analysis, L_P-spaces ([5–7, 11]). This inequality involved exponents in the field $\mathbb{R}$ of real numbers. To the best of our knowledge, there is not works dealing with the subject of exponents in the field $\mathbb{R}$ of elements in uniformly complete Φ-algebras, that is uniformly complete f-algebras with an identity element, except the paper of J. L. Krivine [10], which relies heavily on the representation theory and the Axiom of Choice. In this paper we discuss and give a new and purely algebraic proof of exponents in $\mathbb{R}$ of elements in uniformly complete Φ-algebras. As an application we establish the Young's inequality and we give a short proof of the Hölder's inequality on uniformly complete Φ-algebras.

We take it for granted that the reader is familiar with the notions of vector lattices (or Riesz spaces) and operators between them. For terminology, notations and concepts that are not explained in this paper we refer to the standard monographs [1, 9] and [15].

2. Preliminaries

In order to avoid unnecessary repetitions we will suppose that all vector lattices and ℓ-algebras under consideration are **Archimedean**.

Let us recall some of the relevant notions. Let A be a (real) vector lattice. A vector subspace I of A is called *order ideal* (or *o-ideal*) whenever $|a| \leq |b|$ and $b \in I$ imply $a \in I$. Every o-ideal is a vector sublattice of A. The principal o-ideal generated by $0 \leq e \in A$ is denoted by A_e and it is a sublattice of A. An o-ideal I is called *band* if $J \subset I$ and sup $J = x \in A$ implies $x \in I$. For $B \subset A$, B^d denotes the set $\{x \in A, |x| \wedge |y| = 0, \forall y \in B\}$ and B^d is called the *orthogonal band* of B. The set B^{dd} denotes $\left(B^d\right)^d$ and called the *band generated* by B and B is called *order dense* in A if $B^{dd} = A$.

Let A be a vector lattice, let $0 \leq v \in A$, the sequence $\{a_{n,}\ n = 1, 2, \ldots\}$ in A is called (v) *relatively uniformly convergent* to $a \in A$ if for every real number $\varepsilon > 0$, there exists a natural number n_ε such that $|a_n - a| \leq \varepsilon v$ for all $n \geq n_\varepsilon$. This will be denoted by $a_n \to a\ (v)$. If $a_n \to a\ (v)$ for some $0 \leq v \in A$, then the sequence $\{a_{n,}\ n = 1, 2, \ldots\}$ is called *(relatively) uniformly convergent* to a, which is denoted by $a_n \to a(r.u)$. The notion of (v) *relatively uniformly Cauchy* sequence is defined in the obvious way. A vector lattice is called *relatively uniformly complete* if every relatively uniformly Cauchy sequence in A has a unique limit. Relatively uniformly limits are unique if A is archimedean, see [9, Theorem 63.2].

A linear mapping T defined on a vector lattice A with values in a vector lattice B is called *positive* if $T(A^+) \subset B^+$ (notation $T \in \mathcal{L}^+(A,B)$ or $T \in \mathcal{L}^+(A)$ if $A = B$).

The next paragraph of this section deals with some facts about Dedekind complete vector lattices. The vector lattice A is called *Dedekind complete* if for each non-void majorized set $B \subset A$, sup B exists in A. Every vector lattice A has a *Dedekind completion* $A^\wedge$, this means that there exists a Dedekind complete vector lattice $A^\mathfrak{d}$ containing A as a vector sublattice and such that

$$x' = \sup\left\{x \in A, x \leq x'\right\} = \inf\left\{x \in A, x \geq x'\right\}$$

holds for each $x' \in A^\mathfrak{d}$. By these properties, $A^\mathfrak{d}$ is determined uniquely to within lattice isomorphism. As a consequence, we deduce that A is *coinitial* in $A^\mathfrak{d}$, that is, for all $0 < x' \in A^\mathfrak{d}$ there exists

$0 < x \in A$ such that $0 < x \leq x'$ and A is *cofinal* in $A^{\mathfrak{d}}$, that is, for all $0 \leq y' \in A^{\mathfrak{d}}$ there exists $0 \leq y \in A$ such that $0 < y' \leq y$. For more about this concept, see [9, chapter IV].

In next lines, we recall definitions and some basic facts about f-algebras. For more information about this field, we refer the reader to [1, 3, 4]. A (real) algebra A which is simultaneously a vector lattice such that the partial ordering and the multiplication in A are compatible, so $a, b \in A^+$ implies $ab \in A^+$ is called *lattice-ordered algebra*(briefly *ℓ-algebra*). In an ℓ-algebra A we denote the collection of all nilpotent elements of A by *N(A)*. An ℓ-algebra A is referred to be *semiprime* if $N(A) = \{0\}$. An ℓ-algebra A is called an *f-algebra* if A verifies the property that $a \wedge b = 0$ and $c \geq 0$ imply $ac \wedge b = ca \wedge b = 0$. Any f-algebra is automatically commutative and has positive squares. An f-algebra with an identity element is called Φ-algebra. Every unital f-algebra (i.e., an f-algebra with a unit element) is semiprime. Let A be an f-algebra with unit element e, then for every $0 \leq f \in A$, the increasing sequence $0 \leq f_n = f \wedge ne$ converges (relatively) uniformly to f in A (for details, see e.g., [1, Theorem 8.22]).

3. The Main Results

Let A be a uniformly complete Φ-algebra. It was proven by F. Beukers and C. B. Huijsmans [3] that for every $f \in A_+$ and $n \in \mathbb{N}$, there exists a unique $g \in A_+$ such that $g^n = f$. This element g is called the *nth-root* of f and it is denoted by $f^{\frac{1}{n}}$. It follows easily that for every $f \in A$, the p-power f^p is well defined for every non-negative rational p. Of course, f^p also is defined for a negative p provided that f has an inverse in A. The uniqueness of nth-roots in A together with the commutativity of A guarantees the validity of classical products rules such as $f^p f^q = f^{p+q}$, $(f^p)^q = f^{pq}, \ldots$.

Next we will translate all this notion for the general setting $p \in \mathbb{R}$. In order to hit this mark, we need some prerequisites.

The following proposition is proved by K. Boulabiar [5, Lemma 2], but in order to make this paper self contained, we produce a new approach.

Proposition 3.1. *Let A be a uniformly complete Φ-algebra with e*

as a unit element and let p *be a rational such that* $0 \le p \le 1$. *Then*

$$f^p \le pf + (1-p)\, e$$

for all $f \in A_+$.

Proof. Let $p = \frac{b}{a}$ be a rational such that $0 \le p < 1$ and let $f \in A_+$ such that $f \le me$. Let us define the following sequence

$$f_n = \inf_{\alpha=\frac{k}{n},\ 1\le k\le n} \left\{ \alpha^{-\frac{1}{a}} \left(\frac{b}{a} f + \alpha \left(\frac{a-b}{a} \right) e \right) \right\}. \tag{1}$$

This sequence will turn out to be the natural approximating Cauchy sequence for $f^{\frac{a}{b}}$. We claim that

$$0 \le (f_n)^a - f^b \le \frac{C}{n} g$$

for some $0 \le g \in A$, which means that $(f_n)^a \to f^b$ $(r.u)$, hence $f_n \to f^{\frac{b}{a}}$ $(r.u)$. Indeed,

$$\begin{aligned} (f_n)^a - f^b &= \inf_{\alpha=\frac{k}{n},\ 1\le k\le n} \left\{ \alpha^{-1} \left(\left(\frac{b}{a} f + \alpha \left(\frac{a-b}{a} \right) e \right)^a - \alpha f^b \right) \right\} \\ &= a^{-a} \inf_{\alpha=\frac{k}{n},\ 1\le k\le n} \left\{ \alpha^{-1} \left((bf + \alpha\,(a-b)\, e)^a - a^a \alpha f^b \right) \right\}. \end{aligned}$$

Put $P(f, \alpha e) = (bf + \alpha\,(a-b)\, e)^a - a^a \alpha e f^b$, which is an inhomogeneous polynomial in f and αe. Consider the corresponding inhomogeneous polynomial

$$P(X,1) = (bX + (a-b))^a - a^a X^b \in \mathbb{R}[X].$$

Since $P(1) = P'(1) = 0$, we have $P(X) = (X-1)^2\, G(X)$ for some $G(X) \in \mathbb{R}[X]$. In other words

$$P(f, \alpha e) = (f - \alpha e)^2\, G(f, \alpha e).$$

Moreover

$$G(f, \alpha e) = \beta_0 f^{a-2} + \beta_1 f^{a-2}(\alpha e) + \ldots + \beta_{a-2}(\alpha e)^{a-2}.$$

We remark that $\beta_i \in \mathbb{R},\ 0 \leq i \leq a-2$ and β_i does not depend on α. Since $P(f,\alpha e) = (f-\alpha e)^2 G(f,\alpha e)$, then $|P(f,\alpha e)| = (f-\alpha e)^2 |G(f,\alpha e)|$. Moreover,

$$G(f,\alpha e) \leq |\beta_0| f^{a-2} + |\beta_1| f^{a-2}e + \ldots + |\beta_{a-2}| e^{a-2}$$

and

$$-G(f,\alpha e) \leq |\beta_0| f^{a-2} + |\beta_1| f^{a-2}e + \ldots + |\beta_{a-2}| e^{a-2}.$$

Hence

$$|G(f,\alpha e)| \leq g$$

where $g = |\beta_0| f^{a-2} + |\beta_1| f^{a-2}e + \ldots + |\beta_{a-2}| e^{a-2}$.
We recall that by [4, Proposition 4.1]

$$\inf_{\alpha=\frac{k}{n},\ 1\leq k\leq n} \alpha^{-1}(f-\alpha e)^2 \leq n\frac{1}{n^2}e^2 = \frac{1}{n}e.$$

Therefore

$$\left|(f_n)^a - f^b\right| = |P(f,\alpha e)| \leq a^{-a}\frac{1}{n}g.$$

Hence the sequence $\left\{(f_n)^a - f^b\right\}_{n=1}^{\infty}$ is a uniformly Cauchy sequence in A, which converges to 0. Thus $(f_n)^a \to f^b$ $(r.u)$, hence $f_n \to f^{\frac{b}{a}}$ $(r.u)$. By the formula (1), we deduce that

$$f_n \leq \left(\frac{b}{a}f + \left(\frac{a-b}{a}\right)e\right)$$

and then, as $n \to \infty$

$$f^{\frac{b}{a}} \leq \left(\frac{b}{a}f + \left(\frac{a-b}{a}\right)e\right). \tag{2}$$

Now let $0 \leq f \in A$. Since $f \wedge me \to f$ $(r.u)$, see [1,Theorem 8.22], it follows that

$$(f \wedge me)^{\frac{b}{a}} \leq \left(\frac{b}{a}(f \wedge me) + \left(\frac{a-b}{a}\right)e\right).$$

As $m \to \infty$, then

$$f^{\frac{b}{a}} \leq \left(\frac{b}{a} f + \left(\frac{a-b}{a}\right) e\right)$$

and we are done. □

PROPOSITION 3.2. *Let A be a uniformly complete Φ-algebra with e as a unit element, let p be a real number such that $0 \leq p \leq 1$ and let $(p_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive rationals such that $p_n \to p$. Then, for all $f \in A_+$ such that $f \geq e$, the sequence $\left(f^{p_n}\right)_{n\in\mathbb{N}}$ is a* uniformly Cauchy *sequence and hence it converges in A.*

Proof. If $f \geq e$ then f has an inverse in A, then f^{p_n} has an inverse in A. Since $(p_n)_{n\in\mathbb{N}}$ is sequence of positive rationals such that $p_n \to p \leq 1$, then $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. It follows that, for all real $0 < \varepsilon < 1$, there exists $n_0 \in \mathbb{N}$ such that, $0 \leq p_n - p_m < \varepsilon$, for all $n > m \geq n_0$. Therefore

$$|f^{p_n} - f^{p_m}| = |f^{p_m}| \left|f^{p_n - p_m} - e\right|.$$

By the previous proposition

$$f^{p_n - p_m} \leq (p_n - p_m) f + (1 - (p_n - p_m)) e,$$

then

$$0 \leq f^{p_n - p_m} - e \leq (p_n - p_m)(f - e) \leq \varepsilon (f - e). \tag{3}$$

Moreover since $f \geq e$ and $p_m \leq 1$ then $f^{p_m} \leq f$. Hence

$$|f^{p_n} - f^{p_m}| \leq \varepsilon f (f - e).$$

Therefore the sequence $\left(f^{p_n}\right)_{n\in\mathbb{N}}$ is a uniformly Cauchy sequence and hence it converges in A. □

REMARK 3.3. *Let $0 \leq f \leq e$ let p be a real number such that $0 \leq p \leq 1$ and let $(p_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive rationals such that $p_n \to p$. In general the sequence $\left(f^{p_n}\right)_{n\in\mathbb{N}}$ does not converge uniformly to e. This is illustrated by the following example.*

EXAMPLE 3.4. *Take A the coordinatewise ordered space $\mathbb{R}^2$. The vector lattice A can be supplied with the following multiplication (denoted by $\bullet$):*

$$a \bullet b = (a_1 b_1, a_2 b_2)$$

for $a = (a_1, a_2) \in \mathbb{R}^2$ and $b = (b_1, b_2) \in \mathbb{R}^2$. A straightforward calculation shows that A is an f-*algebra with $e = (1,1)$ as a unit element. Take $e_1 = (1,0)$, then $(e_1)^{\frac{1}{n}} = e_1$. Therefore $(e_1)^{\frac{1}{n}} \to e_1$ $(r.u)$.*

Before continuing with the next result, we recall the following notion.
Let A be a vector lattice and let $0 \leq a \in A$. An element $0 \leq e \in A$ is called a *component* of a if $e \wedge (a - e) = 0$.

PROPOSITION 3.5. *Let A be a uniformly complete Φ-algebra with e as a unit element and let $(p_k)_{k\in\mathbb{N}}$ be an increasing sequence of positive rationals such that $p_k \to p$. Then, for all $f \in A_+$ such that $f \leq e$, the sequence $(f^{p_k})_{k\in\mathbb{N}}$ is a* uniformly Cauchy *sequence and hence it converges in A.*

Proof. Without loss of generality, by the previous theorem, we can assume that A is a Dedekind complete vector lattice, since the multiplication of Φ-algebra can be extended to the Dedekind completion of A in such a manner the Dedekind completion of A becomes a Φ-algebra. By using the Freudenthal spectral Theorem [9, Theorem 40.2.], there exists a sequence $\{s_n : n = 1, 2 \ldots\}$ in A which satisfies $0 \leq s_n \nearrow f\ (u)$, where each element s_n is of the form

$$\sum_{1\leq i\leq n} \alpha_i e_i$$

with real numbers $\alpha_1, \ldots, \alpha_n$ such that $0 \leq \alpha_i \leq 1$ $(i = 1, \ldots, n)$ and mutually disjoint components $e_1, \ldots, e_n$ of e. Therefore

$$\sum_{1\leq i\leq n} \alpha_i e_i \to f\,(r.u)\,.$$

Hence

$$\left(\sum_{1\leq i\leq n} \alpha_i e_i\right)^{p_k} \to f^{p_k}\,(r.u)\,.$$

Since $e_1, \ldots, e_n$ are mutually disjoint components of e, then $(e_i)^{p_k} = e_i$ and

$$\left(\sum_{1\leq i\leq n} \alpha_i e_i\right)^{p_k} = \sum_{1\leq i\leq n} (\alpha_i)^{p_k} e_i.$$

It follows that

$$\sum_{1\leq i\leq n} (\alpha_i)^{p_k} e_i \rightarrow f^{p_k} (r.u).$$

Since $0 \leq \alpha_i \leq 1$ $(i = 1, \ldots, n)$, for all $\varepsilon > 0$, there exits $k_0 \in \mathbb{N}$ such that for all $k_0 \leq k, k' \in \mathbb{N}$, we have $|(\alpha_i)^{p_k} - (\alpha_i)^{p_{k'}}| < \varepsilon$. It follows that

$$\begin{aligned} |f^{p_k} - f^{p_{k'}}| &= \left|\lim_n \sum_{1\leq i\leq n} (\alpha_i)^{p_k} e_i - \lim_n \sum_{1\leq i\leq n} (\alpha_i)^{p_{k'}} e_i\right| \\ &\leq \lim_n \sum_{1\leq i\leq n} |(\alpha_i)^{p_k} - (\alpha_i)^{p_{k'}}| \, e_i \\ &\leq \varepsilon \lim_n \sum_{1\leq i\leq n} e_i \\ &\leq \varepsilon e. \end{aligned}$$

Hence he sequence $(f^{p_k})_{k\in\mathbb{N}}$ is a *uniformly Cauchy* sequence and hence it converges in A. □

Corollary 3.6. *Let A be a uniformly complete Φ-algebra with e as a unit element, let p be a real number such that $0 \leq p \leq 1$ and let $(p_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive rationals such that $p_n \rightarrow p$. Then, for all $f \in A_+$, the sequence $(f^{p_n})_{n\in\mathbb{N}}$ is a* uniformly Cauchy *sequence and hence it converges in A.*

Proof. If $0 \leq f \in A$, then $f = (f \wedge e)(f \vee e)$. Let $f_1 = (f \wedge e)$ and $f_2 = (f \vee e)$. Then for all real $0 < \varepsilon < 1$, there exists $n_0 \in \mathbb{N}$ such that $0 \leq p_n - p_m < \varepsilon$, for all $n > m \geq n_0$. Since $(f_1)^{p_m}, (f_1)^{p_n} \leq e$, therefore

$$\begin{aligned} |f^{p_n} - f^{p_m}| &= |(f_1 f_2)^{p_n} - (f_1 f_2)^{p_m}| \\ &= (f_2)^{p_m} \left|(f_1)^{p_n} (f_2)^{p_n - p_m} - (f_1)^{p_m}\right| \\ &\leq f_2 \left|(f_1)^{p_n} (f_2)^{p_n - p_m} - (f_1)^{p_m}\right| \end{aligned}$$

$$\begin{aligned}
&= f_2 \left|(f_1)^{p_n} (f_2)^{p_n - p_m} - (f_1)^{p_n} + (f_1)^{p_n} - (f_1)^{p_m}\right| \\
&\leq f_2 \left|(f_1)^{p_n} (f_2)^{p_n - p_m} - (f_1)^{p_n}\right| + f_2 \left|(f_1)^{p_n} - (f_1)^{p_m}\right| \\
&= f_2 (f_1)^{p_n} \left|(f_2)^{p_n - p_m} - e\right| + f_2 \left|(f_1)^{p_n} - (f_1)^{p_m}\right| \\
&\leq f_2 (f_1) \left|(f_2)^{p_n - p_m} - e\right| + f_2 \left|(f_1)^{p_n} - (f_1)^{p_m}\right| \\
&\leq f_2 \left|(f_2)^{p_n - p_m} - e\right| + f_2 \left|(f_1)^{p_n} - (f_1)^{p_m}\right|.
\end{aligned}$$

By the first step (inequality 3),

$$\left|(f_2)^{p_n - p_m} - e\right| \leq \varepsilon (f_2 - e)$$

and by the previous proposition

$$\left|(f_1)^{p_n} - (f_1)^{p_m}\right| \leq \varepsilon e.$$

Hence the sequence $(f^{p_n})_{n\in\mathbb{N}}$ is a uniformly Cauchy sequence and hence it converges in A, which gives the desired result. □

Since any sequence $(p_n))_{n\in\mathbb{N}}$ of positive rationals such that $p_n \to p$, is a Cauchy sequence. It follows that, for all real $0 < \varepsilon < 1$, there exists $n_0 \in \mathbb{N}$ such that, $0 \leq |p_n - p_m| < \varepsilon$, for all $n > m \geq n_0$. Since also $\mathbb{R}$ is totally ordered, we can assume that $|p_n - p_m| = p_n - p_m$ or $|p_n - p_m| = p_m - p_n$. Hence we deduce the following result.

Corollary 3.7. *Let A be a uniformly complete Φ-algebra with e as a unit element, let p be a real number such that $0 \leq p \leq 1$ and let $(p_n)_{n\in\mathbb{N}}$ be a sequence of positive rationals such that $p_n \to p$. Then, for all $f \in A_+$, the sequence $(f^{p_n})_{n\in\mathbb{N}}$ is a* uniformly Cauchy *sequence and hence it converges in A.*

Thus we have gathered all prerequisites to the first main result of this paper.

Theorem 3.8. *Let A be a uniformly complete Φ-algebra with e as a unit element and let p be a real number. Then the element f^p is* well defined *in A, for all $f \in A_+$.*

Proof. Let $f \in A_+$. If $p \geq 0$ let us take $E(p)$ be the entire part of p. Then $p = E(p) + r$, where $0 \leq r \leq 1$. Then there exists a sequence $(r_n)_{n\in\mathbb{N}}$ of positive rationals such that $r_n \to r$. Therefore, by the previous result, $(f^{r_n})_{n\in\mathbb{N}}$ is a uniformly Cauchy sequence and hence it converges to a positive element g in A. Moreover, since $f^{r_n+E(p)} = f^{r_n} f^{E(p)}$ and since $r_n + E(p) \to p$, then we define f^p as $f^r f^{E(p)}$. The case where $p < 0$, this means that f has an inverse f^{-1} in A. Then we define f^p as $\left(f^{-1}\right)^{-p}$ and the proof is complete. $\square$

The classical Hölder's inequality states: Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_p(\mu)$ and $g \in L_q(\mu)$, then $fg \in L_1(\mu)$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q .$$

In [5], K. Boulabiar proved that if A is a uniformly complete Φ-algebra, if T is a positive linear functional and if p, q are rational numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$T(|fg|) \leq (T(|f|^p))^{\frac{1}{p}} (T(|g|^q))^{\frac{1}{q}}$$

for all $f, g \in A$. The proof relies heavily on a restricted Young's inequality on Φ-algebras ($fg \leq \frac{1}{p} f^p + \frac{1}{q} f^q$, p, q are rational numbers such that $\frac{1}{p} + \frac{1}{q} = 1$). Moreover, in [5] the author remarks that one can deduce the Young's inequality in more general setting, that is for all p, q such that $\frac{1}{p} + \frac{1}{q} = 1$, via Krivine's approach [10], which relies heavily on representation theory and the Axiom of Choice. In the sequel we will give a constructive approach to establish the Young's inequality.

Proposition 3.9. *Let A be a uniformly complete Φ-algebra with e as a unit element and let $p \in \mathbb{R}$ such that $0 \leq p \leq 1$. Then*

$$f^p \leq pf + (1-p)e$$

for all $f \in A_+$.

Proof. Let $p \in \mathbb{R}$ such that $0 \leq p \leq 1$, then there exists a sequence $(p_n)_{n\in\mathbb{N}}$ of positive rationals such that $p_n \to p$ and $0 \leq p_n \leq 1$. Hence by Proposition 3.1, for all $f \in A_+$

$$f^{p_n} \leq p_n f + (1-p_n)e.$$

Since $f^{p_n} \to f^p(r.u)$, $p_n f \to pf\,(r.u)$ and $(1-p_n)\,e \to (1-p)\,e$ $(r.u)$, it follows that

$$f^p \leq pf + (1-p)\,e$$

for all $f \in A_+$, as required. □

All the preparations are done for our first main result in this paper.

THEOREM 3.10. *(Young's inequality) Let A be a uniformly complete Φ-algebra with e as A unit element and let p,q be real numbers such that $1 < p, q$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$fg \leq \frac{1}{p} f^p + \frac{1}{q} g^q$$

for all $f, g \in A_+$.

Proof. Let p, q be real numbers such that $1 < p, q$, $\frac{1}{p} + \frac{1}{q} = 1$. Then there exist two sequence $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ of positive rationals such that $1 < p_n \to p$, $1 < q_n \to q$ and $\frac{1}{p_n} + \frac{1}{q_n} = 1$. We assume that g has an inverse in A. Then, by the previous proposition

$$fg^{-\frac{q_n}{p_n}} \leq \frac{1}{p_n} f^{p_n} g^{-q_n} + \frac{1}{q_n} e.$$

Multiplying this inequality by g^{q_n} we have

$$fg^{q_n - \frac{q_n}{p_n}} \leq \frac{1}{p_n} f^{p_n} + \frac{1}{q_n} g^{q_n}.$$

Since $\frac{1}{p_n} + \frac{1}{q_n} = 1$, then $q_n - \frac{q_n}{p_n} = 1$. Hence

$$fg \leq \frac{1}{p_n} f^{p_n} + \frac{1}{q_n} g^{q_n}.$$

Moreover since $f^{p_n} \to f^p(r.u)$ and $g^{q_n} \to g^q(r.u)$, then

$$fg \leq \frac{1}{p} f^p + \frac{1}{q} g^q. \tag{4}$$

Now let $g \in A_+$, by the Birkhoff's inequality the sequence $\left(g \vee \frac{e}{n}\right)_{n\in\mathbb{N}}$ is uniformly convergent to g. Moreover $g \vee \frac{e}{n} \geq \frac{e}{n}$, then $g \vee \frac{e}{n}$ has an inverse in A. It follows, by the inequality (4), that

$$f\left(g \vee \frac{e}{n}\right) \leq \frac{1}{p}f^p + \frac{1}{q}\left(g \vee \frac{e}{n}\right)^q.$$

Now since $g \vee \frac{e}{n} \to g(r.u)$ then $f\left(g \vee \frac{e}{n}\right) \to fg(r.u)$ and $\left(g \vee \frac{e}{n}\right)^q \to g^q(r.u)$. By the way

$$fg \leq \frac{1}{p}f^p + \frac{1}{q}g^q,$$

which gives the desired result. □

The classical Cauchy-Schwartz's inequality states:If $f, g \in L_2(\mu)$ then $fg \in L_1(\mu)$ and

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2 .$$

Hence, we check that the Cauchy-Schwartz's inequality is a natural deduction from the Hölder's inequality (let us take $p = 2$).

There are known equivalences of Lyapunov's inequality to the Hölder's inequality ([12, pp. 457–462]) but it is interesting that the Hölder's inequality is also equivalent to the Cauchy-Schwartz's inequality. This observation follows implicitly from the announcement by A. W. Marshall and I. Olkin [12, pp. 457] and in Y. C. Li and S. Y. Shaw [11], it was proven that the Cauchy-Schwartz's and the Hölder's inequalities are equivalent on L_P-spaces. Next, we will give a constructive and direct manner to prove the Hölder's inequality by using the Cauchy-Schwartz's inequality. We indebted ourselves to Y. C. Li and S. Y. Shaw [11]. In order to reach our aim, we need, the following result.

Lemma 3.11. *Let A be a uniformly complete Φ-algebra with e as a unit element, let p be a real number, let $f \in A_+$ and let $T : A \to \mathbb{R}$ is a positive functional. Then, the following map $\Psi : \mathbb{R}_+ \to \mathbb{R}$, $p \mapsto T(f^p)$ is continuous.*

Proof. Let $f \in A_+$ and let $p \in \mathbb{R}_+$ then there exists a sequence $(p_n)_{n\in\mathbb{N}}$ of positive real numbers such that $p_n \to p$. A simple deduction from Theorem 3.8, we have $f^{p_n} \to f^p\,(r.u)$. Since $T: A \to \mathbb{R}$ is a positive functional, we deduce that $T(f^{p_n}) \to T(f^p)$, as required. □

We are now in position to prove the second main result of the present work.

Theorem 3.12. *(Hölder's inequality) Let A be a uniformly complete Φ-algebra with e as a unit element, let p,q be real numbers such that $\frac{1}{p}+\frac{1}{q}=1$ and let $T : A \to \mathbb{R}$ is a non trivial positive functional. Then, the Hölder's inequality*

$$T(|fg|) \leq (T(|f|^p))^{\frac{1}{p}} (T(|g|^q))^{\frac{1}{q}}$$

holds for all $f, g \in A$.

Proof. It is sufficient to assume that $f, g \geq 0$. The inequality is obvious if $T(fg) = 0$. We assume that $T(fg) > 0$ and f, g have inverses in A. Define the function

$$H(t) = T\left(f^{pt}g^{q(1-t)}\right) = T\left((g^q)\left(f^p g^{-q}\right)^t\right)$$

for all $t \in [0,1]$. By the Cauchy-Schwartz's inequality in the Φ-algebra A, (see [8, Proposition 3.3]), we deduce

$$\begin{aligned} 0 &< T(fg) = T\left(\left((g^q)^{\frac{1}{2}}\left(f^p g^{-q}\right)^{\frac{t}{2}}\right)\left(f^{1-\frac{tp}{2}}g^{1-\frac{1}{2}q(1-t)}\right)\right) \\ &\leq \left[T\left((g^q)\left(f^p g^{-q}\right)^t\right)\right]^{\frac{1}{2}}\left[T\left(f^{2-tp}g^{2-q(1-t)}\right)\right]^{\frac{1}{2}} \\ &= H(t)^{\frac{1}{2}}\, T\left(f^{2-tp}g^{2-q(1-t)}\right)^{\frac{1}{2}}. \end{aligned}$$

Hence $H(t) > 0$. It is known ([14, Chapter VII] that a function $k : (a,b) \to \mathbb{R}$ is convex if and only if k is continuous and midconvex on (a,b). Hence, by the previous lemma, H is continuous on $(0,1)$. Moreover, for all $\alpha, \beta \in (0,1)$, we have

$$\begin{aligned} H\left(\frac{1}{2}\alpha+\frac{1}{2}\beta\right) &= T\left((g^q)\left(f^p g^{-q}\right)^{\frac{1}{2}\alpha+\frac{1}{2}\beta}\right) \\ &= T\left(\left[(g^q)\left(f^p g^{-q}\right)^{\alpha}\right]^{\frac{1}{2}}\left[(g^q)\left(f^p g^{-q}\right)^{\beta}\right]^{\frac{1}{2}}\right). \end{aligned}$$

Hence, by the Cauchy-Schwartz's inequality in the Φ-algebra A, (see [8, Proposition 3.3]), we deduce

$$H\left(\frac{1}{2}\alpha+\frac{1}{2}\beta\right)\leq H\left(\alpha\right)^{\frac{1}{2}}H\left(\beta\right)^{\frac{1}{2}}.$$

Thus we have

$$\ln H\left(\frac{1}{2}\alpha+\frac{1}{2}\beta\right)\leq\frac{1}{2}\ln H\left(\alpha\right)+\frac{1}{2}\ln H\left(\beta\right),$$

that is $\ln H$ is midconvex on (0.1). Then $\ln H$ is convex on (0.1). It follows that

$$\begin{aligned}\ln H\left(\frac{1}{p}t+\frac{1}{q}\left(1-t\right)\right) &\leq \frac{1}{p}\ln H\left(t\right)+\frac{1}{q}\ln H\left(\left(1-t\right)\right)\\ &= \ln H\left(t\right)^{\frac{1}{p}}+\ln H\left(\left(1-t\right)\right)^{\frac{1}{q}}\\ &= \ln\left(H\left(t\right)^{\frac{1}{p}}H\left(\left(1-t\right)\right)^{\frac{1}{q}}\right)\end{aligned}$$

so that

$$H\left(\frac{1}{p}t+\frac{1}{q}\left(1-t\right)\right)\leq H\left(t\right)^{\frac{1}{p}}H\left(\left(1-t\right)\right)^{\frac{1}{q}}$$

for all $t\in(0.1)$. Since H is continuous on (0.1) and convex on $[0.1]$, we have

$$\begin{aligned}H\left(\frac{1}{p}\right) &= \lim_{t\uparrow1}H\left(\frac{1}{p}t+\frac{1}{q}\left(1-t\right)\right)\\ &\leq \limsup_{t\uparrow1}H\left(t\right)^{\frac{1}{p}}\limsup_{t\uparrow1}H\left(\left(1-t\right)\right)^{\frac{1}{q}}\\ &\leq H\left(1\right)^{\frac{1}{p}}H\left(0\right)^{\frac{1}{q}}.\end{aligned}$$

Therefore

$$T\left(fg\right)\leq\left(T\left(f^{p}\right)\right)^{\frac{1}{p}}\left(T\left(g^{q}\right)\right)^{\frac{1}{q}}. \tag{5}$$

Now let $f,g\in A_{+}$. By Birkhoff's inequality, the sequences $\left(f\vee\frac{e}{n}\right)_{n\in\mathbb{N}},\left(g\vee\frac{e}{n}\right)_{n\in\mathbb{N}}$ are uniformly convergent to f,g respectively.

Moreover $\left(g \vee \frac{e}{n}\right), \left(f \vee \frac{e}{n}\right) \geq \frac{e}{n}$, then $\left(f \vee \frac{e}{n}\right), \left(g \vee \frac{e}{n}\right)$ have inverses in A. By the inequality (5), we have

$$T\left(\left(f \vee \frac{e}{n}\right)\left(g \vee \frac{e}{n}\right)\right) \leq \left(T\left(\left(f \vee \frac{e}{n}\right)^p\right)\right)^{\frac{1}{p}} \left(T\left(\left(g \vee \frac{e}{n}\right)^q\right)\right)^{\frac{1}{q}}.$$

Since $\left(f \vee \frac{e}{n}\right)\left(g \vee \frac{e}{n}\right) \rightarrow fg \ (r.u)$, $\left(f \vee \frac{e}{n}\right)^p \rightarrow f^p \ (r.u)$ and $\left(g \vee \frac{e}{n}\right)^q \rightarrow g^q \ (r.u)$. Moreover by the fact that T is a positive operator, we deduce that

$$T(fg) \leq (T(f^p))^{\frac{1}{p}} (T(g^q))^{\frac{1}{q}}$$

and the proof is complete. □

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