# Variations on Hartogs and Henkin-Tumanov Theorems

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SUMMARY. - *There are equivalent characterizations for holomorphic functions defined on open sets of $\mathbb{C}^n$; first of all, they can be represented locally as sums of convergent power series. It is obvious that a holomorphic function of several complex variables is separately holomorphic in each variable. Just separating variables, a lot of the well-known properties of holomorphic functions of one complex variable, as the integral Cauchy formula, have a corresponding version in several complex variables; for separation of variables, we need the function to be continuous. Surprisingly, a function which is separately holomorphic, is indeed $C^0$ and even $C^1$ and therefore holomorphic (Hartogs Theorem, 1906).*

*This short note deals with the problem of separate analyticity and extends the discussion to the case of separately CR functions defined on CR manifolds. We present our result of [5] and explain how it is related to the former literature. In particular, we explain its link with former results by Henkin and Tumanov of 1983 and by Hanges and Treves of 1983.*

## 1. Introduction

In complex analysis in several variables, the study of the properties of holomorphic functions on domains of $\mathbb{C}^n$ shows great differences from the case of one complex variable. In particular, the theme

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of separate analyticity has meaning only in the setting of several complex variables.

Let us recall some properties of holomorphic functions in the case of one complex variable: the real and imaginary parts of an analytic function are harmonic functions, which are conjugate to each other; a harmonic function is uniquely determined by its boundary values; we can construct a harmonic function with prescribed boundary values and construct locally its harmonic conjugate, which is unique up to an additive constant.

In actual research, several complex variables play a more crucial role. Their theory is quite difficult to treat, compared with the theory of one complex variable; the real and imaginary parts of an analytic function are now pluriharmonic; this fact imposes a restriction stronger than being merely harmonic, and it is not always possible to construct a pluriharmonic function with prescribed boundary values on a given part of the boundary.

The most relevant phenomenon, which makes the difference between the cases of one and several variables, is connected to the concept of domain of holomorphy (this is a domain $\Omega$ in $\mathbb{C}^n$ which is the natural domain of a holomorphic function, so there does not exist $\Omega_1 \supset \Omega$, where all the holomorphic functions on $\Omega$ extend). In $\mathbb{C}$, all the domains are domains of holomorphy, while it is not true that every domain in $\mathbb{C}^n$ is the natural domain of a holomorphic function.

On a domain of $\mathbb{C}$, we can easily construct a holomorphic function, which is singular at a point of the boundary, while this is not always possible in $\mathbb{C}^n$. In fact, there exist domains $\widetilde{\Omega}$ in $\mathbb{C}^n$ such that all holomorphic functions holomorphically extend to $\widetilde{\Omega}_1 \supset \widetilde{\Omega}$.

The note is divided into four parts: in the first part, after the definition of holomorphic and separately holomorphic function on a domain of $\mathbb{C}^n$, we introduce CR functions on CR submanifolds of $\mathbb{C}^n$. The second part is used to present some properties of holomorphic functions in several complex variables, as Cauchy integral formula, to give the first results on separate analyticity with the hypothesis of continuity and boundedness on compacts.

Then, the third part introduces Hartogs Theorem (1906), which is really strong, because it does not require any hypothesis for a sep-

arately holomorphic function to be jointly holomorphic; and finally, in the last part, we explain the setting of separately CR functions and we regain and generalize a result of Henkin and Tumanov (1983), using the technique of approximation.

## 2. Holomorphic functions and CR functions

We want to consider functions $f : \mathbb{C}^n \to \mathbb{C}$. Let

$$(x, y) \mapsto (z, \bar{z}) = (x + iy, x - iy)$$

be the identification of $\mathbb{R}^{2n}$ with the diagonal of $\mathbb{C}^n \times \bar{\mathbb{C}}^n$ and use $(z, \bar{z})$ as coordinates in $\mathbb{C}^n$.

DEFINITION 2.1. *A function $f$, defined on a domain $\Omega$ in $\mathbb{C}^n$, is holomorphic if it is $C^1$ and satisfies the differential system $\partial_{\bar{z}_j} f = 0$, $\forall j = 1, ..., n$.*

DEFINITION 2.2. *A function $f$, defined on a domain $\Omega$ in $\mathbb{C}^n$, is separately holomorphic if it satisfies the differential system $\partial_{\bar{z}_j} f = 0$, $\forall j = 1, ..., n$.*

This differential system is the well-known Cauchy-Riemann system. It is an immediate remark that a holomorphic function is separately holomorphic; what we want to show is that the hypothesis of separate analyticity suffices to conclude that the function is $C^1$, so it is jointly holomorphic.

We want to introduce CR manifolds and CR functions as generalizations of complex manifolds and holomorphic functions.

The starting point is to notice that, given a smooth submanifold $M$ of $\mathbb{C}^n$, its tangent space at a point $z \in M$ is not invariant, in general, under the moltiplication for $i$; so it makes sense to look for the largest $i$-invariant subspace of $T_z M$.

DEFINITION 2.3. *For a point $z \in M$, the complex tangent space of $M$ at $z$ is the vector space $T_z^{\mathbb{C}} M = T_z M \cap i T_z M$.*

While $TM$ is a bundle, it is not always true that the dimension of $T^{\mathbb{C}} M$ is constant.

DEFINITION 2.4. *M is called a CR submanifold of $\mathbb{C}^n$ if the dimension of $T^{\mathbb{C}}M$ is constant.*

EXAMPLE 2.5. In $\mathbb{C}^n$, any complex submanifold is a CR submanifold, because, for a complex submanifold $M$, the real tangent space is already $i$-invariant, so $T_zM = T_z^{\mathbb{C}}M$; another example of CR submanifold is the class of real hypersurfaces in $\mathbb{C}^n$. Instead, if we consider the manifold $M = \{z \in \mathbb{C}^n : |z| = 1, \mathsf{Im}\, z_1 = 0\}$, which is the equator of the unit sphere in $\mathbb{C}^n$, this is not a CR submanifold of $\mathbb{C}^n$, because at every point $z \neq (\pm 1, 0, \ldots, 0)$ in $M$ we have $dim_{\mathbb{C}}T_z^{\mathbb{C}}M = n - 2$, while, at $z = (\pm 1, 0, \ldots, 0) \in M$, $T_zM$ is $i$-invariant, so $dim_{\mathbb{C}}T_z^{\mathbb{C}}M = dim_{\mathbb{C}}T_zM = n - 1$.

We give other definitions

- $T^{1,0}(\mathbb{C}^n) := Span\{\partial_{z_j}\}$ and $T^{0,1}(\mathbb{C}^n) := Span\{\partial_{\bar{z}_j}\}$.

- $M$ is called totally real if $T_z^{\mathbb{C}}M = \{0\}$, for every $z \in M$.

- $M$ is called generic if $T_zM + iT_zM = T_z\mathbb{C}^n$, for every $z \in M$.

- $T^{1,0}M := T^{1,0}\mathbb{C}^n \cap (\mathbb{C} \otimes TM)$ and $T^{0,1}M := T^{0,1}\mathbb{C}^n \cap (\mathbb{C} \otimes TM)$.

Now, for a CR submanifold $M$ of $\mathbb{C}^n$, it is possible to introduce the notion of CR function on $M$. Let $M$ be a generic CR submanifold of $\mathbb{C}^n$, defined by a system $\rho_1 = 0, \ldots, \rho_d = 0$ of independent equations. The following two definitions are equivalent

DEFINITION 2.6. *A $C^1$ function $f : M \to \mathbb{C}$ is CR if $\bar{L}f = 0$, for every $\bar{L} \in T^{0,1}M$.*

DEFINITION 2.7. *A $C^1$ function $f : M \to \mathbb{C}$ is CR if $\bar{\partial}\widetilde{f} \wedge \bar{\partial}\rho_1 \wedge \ldots \wedge \bar{\partial}\rho_d = 0$ on $M$, where $\widetilde{f} : \mathbb{C}^n \to \mathbb{C}$ is any $C^1$ extension of $f$.*

CR functions on CR manifolds are analogous to holomorphic functions on complex manifolds, though there are relevant differences, as the fact that CR functions are not always smooth. For the analogies, the first is that the restriction of a holomorphic function to a CR submanifold is a CR function; in particular, if $M = \mathbb{C}^n$, holomorphic functions are CR, while, for the converse, we need $M$ and $f$ to be $C^\omega$. In general, the class of CR functions is strictly larger than the class of restrictions of holomorphic functions. In other terms, CR functions not always extend as holomorphic functions.

EXAMPLE 2.8. $M = \mathbb{R} \times \mathbb{R}$ is a totally real and generic submanifold of $\mathbb{C}^2$ and $T^{0,1}M = \{0\}$, so all the functions $f(x_1, x_2)$ of class $C^1$ on $M$ are CR. $M = \mathbb{R} \times \mathbb{C}$ is a generic submanifold of $\mathbb{C}^2$, such that $T^{0,1}M$ is spanned by the vector field $\partial_{\bar{z}_2}$. Thus every function $f(x_1, z_2)$ of class $C^1$ on $M$, which satisfies $\partial_{\bar{z}_2}f = 0$, is CR. These functions are separately holomorphic in $z_2$ and the holomorphic extension needs $f$ to be $C^\omega$. Finally, $M = \mathbb{C} \times \mathbb{C}$ is a complex submanifold and $T^{0,1}M$ is spanned by $\partial_{\bar{z}_1}$ and $\partial_{\bar{z}_2}$; thus, CR functions and holomorphic functions coincide in this case.

## 3. Properties of holomorphic functions of several complex variables

THEOREM 3.1 (Cauchy integral formula on polydiscs). *Let $f$ be a continuous function on the closure of a polydisc $P = D_1 \times \ldots \times D_n \subset \mathbb{C}^n$, which is, for any $j$, a holomorphic function of $z_j$, when the other variables $z_k$, for $k \neq j$, are fixed. Then, for any $z \in P$, we have*

$$f(z) = (2\pi i)^{-n} \int_{\partial_0 P} \frac{f(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \wedge \ldots \wedge d\zeta_n,$$

*where $\partial_0 P = \partial D_1 \times \ldots \times \partial D_n$.*

This is the generalization of Cauchy formula on the complex plane, which is a consequence of Stokes formula. For $f \in C^1(\bar{\Omega})$, where $\Omega$ is a bounded open set of $\mathbb{C}$, it says that $\int_{\partial\Omega} f \, dz = \int\int_\Omega df \wedge dz = \int\int_\Omega \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz$. In particular, if $f$ is $C^1(\bar{\Omega})$ and analytic, then $\int_{\partial\Omega} f \, dz = 0$.

The $C^1$-regularity in each $z_j$ is needed for Stokes formula, and the joint $C^0$-regularity is needed for Fubini Theorem.

COROLLARY 3.2. *If $f$ is $C^0(\Omega)$ and separately holomorphic in each $z_j$, when the other variables are fixed, then $f$ is $C^\infty(\Omega)$. (In particular $f$ is holomorphic on $\Omega$.)*

It suffices to consider Cauchy formula on a polydisc contained in $\Omega$; the integrand is $C^\infty$ and analytic in $z$ when $(\zeta, z) \in \partial_0 P \times P$, so we can derive under the integral.

THEOREM 3.3 (Cauchy inequalities). *Let $f$ be holomorphic on $P$ and continuous on $\bar{P}$. Then*

$$|f^{\alpha}(z_o)| \leq \frac{\alpha!}{r^{\alpha}}\sup{}_{\partial_0 P(z_o,r)}|f|.$$

COROLLARY 3.4. *Let $\{f_n\}$ be a sequence of holomorphic functions on $\Omega$, and $\{f_n\}$ converges uniformly to $f$ on compact sets of $\Omega$; then $f \in hol(\Omega)$ and $\{\partial^{\alpha} f_n\}$ converges uniformly to $\partial^{\alpha} f$ on compact sets of $\Omega$.*

COROLLARY 3.5 (Stjelties-Vitali). *Let $\{f_n\}$ be a sequence of holomorphic functions on $\Omega$, uniformly bounded on compact sets of $\Omega$; then, there exists a subsequence $\{f_{n_k}\}$ uniformly convergent on compact sets of $\Omega$, and its limit is holomorphic.*

Another result on separate analyticity is then given through the hypothesis of boundedness on compact sets, using the technique of the Theorem of Stjelties-Vitali: for holomorphic functions, uniform boundedness is equivalent to equicontinuity.

PROPOSITION 3.6. *If $f$ is separately holomorphic and bounded on compact sets of $\Omega$, then $f$ is holomorphic on $\Omega$.*

## 4. Hartogs Theorem

THEOREM 4.1 (Hartogs, 1906). *If $f : \Omega \to \mathbb{C}$ is separately holomorphic, then it is holomorphic. (We do not need any hypothesis on the initial regularity of $f$; this is $C^1$ as a consequence.)*

REMARK 4.2. *A corresponding result for real analytic functions is false, as the following example shows*

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = \begin{cases} xy/(x^2 + y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

*$f$ is separately $C^{\omega}$ in $x$ (when $y$ is fixed) and in $y$ (when $x$ is fixed), $f$ is bounded on the plane, but $f$ is not continuous at (0,0).*

The proof of Hartogs Theorem needs a result on subharmonic functions, which is known as Hartogs lemma.

LEMMA 4.3 (Hartogs lemma). *Let $v_k$ be a sequence of subharmonic functions, which are uniformly bounded on any compact subset of $\Omega$. Let $\lim \sup_{k \to \infty} v_k(z) \leq C, \quad \forall z \in \Omega$; then, $\forall \epsilon > 0$ and $\forall K \subset\subset \Omega$, there is $k_0$ such that*

$$\sup_{z \in K} v_k(z) \leq C + \epsilon, \quad \forall k \geq k_0.$$

The proof is an application of Fatou's lemma in the integrals which enter in the submean property. For sequences of functions which admit integral representations, or estimates by integrals like submeans, in case they have a uniform bound, then the pointwise "lim sup" enters into the integrals and becomes "uniform".

The proof of Hartogs Theorem can be divided into two parts: the first is an application of Baire Theorem, the second uses Hartogs lemma.

*Proof.* The statement is local and can be proved by adding, one by one, the directions of separate analyticity: so we can consider the bidisc $\bar{\Delta} \times \bar{\Delta} \subset\subset \Omega$ in $\mathbb{C}^2$ and prove these two steps

**STEP 1: Analyticity on $\Delta_\epsilon \times \Delta$.** We prove that $f$, which is separately holomorphic in $z_1$ and $z_2$, is holomorphic on $\Delta_\epsilon \times \Delta$.

Let us define $E_l := \{z_1 \in \Delta : \sup_{z_2 \in \Delta} |f(z_1, z_2)| \leq l\}$. $E_l$ is closed and $\cup_l E_l = \Delta$. By Baire Theorem, there exists $l_0$ such that $\text{Int}(E_{l_0}) \neq \varnothing$; so $f$ is holomorphic on $\text{Int}(E_{l_0}) \times \Delta$ and, repeating the same construction with different sets $E_l$ on any open subset of $\Delta$, we can say that $f$ is holomorphic on $B \times \Delta$, for an open dense subset $B \subset \Delta$. Also, we can assume, without loss of generality, that $0 \in B$, so $f$ is holomorphic on the strip $\Delta_\epsilon \times \Delta$.

**STEP 2: Analyticity on $\Delta \times \Delta$.** At this point, we can even forget that $f$ is separately holomorphic in $z_2$, when $z_1$ is outside $\Delta_\epsilon$.

So we prove that, if $f$ is holomorphic on $\Delta_\epsilon \times \Delta$, and separately holomorphic for $z_1 \in \Delta$ when $z_2$ is fixed, then $f$ is holomorphic on $\Delta \times \Delta$.

We consider the Taylor series of $f$ with respect to $z_1$, at $z_1 = 0$:

$$f(z_1, z_2) = \sum_k \frac{\partial_{z_1}^k f(0, z_2)}{k!} z_1^k;$$

it converges uniformly for $z_2 \in \Delta$ and normally for $z_1 \in \Delta_\epsilon$. We define

$$v_k(z_2) := \left( \frac{|\partial_{z_1}^k f(0, z_2)|}{k!} \right)^{\frac{1}{k}}.$$

Cauchy inequalities yield, by the assumption of separate analyticity of $f$ in $z_1 \in \Delta$, for fixed $z_2$, $\limsup_{k \to \infty} v_k(z_2) \leq 1$; on the other hand, by the assumption of analyticity of $f$ on $\Delta_\epsilon \times \Delta$, they yield $\limsup_k \sup_{z_2 \in \Delta} v_k \leq \epsilon^{-1}$. By Hartogs lemma, the pointwise estimate in $z_2$ becomes uniform

$$\limsup_k \sup_\Delta v_k \leq 1.$$

Hence, the power series in $z_1$, with holomorphic coefficients in $z_2$, converges normally for $z_1 \in \Delta$ and uniformly for $z_2 \in \Delta$, so the sum is a holomorphic function on $\Delta \times \Delta$. $\qquad\qquad\square$

REMARK 4.4. *Note that Hartogs Theorem consists only in Step 2; Step 1 is a preliminary by Baire. In the second step we prove two important results: the uniformity in $\Delta$ and the propagation. When we say that "we can even forget that $f$ is separately holomorphic in $z_2$, when $z_1$ is outside $\Delta_\epsilon$", even if we suppose that $f$ is continuous, it is not easy to prove the analyticity on the bidisc: the problem has become a problem of propagation.*
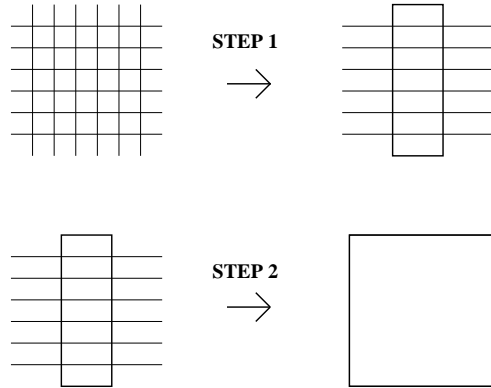
We can appreciate the proof through this figure



Figure 1: Hartogs Theorem

## 5. Separately CR functions

The first remark of this section is an application of Hartogs Theorem; we can easily prove it just iterating the technique of "doubling" the radius of convergence.

REMARK 5.1. *Let $f$ be separately holomorphic on $\Delta^+ \times \Delta = \{(z_1, z_2) \in \Delta \times \Delta : \operatorname{Re} z_1 > 0\}$ and holomorphic on $\Delta_\epsilon^+ \times \Delta = \{(z_1, z_2) \in \Delta^+ \times \Delta : |z_1| < \epsilon\}$; then, $f$ is holomorphic on $\Delta^+ \times \Delta$.*

When the leaves of the foliation are complex curves, the problem changes again; it is a problem of propagation, as Step 2 was, for a non-holomorphic foliation; in full generality, the validity of the statement is an open question. If we suppose that $f$ is $C^0$, we have the following statement

*Let $\{\gamma_\lambda\}_{\lambda \in \Lambda}$ be a foliation of $\Delta^+ \times \Delta$ by complex curves, such that $\gamma_\lambda \cap (\Delta_\epsilon^+ \times \Delta) \neq \varnothing, \forall \lambda \in \Lambda$. Let $f$ be a $C^0$ function on $\Delta^+ \times \Delta$, such that $f$ is holomorphic on $\Delta_\epsilon^+ \times \Delta$ and $f_{|\gamma_\lambda}$ is holomorphic, $\forall \lambda \in \Lambda$; then, $f$ is holomorphic on $\Delta^+ \times \Delta$.*

The result can be proved in several ways. (For instance, it is also a corollary of the subsequent Theorem 5.6.)
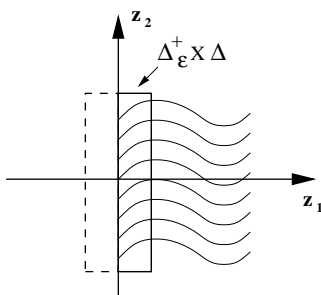
Figure 2: Foliation by complex curves.

The former statement can be generalized in many directions: first, in replacing the open set $\Delta^+ \times \Delta$ by a CR manifold $M$, and $\Delta_\epsilon^+ \times \Delta$ by an open set $M_\epsilon \subset M$ (that can even be shrunk to a proper submanifold $N$), and also in replacing the foliation $\{\gamma_\lambda\}$ of complex curves by a foliation $\{\gamma_\lambda\}$ of CR manifolds of CR dimension 1.

THEOREM 5.2 (Mascolo). *Let $M$ be a CR connected manifold with boundary $N$, foliated by a family $\{\gamma_\lambda\}$ of CR manifolds of CR dimension 1, issued from $N$, with $T^\mathbb{C}\gamma_\lambda$ transversal to $TN$ at any common point of $\gamma_\lambda \cap N$. Let $f$ be a $C^0$ function on $M$, which is CR along $N$, CR and $C^1$ along each $\gamma_\lambda$. Then, $f$ is CR all over $M$.*

REMARK 5.3. *Since $M$ is CR, then $N$ is also CR. In fact, its CR codimension is always $\leq 1$, but it is in fact $\equiv 1$ because we have a foliation by leaves whose complex structure is transversal to $N$.*

REMARK 5.4. *$M$ is a CR manifold of $\mathbb{C}^n$; by a projection $\mathbb{C}^n \to T_zM + iT_zM$, which is a diffeomorphism when restricted to $M$, we can assume without loss of generality that $M$ is generic.*

An idea of the proof, which is divided into two steps, is to select a totally real manifold $E_o \subset N$, invariant under the foliation $\{L_\lambda \cap N\}$, and define an approximation of $f$ by entire functions $\{f_\alpha\}$, defined by

$$f_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \int_{+E_o} f(\xi)e^{-\alpha(\xi-z)^2}d\xi_1 \wedge \ldots \wedge d\xi_n.$$

By deforming the manifold $E_o$, we see that the above sequence provides in fact a uniform approximation of $f$ in a neighbourhood $M_\epsilon$ of

$E_o$ in $M$; hence $f$ is CR in that neighbourhood. (The same method of polynomial approximation was first exploited by Tumanov in [6], in proving that a given function is CR.) A repeated use of this technique and a connectedness argument yield the global theorem.
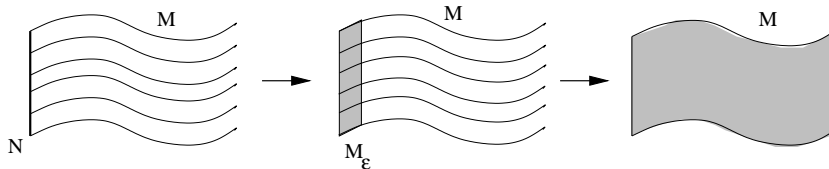


Figure 3: CR extension from $N$ to $M_\epsilon$, and global extension.

Our Theorem is a generalization of the following Theorem by Henkin and Tumanov

THEOREM 5.5 (Henkin-Tumanov, 1983). *Let $\{\gamma_\lambda\}$ be a foliation of $M$ by complex curves which are transversal to $N$ at any common point of $\gamma_\lambda \cap N$, and let $f$ be $C^0$ on $M$, CR on $N$ and holomorphic along each $\gamma_\lambda$; then, $f$ is CR on $M$.*

In turn, Theorem 5.5 is related to the following

THEOREM 5.6 (Hanges-Treves, 1983). *Let $M$ be a hypersurface of $\mathbb{C}^n$, $\Omega$ one side of $M$, $\gamma$ a complex curve of $M$, $z_o$ a point of $\gamma$, $f$ a holomorphic function on $\Omega$, such that $|f(z)| \leq C dist(z, \partial\Omega)^{-N}$ for suitable $C$ and $N$. Then, if $f$ extends across $M$ at $z_o$, it also extends at any other point $z_1 \in \gamma$.*

On one hand, Theorem 5.5 is far more general, because what is propagated is the property of $f$ of being CR, not necessarily holomorphic. On the other hand, in Theorem 5.6 we do not have any foliation by complex curves: there is only one leaf, which has the property of "being a propagator". (Cf. [1] and [2] for a more detailed account on propagation; note that the techniques of [1] and [2] apply also to a function $f$ which is not tempered at $\partial\Omega$.)

EXAMPLE 5.7. In $\mathbb{C}^4$ we consider $M = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : y_1 \geq -(|z_2|^2 + |z_3|^2 + |z_4|^2)\}$, with boundary $N = \{z = (z_1, z_2, z_3, z_4) \in$

$\mathbb{C}^4 : y_1 = -(|z_2|^2 + |z_3|^2 + |z_4|^2)\}$. Let $(\gamma_{a,b,c})_{a,b,c\in\mathbb{R}}$ be manifolds in $\mathbb{C}^4$ defined by

$$\begin{cases} y_2 = |z_1|^2 + a \\ y_3 = |z_1|^2 + b \\ y_4 = |z_1|^2 + c. \end{cases}$$

The setting of this example is good for our Theorem; the use of Henkin-Tumanov Theorem is not possible, because the $\gamma_{a,b,c}$ are strictly pseudoconvex, so they can not be foliated by complex curves.

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