

# Postulation Problems for Vector Bundles

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*A Fabio, con tanto affetto*

**SUMMARY.** - *We treat the general problem of studying the postulation of a subscheme of a projective variety with respect to a vector bundle of rank  $r$ , and we recall some results already stated for particular varieties and bundles, rewriting them in a general set-up. We also give an example of the use of these techniques proving that a generic union of 2-jets in  $\mathbb{P}^2$  has the expected resolution.*

## 1. Introduction

Let  $X$  be a closed subscheme of  $\mathbb{P}^n$  with homogeneous ideal  $I_X$  and ideal sheaf  $\mathcal{I}_X$ ; if we wish to understand the geometry of  $X$  as embedded scheme, one of the first points of interest is the knowledge of the dimension of the linear system of hypersurfaces of given degree containing  $X$ , and how many independent relations there are among these hypersurfaces, and how these relations are related, and so on. In other words, we would like to know the minimal free resolution of  $\mathcal{I}_X$ :

$$0 \rightarrow E_{n-1} \xrightarrow{f_{n-1}} \dots \rightarrow E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} \mathcal{I}_X \rightarrow 0$$

where for each  $p = 0, \dots, n-1$ ,  $E_p$  is a free vector bundle:  $E_p \cong \bigoplus_q (B_{p,q} \otimes \mathcal{O}_{\mathbb{P}^n}(-q))$ ,  $B_{p,q}$  being a finite dimensional vector space

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different from 0 only for a finite numbers of  $q$ 's, and the arrows are given by matrices of homogeneous polynomials which are zero or of strictly positive degree.

Hence the Betti numbers of  $X$ ,  $b_{p,q} = \dim B_{p,q}$ , count how many copies of  $\mathcal{O}_{\mathbb{P}^n}(-q)$  appear at the step  $p$  of the resolution; the minimal free resolution (up to isomorphisms) is known when the Betti numbers are known.

We say that  $X$  has *good postulation* if the restriction maps

$$H^0(\mathcal{O}_{\mathbb{P}^n}(k)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(k)|_X)$$

have maximal rank, i.e. are injective or surjective, for each  $k \geq 0$ . Notice that if the integers  $h^0(\mathcal{O}_{\mathbb{P}^n}(k)|_X)$  are known, for example if  $X$  is a 0-dimensional scheme or a non special curve of given degree and genus, and if  $X$  has good postulation, then we also know  $h^0(\mathcal{I}_X(k))$ .

If  $v$  is the minimum degree of an hypersurface containing  $X$ , and  $\mathcal{I}_X$  is  $(v+1)$ -regular, i.e.  $H^i(\mathcal{I}_X(v+1-i)) = 0$  for all  $i > 0$ , then  $X$  has good postulation, and for each  $p$  there are at most two non zero Betti numbers (see [5] Section 20.5, Exercise 20.20, and [18] p.99):

$$E_p \cong B_{p,-v-p} \otimes \mathcal{O}_{\mathbb{P}^n}(-v-p) \oplus B_{p,-v-p-1} \otimes \mathcal{O}_{\mathbb{P}^n}(-v-p-1).$$

Some of these  $b_{p,q}$  may still vanish.

For example, let us consider the map  $f_0$ ; if  $\alpha_1, \dots, \alpha_k$  is a minimal system of generators for the homogeneous ideal  $I_X$ , then  $f_0 = (\alpha_1 \dots \alpha_k)$ . A minimal system of generators for  $(I_X)_{k+1}$  is given by a basis of  $\text{coker} \mu_k$ , where  $\mu_k$  denotes the multiplication map:

$$\mu_k : H^0(\mathcal{I}_X(k)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \rightarrow H^0(\mathcal{I}_X(k+1)),$$

hence in degree  $v$  we find a basis for  $\text{coker} \mu_{v-1} = H^0(\mathcal{I}_X(v)) \neq 0$ . If  $X$  has good postulation, i.e., if  $H^1(\mathcal{I}_X(v)) = 0$ , then by Castelnuovo-Mumford Lemma (see [18] p.99) the maps  $\mu_k$  with  $k \geq v+1$  are surjective, hence the only other  $\text{coker} \mu_k$  which can be different from zero is  $\text{coker} \mu_v$ . Now assume that the maps  $\mu_k$  are all of maximal rank, in which case we say that  $\mathcal{I}_X$  is *minimally generated*; then if  $(n+1)h^0(\mathcal{I}_X(v)) \geq h^0(\mathcal{I}_X(v+1))$ ,  $\mu_v$  is surjective, so that  $b_{0,-v-1} = 0$ .

What happens is that if all the kernels of the arrows in the minimal free resolution are minimally generated, then for all but maybe

one  $p$  there is only one non zero Betti number; that is, the  $E_p$ 's are as small as possible (see for example [17], Section 2). In this case we say that  $X$  has the *expected resolution*. It is the expected one because there are quite a lot of conjectures and some theorems saying that under specific assumptions the generic scheme  $X$  of  $\mathbb{P}^n$  has a minimal free resolution of this form; here generic means generic in an irreducible component of its Hilbert scheme.

Let us set  $\Omega_{\mathbb{P}^n}^p := \bigwedge^p \Omega_{\mathbb{P}^n}$ , where  $\Omega_{\mathbb{P}^n}$  is the cotangent bundle of  $\mathbb{P}^n$ . When  $X$  has good postulation,  $\mathcal{I}_X$  is minimally generated if and only if the restriction maps

$$H^0(\Omega_{\mathbb{P}^n}(t)) \rightarrow H^0(\Omega_{\mathbb{P}^n}(t)|_X)$$

are of maximal rank for all  $t$ ; this can be easily seen (see for example [16] 1.1) using the Euler sequence:  $0 \rightarrow \Omega_{\mathbb{P}^n}(1) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow 0$ , and is a particular case of the use of Koszul cohomology.

In fact, taking exterior powers in the Euler sequence leads to the exact sequences of vector bundles:

$$0 \rightarrow \Omega_{\mathbb{P}^n}^p(p) \rightarrow \bigwedge^p (H^0(\mathcal{O}_{\mathbb{P}^n}(1))) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \Omega_{\mathbb{P}^n}^{p-1}(p) \rightarrow 0,$$

which remain exact if tensored by  $\mathcal{I}_X(t)$ . Using these sequences and the Koszul cohomology groups of the sheaf  $\mathcal{I}_X$  on  $\mathbb{P}^n$ , it is possible to prove (see [8], [9]) that

$$\begin{aligned} B_{p,p+q} &\cong \ker(H^1(\Omega_{\mathbb{P}^n}^p(p) \otimes \mathcal{I}_X(q-1))) \\ &\rightarrow \bigwedge^{p+1} (H^0(\mathcal{O}_{\mathbb{P}^n}(1))) \otimes H^1(\mathcal{I}_X(q-1)). \end{aligned}$$

If for example  $X$  is a general union of points in  $\mathbb{P}^n$ , the Koszul cohomology allows to see that the Minimal Resolution Conjecture (see [17]), which says that  $X$  has the expected resolution, is reducible to the following problem: prove that the restriction maps:

$$H^0(\Omega_{\mathbb{P}^n}^p(t)) \rightarrow H^0(\Omega_{\mathbb{P}^n}^p(t)|_X)$$

have maximal rank for each  $p = 0, \dots, n$  and for each  $t \geq 0$  (see [13], Introduction and Section 9).

Hence the understanding of the complex of relations among the equations defining a closed subscheme  $X$  of  $\mathbb{P}^n$  amounts to studying the postulation of  $X$  with respect not only to line bundles but also to vector bundles of higher rank.

This method has been introduced by A.Hirschowitz in a letter to R.Hartshorne, and then used to prove expected resolution conjectures for curves (see [14], [16]) or for 0-dimensional schemes (see for example [13], [15], [3] and references therein).

In Section 2 we treat the general problem of studying the postulation of a subscheme of a projective variety with respect to a vector bundle of rank  $r$ , and we recall some results already stated for particular varieties and bundles, rewriting them in a general set-up.

In Section 3 we give an example of the use of these techniques proving that a generic union of 2-jets in  $\mathbb{P}^2$  has the expected resolution; here 2-jet means curvilinear scheme of length 2.

We work on an algebraically closed field  $K$  of characteristic 0.

## 2. Postulation problems with respect to vector bundles

Let  $S$  be a smooth projective variety,  $E$  a rank  $r$  vector bundle on  $S$ ,  $Z$  a closed subscheme of  $S$ ; let  $\rho$  be the restriction map in the natural sequence:

$$0 \rightarrow E \otimes \mathcal{I}_Z \rightarrow E \xrightarrow{\rho} E|_Z \rightarrow 0 \quad (\dagger).$$

We may want to study *the postulation of  $Z$  with respect to  $E$* , i.e. the rank of the linear map:

$$H^0(E) \xrightarrow{H^0(\rho)} H^0(E|_Z).$$

For example, if  $S = \mathbb{P}^n$  and  $E = \mathcal{O}_{\mathbb{P}^n}(t)$ , this boils down to the classical problem of how many independent hypersurfaces of a given degree contain the scheme  $Z$ , problem which has been intensively studied for many different types of  $Z$  and with various methods.

We say that  $Z$  *imposes independent conditions to the sections of  $E$*  if the map  $H^0(\rho)$  is of maximal rank. Hence  $Z$  imposes independent conditions to the sections of  $E$  if  $\dim \ker H^0(\rho) = h^0(E \otimes \mathcal{I}_Z) = \max \{0, h^0(E) - h^0(E|_Z)\}$ .

We say that  $Z$  is  $E$ -settled if the map  $H^0(\rho)$  is bijective.

Quite a big effort has been done in the last twenty years in order to prove that the generic scheme of this or that irreducible component of the Hilbert scheme of  $\mathbb{P}^n$  has good postulation, which means, as we have seen, that  $Z$  imposes independent conditions to the sections of  $\mathcal{O}_{\mathbb{P}^n}(t)$  for any  $t$  (for results about the postulation of subschemes of small dimension in  $\mathbb{P}^n$ , see for example [2], [1], [10], [19] and references therein). One of the standard methods used to afford this problem is the Horace method (see [12]), which consists in cutting with an hypersurface, get rid of what happens on the hypersurface and work with the scheme and the bundle which are left; when the conditions are good, doing this inductively permits to conclude. In other words, if  $H$  is an effective divisor on  $S$ , we use the following exact sequence, called *the residual sequence*:

$$0 \rightarrow \mathcal{I}_{\text{res}_H Z}(-H) \rightarrow \mathcal{I}_Z \rightarrow \mathcal{I}_{Z \cap H, H} \rightarrow 0; \quad (*)$$

here  $\text{res}_H Z$  denotes the *the residual scheme of  $Z$  with respect to  $H$* , that is, the scheme having homogeneous ideal  $(I_Z : I_H)$ .

Assume we are studying the postulation, with the purpose of proving that it is good, of a certain class of schemes with respect to  $\mathcal{O}_{\mathbb{P}^n}(t)$  for all  $t \geq 0$ ; for example, generic unions of  $n$  fat points of given multiplicity, say 2, for any  $n$ , or the generic rational curve of degree  $d$  for any  $d$  and so on. Usually, one reduces the problem to proving a settled case for each  $t$ . For example, in the first case, assume that for each  $t$  we are able to build a scheme  $X_t$  which consists of the maximum possible number, say  $n_t$ , of fat double points, plus a scheme  $R_t$ , the *remainder scheme*, which is contained in a fat double point, and such that the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(t)|_{X_t})$$

is bijective (the length of  $X_t$  is hence equal to  $h^0(\mathcal{O}_{\mathbb{P}^n}(t))$ ).

Then it is possible and easy to show that for the generic union  $Y$  of  $n$  fat double points the map  $H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(t)|_Y)$  is injective when  $n > n_t$  and surjective when  $n \leq n_t$ . This is done adding or taking away points from  $X_t$  until we get  $Y$ , in other words using the cohomology of the following exact sequences tensored by

$\mathcal{O}_{\mathbb{P}^n}(t)$ :

$$0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{I}_{X_t} \rightarrow \mathcal{I}_{X_t, Y} \rightarrow 0 \quad \text{when } Y \supseteq X_t,$$

$$0 \rightarrow \mathcal{I}_{X_t} \rightarrow \mathcal{I}_Y \rightarrow \mathcal{I}_{Y, X_t} \rightarrow 0 \quad \text{when } Y \subseteq X_t$$

(see for example the proof of Proposition 3.2 in the following section).

So let us see how we use the exact sequence (\*) for the proof of the settled cases. If  $H$  is for example an hyperplane, the cohomology of the residual sequence tensored by  $\mathcal{O}_{\mathbb{P}^n}(t)$  gives:

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(t-1) \otimes \mathcal{I}_{\text{res}_H X_t}) \rightarrow \\ H^0(\mathcal{O}_{\mathbb{P}^n}(t) \otimes \mathcal{I}_{X_t}) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(t)|_H \otimes \mathcal{I}_{X_t \cap H, H}) \rightarrow \dots \end{aligned}$$

so if we are able to prove  $H^0(\mathcal{O}_{\mathbb{P}^n}(t)|_H \otimes \mathcal{I}_{X_t \cap H, H}) = 0$ , we see that  $H^0(\mathcal{O}_{\mathbb{P}^n}(t-1) \otimes \mathcal{I}_{\text{res}_H X_t}) \cong H^0(\mathcal{O}_{\mathbb{P}^n}(t) \otimes \mathcal{I}_{X_t})$  and we can go on using induction on the degree. On the other hand in order to have results for  $H^0(\mathcal{O}_{\mathbb{P}^n}(t)|_H \otimes \mathcal{I}_{X_t \cap H, H})$  it is possible to use induction on the dimension, since  $\mathcal{O}_{\mathbb{P}^n}(t)|_H$  is well known, it is just  $\mathcal{O}_{\mathbb{P}^{n-1}}(t)$ .

Now let's turn to the higher rank cases. Obviously, the sequence (\*) remains exact if we tensor it by a vector bundle  $E$  of any rank:

$$0 \rightarrow E \otimes \mathcal{I}_{\text{res}_H Z}(-H) \rightarrow E \otimes \mathcal{I}_Z \rightarrow E|_H \otimes \mathcal{I}_{Z \cap H, H} \rightarrow 0 \quad (**)$$

but if we are dealing with a postulation problem with respect to  $E$  with rank  $E > 1$  it becomes more complicated to use it.

An easy example is the following: if  $S = \mathbb{P}^2$ , a divisor  $H$  which is easy to manage is a line, or a smooth conic, since every vector bundle restricted to a smooth rational curve splits. If rank  $E = 1$ , i.e.  $E = \mathcal{O}_{\mathbb{P}^2}(t)$ , we find  $E|_H \cong \mathcal{O}_{\mathbb{P}^1}(t)$  and respectively  $E|_H \cong \mathcal{O}_{\mathbb{P}^1}(2t)$ , and it is not difficult to decide whether a certain subscheme  $Z \cap H$  imposes independent conditions to the sections of these bundles. On the other hand, if  $E$  has rank  $r$ ,  $E$  will surely split on  $H \cong \mathbb{P}^1$  but it may split as  $\mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(m_r)$  with for example  $0 < m_1 < m_2 < \dots < m_r$ , so that if the length of our scheme  $Z \cap H$  is, say,  $m_r + 1$ , it imposes independent conditions to the sections of the last summand but not to those of the other ones. In fact, it does not exist any subscheme of  $H \subseteq \mathbb{P}^2$  which is  $E|_H$ -settled, since there is

nothing smaller than a point in  $\mathbb{P}^2$ , and a point “counts”  $r$  times for  $E$ .

One way to afford the problem in the higher rank case is to use the following set up: let  $\pi : \mathbb{P}(E) \rightarrow S$  be the canonical projection,  $\pi^{-1}Z$  the inverse image of the subscheme  $Z$ , that is, the closed subscheme  $Z \times_S \mathbb{P}(E)$  of  $\mathbb{P}(E)$  having as its ideal sheaf the inverse image ideal sheaf of  $Z$ , and  $r$  the restriction map in the natural sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \mathcal{I}_{\pi^{-1}Z} \rightarrow \mathcal{O}_{\mathbb{P}(E)}(1) \xrightarrow{r} \mathcal{O}_{\mathbb{P}(E)}(1)|_{\pi^{-1}Z} \rightarrow 0 \quad (\dagger\dagger)$$

Let  $\mathcal{L}$  be an invertible sheaf on  $S$ . Then we have:

PROPOSITION 2.1. *In the previous notations, studying the postulation of  $Z$  with respect to  $E \otimes \mathcal{L}$  is equivalent to studying the postulation of  $\pi^{-1}Z$  with respect to  $\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L}$ .*

In fact, compare the cohomology sequence of  $(\dagger) \otimes \mathcal{L}$ :

$$0 \rightarrow H^0(E \otimes \mathcal{L} \otimes \mathcal{I}_Z) \rightarrow H^0(E \otimes \mathcal{L}) \xrightarrow{\alpha} H^0((E \otimes \mathcal{L})|_Z) \rightarrow \dots \quad (\circ)$$

with the cohomology sequence of  $(\dagger\dagger) \otimes \pi^*\mathcal{L}$ :

$$0 \rightarrow H^0(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L} \otimes \mathcal{I}_{\pi^{-1}Z}) \rightarrow H^0(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L}) \xrightarrow{\beta} \xrightarrow{\beta} H^0((\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L})|_{\pi^{-1}Z}) \rightarrow \dots \quad (\circ\circ)$$

As in [6], where this is done for a particular choice of  $S$ ,  $E$  and  $\mathcal{L}$  (see Lemma 3.2 therein), it is now enough to notice that the dimensions of the domains, the codomains and the kernels of the maps  $\alpha$  and  $\beta$  are equal. The last assertion follows from the fact that  $\pi_*(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L}) \cong E \otimes \mathcal{L}$  (see for example [11] Exercise III.8.3 and III.8.4), so that their  $H^0$  are isomorphic, and the following lemma, which is Lemma 2.1 in [16]:

LEMMA 2.2. *Let  $S$  be a smooth projective variety,  $\mathcal{L}$  an invertible sheaf on  $S$ ,  $E$  a rank  $r$  vector bundle on  $S$ ,  $\pi : \mathbb{P}(E) \rightarrow S$  the canonical projection and  $Z$  a closed subscheme of  $S$ . Then,*

$$\begin{aligned} \pi_*(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L} \otimes \mathcal{I}_{\pi^{-1}Z}) &\cong E \otimes \mathcal{L} \otimes \mathcal{I}_Z \\ \text{and } \pi_*((\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L})|_{\pi^{-1}Z}) &\cong (E \otimes \mathcal{L})|_Z. \end{aligned}$$

It is also worth noticing that by [11] Ex. III.8.4,  $R^i p_* \mathcal{O}_{\mathbb{P}(E)}(1) = 0$  for  $i > 0$ , hence by projection formula  $R^i \pi_*(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* \mathcal{L}) \cong R^i \pi_* \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \mathcal{L} = 0$  for  $i > 0$ , so that by [11] Ex. III.8.1 there are natural isomorphisms  $H^i(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* \mathcal{L}) \cong H^i(E \otimes \mathcal{L})$  for all  $i \geq 0$ . Hence in particular if  $H^1(E \otimes \mathcal{L}) = 0$ , also  $H^1(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* \mathcal{L}) = 0$  and the cokernels in (◦) and in (◦◦) are respectively  $H^1(E \otimes \mathcal{L} \otimes \mathcal{I}_Z)$  and  $H^1(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* \mathcal{L} \otimes \mathcal{I}_{\pi^{-1}Z})$ .

Up to now we have improved the situation with respect to the rank, since instead of working on  $S$  with a rank  $r$  vector bundle we work with a line bundle, but the dimension of  $S$ , as well as that of our scheme  $Z$ , has increased by 1 since we now have to take care of  $\pi^{-1}Z$  inside  $\mathbb{P}(E)$ . There is nothing to do for the ambient variety, but we can do better for  $\pi^{-1}Z$ ; for example, if  $Z$  is 0-dimensional, we have the following Lemma 2.3. We first need some notations:

NOTATION 1. Let  $S$  be a smooth projective variety,  $E$  a rank  $r$  vector bundle on  $S$ ,  $\pi : \mathbb{P}(E) \rightarrow S$  the canonical projection. Let  $U$  be an open subset in  $S$  such that  $E|_U \cong E_1 \oplus \dots \oplus E_r$ ,  $E_i \cong \mathcal{O}_U$  for  $i = 1, \dots, r$ , and  $Z \subseteq U$  a closed subscheme of  $U$ . We set:

$$Z_i := \pi^{-1}(Z) \cap \mathbb{P}(E_i) \quad \text{for } i = 1, \dots, r, \quad \widehat{Z} := Z_1 \cup \dots \cup Z_r,$$

(hence  $\pi$  gives isomorphisms  $Z_i \cong Z$  for  $i = 1, \dots, r$ ).

Now we can state the following Lemma 2.3; this is nothing else than [7] Lemma 2.2, which refers to a particular choice of  $S$  and  $E$ , rewritten in the general case, and the proof of the former goes exactly as the proof of the latter, taking care of the obvious generalizations. The essential thing here is that a fiber of  $\mathbb{P}(E)$  is just a  $\mathbb{P}^{r-1}$ , and we are dealing with the sheaf  $\mathcal{O}_{\mathbb{P}(E)}(1)$  which is  $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$  restricted on a fiber; hence to ask that one of its sections vanishes on the whole fiber  $\pi^{-1}(P)$  (here  $P$  is a point of  $S$ ), or on the  $r$  independent points of the fiber  $\pi^{-1}(P) \cap \mathbb{P}(E_i)$ , is the same.

In the next lemma  $Z$  is a 0-dimensional scheme; notice that in this case  $\text{length}(\widehat{Z}) = r \text{length}(Z)$ .

If  $Z$  is a variety of dimension  $\geq 1$  such that  $E|_Z$  splits as  $F_1 \oplus \dots \oplus F_r$  (where the  $F_i$ 's are line bundles on  $Z$ ), it is easy to see that the vanishing of the sections of  $E$  along  $\pi^{-1}(Z)$  is equivalent to the



vanishing of the sections of  $\mathcal{O}_{\mathbb{P}(E)}(1)$  along  $\mathbb{P}(F_1) \cup \dots \cup \mathbb{P}(F_r)$ , but each one of these copies of the variety  $Z$  may behave differently.

For example, if  $Z$  is a line inside  $\mathbb{P}^2$  and  $E$  is the cotangent bundle of  $\mathbb{P}^2$ , then  $E|_Z = \Omega_Z \oplus F$  where  $F \cong \mathcal{O}_Z(-1)$ ; if  $\mathcal{L} := \mathcal{O}_{\mathbb{P}^2}(2)$  and we want to study the postulation of  $\pi^{-1}(Z)$  with respect to  $\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L}$ , we have to consider that  $(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L})|_{\mathbb{P}(\Omega_Z)} \cong \Omega_Z(2) \cong \mathcal{O}_Z$ , while  $(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L})|_{\mathbb{P}(F)} \cong \mathcal{O}_Z(1)$  (see [16], 3.2.1 and [11], V.2.6).

LEMMA 2.3. *Let  $S, E, \pi$  be as in Notation 1, let  $\mathcal{L}$  be an invertible sheaf on  $S$ , and let  $Z$  be a 0-dimensional subscheme of  $S$ . Let  $U$  be an open subset of  $S$  as in Notation 1, and such that  $Z \subseteq U$ . Then*

$$H^0(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L} \otimes \mathcal{I}_{\pi^{-1}(Z)}) \cong H^0(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L} \otimes \mathcal{I}_{\hat{Z}}).$$

Let's go back to the problem we were speaking above in the example on  $\mathbb{P}^2$  with rank  $E = r > 1$ , that is, the problem of breaking up a point of  $\mathbb{P}^2$  into smaller pieces, in order to have something which counts 1, and not  $r$ , for  $E$ . For example, if  $h^0(E) = s > 0$  with  $r$  not dividing  $s$ , and if we wish to build a, say, generic union of points having the numerical prerequisites for being  $E$ -settled, it is not possible to look for it in  $\mathbb{P}^2$  but we can do the following: divide  $s$  by  $r$ ,  $s = nr + i$  with  $0 < i < r$ , consider  $p_1, \dots, p_{n+1}$  generic points in  $\mathbb{P}^2$  and take the generic union  $X$  in  $\mathbb{P}(E)$  of  $\pi^{-1}(p_1), \dots, \pi^{-1}(p_n)$  with  $i$  generic points  $q_1, \dots, q_i$  in the fiber  $\pi^{-1}(p_{n+1}) \cong \mathbb{P}^{r-1}$ ; this is a  $(r - 1)$ -dimensional subscheme of  $\mathbb{P}(E)$ , and by Lemma 2.2

$$\begin{aligned} h^0(\mathcal{O}_{\mathbb{P}(E)}(1)|_X) &= \sum_{j=1}^n h^0(\mathcal{O}_{\mathbb{P}(E)}(1)|_{\pi^{-1}(p_j)}) + h^0(\mathcal{O}_{\mathbb{P}(E)}(1)|_{q_1 \cup \dots \cup q_i}) \\ &= \sum_{j=1}^n h^0(E|_{p_j}) + i = nr + i = s. \end{aligned}$$

If you prefer to have a 0-dimensional scheme in  $\mathbb{P}(E)$ , just take the union of  $r$  generic points in each fiber  $\pi^{-1}(p_1), \dots, \pi^{-1}(p_q)$  with  $i$  generic points in  $\pi^{-1}(p_{q+1})$ ; Lemma 2.3 above assures that this is equivalent to the other choice.

### 3. The minimal free resolution for generic unions of 2-jets in $\mathbb{P}^2$

If  $X$  is a 0-dimensional scheme, we denote the length of  $X$  by  $l(X)$ .

In the following a 2-jet is a curvilinear scheme of length 2 in  $\mathbb{P}^2$ ; hence in affine coordinates  $x, y$  a 2-jet has an ideal of the form  $(x^2, y)$ , and consists of a point plus a tangent direction.

In this section  $Z$  will denote a generic union of 2-jets  $Z_1, \dots, Z_m$  in  $\mathbb{P}^2$ , so that  $l(Z) = 2m$ .

In [4] it is proved that such a  $Z$  has good postulation, i.e., for each  $k \geq 0$ ,  $h^0(\mathcal{I}_Z(k)) = \max\{0, h^0(\mathcal{O}_{\mathbb{P}^2}(k)) - h^0(\mathcal{O}_Z(k))\} = \max\{0, \binom{k+2}{2} - 2m\}$ . Here we prove, as an illustration of the techniques of Section 2, that it is also minimally generated.

Notice that good postulation plus minimal generation gives, the ambient space being  $\mathbb{P}^2$ , the entire minimal free resolution, which is hence the expected one.

In the following we use the results of Section 2 with ambient variety  $\mathbb{P}^2$  and vector bundle  $\Omega := \Omega_{\mathbb{P}^2}$ , the cotangent bundle of  $\mathbb{P}^2$ ; hence we work in  $\mathbb{P}(\Omega)$  with the projection  $\pi : \mathbb{P}(\Omega) \rightarrow \mathbb{P}^2$ . We need some more

NOTATION 2. For each  $n \geq 0$  we set:

$$\mathcal{E}_n := \mathcal{O}_{\mathbb{P}(\Omega)}(1) \otimes \pi^{-1}\mathcal{O}_{\mathbb{P}^2}(n).$$

We recall that  $h^0(\mathcal{E}_{k+1}) = h^0(\Omega(k+1)) = k(k+2)$ .

For any  $k \geq 0$  we write the number of global sections of  $\mathcal{E}_{k+1}$  modulo 2, i.e.  $k(k+2) = 2l + \epsilon$ ,  $\epsilon = 0, 1$ .

If  $k = 2l$ ,  $k(k+2) \equiv 0 \pmod{4}$ ;  $Y_k$  will denote a generic union in  $\mathbb{P}^2$  of  $\frac{k(k+2)}{4}$  2-jets, hence  $l(Y_k) = \frac{k(k+2)}{2}$ . If  $k = 2l + 1$ ,  $k(k+2) \equiv 3 \pmod{4}$ ;  $Y_k$  will denote a generic union in  $\mathbb{P}^2$  of  $\lfloor \frac{k(k+2)}{4} \rfloor$  2-jets, hence  $l(Y_k) = \frac{k(k+2)-3}{2}$ .

Let  $P$  be a point in  $\mathbb{P}^2$ ,  $A, B$  two distinct points in the fiber  $\pi^{-1}(P)$ , and  $\eta(A)$  a nilpotent of length 2, supported on  $A$  and not contained in  $\pi^{-1}(P)$ . For any  $k \geq 0$ , the remainder scheme  $R_k$  is defined to be the empty set for  $k$  even, and  $\eta(A) \cup B$  for  $k$  odd.

For any  $k \geq 0$ ,  $Z_k$  will denote the generic union in  $\mathbb{P}(\Omega)$  of  $\widehat{Y}_k$  (recall Notation 1) with  $R_k$ . Hence the 0-dimensional scheme  $Z_k$  has

length  $k(k + 2)$ , so that if  $H^0(\mathcal{I}_{Z_k} \otimes \mathcal{E}_{k+1}) = 0$  then the restriction map  $H^0(\mathcal{E}_{k+1}) \rightarrow H^0(\mathcal{E}_{k+1}|_{Z_k})$  is bijective and  $Z_k$  is  $\mathcal{E}_{k+1}$ -settled.

PROPOSITION 3.1. *For any  $k \geq 0$  we have  $H^0(\mathcal{E}_{k+1}) \otimes \mathcal{I}_{Z_k} = 0$ .*

*Proof.* We use induction on  $k$  with step 2.

The initial cases are: for  $k$  even,  $k = 0$  for which the statement is trivially true, since  $H^0(\mathcal{E}_1) = 0$ ; for  $k$  odd,  $k = 1$  which is proved in [15] lemma 2.2 case A(1).

Now we assume the assertion true for  $k - 2$ . Let  $C$  be a smooth conic in  $\mathbb{P}^2$ . We consider a specialization  $Y_k^*$  of  $Y_k$  (and the corresponding specialization  $Z_k^*$  of  $Z_k$ ) obtained specializing  $k$  among the 2-jets on  $C$ , so that  $l(Y_k^* \cap C) = 2k$ .

We now consider the exact sequence:

$$0 \rightarrow \mathcal{E}_{k-1} \otimes \mathcal{I}_{res_{\pi^{-1}C}Z_k^*} \rightarrow \mathcal{E}_{k+1} \otimes \mathcal{I}_{Z_k^*} \rightarrow \mathcal{E}_{k+1}|_{\pi^{-1}C} \otimes \mathcal{I}_{Z_k^* \cap \pi^{-1}C, \pi^{-1}C} \rightarrow 0.$$

We have:  $h^0(\mathcal{E}_{k+1}|_{\pi^{-1}C} \otimes \mathcal{I}_{Z_k^* \cap \pi^{-1}C, \pi^{-1}C}) = h^0(\Omega(k + 1)|_C \otimes \mathcal{I}_{Y_k^* \cap C, C})$  by 2.2 taking into account that  $\mathcal{O}_{\mathbb{P}(\Omega)}(1)|_{\pi^{-1}C} \cong \mathcal{O}_{\mathbb{P}(\Omega|_C)}(1)$ ; since  $\Omega|_C \cong \mathcal{O}_{\mathbb{P}^1}(-3)^2$ , we have  $h^0(\Omega(k + 1)|_C \otimes \mathcal{I}_{Y_k^* \cap C, C}) = h^0(\mathcal{O}_{\mathbb{P}^1}(2k - 1)^2 \otimes \mathcal{I}_{Y_k^* \cap C, C}) = 0$ .

Moreover,  $res_C Y_k^*$  is a generic union of 2-jets, with  $l(res_C Y_k^*) = l(Y_k^*) - l(Y_k^* \cap C) = l(Y_k^*) - 2k$ , and  $res_{\pi^{-1}C} Z_k^* = \widehat{res_C Y_k^*} \cup R_k$ , so that  $l(res_{\pi^{-1}C} Z_k^*) = 2l(res_C Y_k^*) + l(R_k) = (k - 2)k$ . Hence,  $h^0(\mathcal{E}_{k-1} \otimes \mathcal{I}_{res_{\pi^{-1}C}Z_k^*}) = 0$  by induction assumption, so  $h^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{Z_k^*}) = 0$  and by semicontinuity we conclude  $h^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{Z_k}) = 0$ .  $\square$

PROPOSITION 3.2. *Let  $Y$  be a generic union of  $m$  2-jets  $Z_1, \dots, Z_m$  in  $\mathbb{P}^2$ . Then the restriction map  $\psi_k : H^0(\mathcal{E}_{k+1}) \rightarrow H^0(\mathcal{E}_{k+1}|\hat{Y})$ , or, equivalently, the restriction map  $\rho_k : H^0(\Omega(k + 1)) \rightarrow H^0(\Omega(k + 1)|_Y)$  is injective if  $\lfloor \frac{k(k+2)}{4} \rfloor < m$  and surjective if  $\lfloor \frac{k(k+2)}{4} \rfloor \geq m$ .*

*Proof.* Let  $\lfloor \frac{k(k+2)}{4} \rfloor < m$ ; hence, we can assume  $Z_k \subseteq \hat{Y}$  (if  $k$  is odd, and  $X$  is one of the 2-jets of  $Y$ , we can assume that the scheme  $R_k$  is contained in  $\hat{X}$ ). Hence we have the exact sequence (where  $\mathcal{I}_{Z_k, \hat{Y}}$  is the ideal sheaf of  $Z_k$  in  $\hat{Y}$ ):

$$0 \rightarrow \mathcal{E}_{k+1} \otimes \mathcal{I}_{\hat{Y}} \rightarrow \mathcal{E}_{k+1} \otimes \mathcal{I}_{Z_k} \rightarrow \mathcal{E}_{k+1} \otimes \mathcal{I}_{Z_k, \hat{Y}} \rightarrow 0$$

which, since  $H^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{Z_k}) = 0$  by 3.1, gives  $H^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{\hat{Y}}) = 0$ , i.e.  $\psi_k$  is injective for  $\hat{Y}$  or equivalently (see Section 2)  $\rho_k$  is injective for  $Y$ .

Now let  $\lfloor \frac{k(k+2)}{4} \rfloor \geq m$ ; we can take away  $\lfloor \frac{k(k+2)}{4} \rfloor - m$  2-jets from  $Y_k$  and we are left with  $Y$ ; hence, we can assume  $\hat{Y} \subseteq Z_k$ . So we get the exact sequence

$$0 \rightarrow \mathcal{E}_{k+1} \otimes \mathcal{I}_{Z_k} \rightarrow \mathcal{E}_{k+1} \otimes \mathcal{I}_{\hat{Y}} \rightarrow \mathcal{E}_{k+1} \otimes \mathcal{I}_{\hat{Y}, Z_k} \rightarrow 0.$$

Since  $h^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{Z_k}) = 0$ , we get

$$\begin{aligned} h^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{\hat{Y}}) &\leq h^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{\hat{Y}, Z_k}) \\ &= h^0(\mathcal{E}_{k+1} \otimes \mathcal{O}_{Z_k}) - h^0(\mathcal{E}_{k+1} \otimes \mathcal{O}_{\hat{Y}}) \\ &= k(k+2) - 4m. \end{aligned}$$

On the other hand the exact sequence

$$0 \rightarrow \mathcal{E}_{k+1} \otimes \mathcal{I}_{\hat{Y}} \rightarrow \mathcal{E}_{k+1} \rightarrow \mathcal{E}_{k+1}|_{\hat{Y}} \rightarrow 0$$

gives  $h^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{\hat{Y}}) \geq h^0(\mathcal{E}_{k+1}) - h^0(\mathcal{E}_{k+1}|_{\hat{Y}}) = k(k+2) - 4m$ , so that we have the equality  $h^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{\hat{Y}}) = h^0(\mathcal{E}_{k+1}) - h^0(\mathcal{E}_{k+1}|_{\hat{Y}})$  and the map  $\psi_k$  is surjective for  $\hat{Y}$  or equivalently (see Section 2)  $\rho_k$  is surjective for  $Y$ .  $\square$

Now the property of having the expected resolution for a generic union of 2-jets follows from Proposition 3.2 together with what we have seen in Section 1:

**COROLLARY 3.3.** *Let  $Z$  be a generic union of  $m$  2-jets  $Z_1, \dots, Z_m$  in  $\mathbb{P}^2$ . Then, for any  $m > 0$ ,  $Z$  is minimally generated and has the expected minimal free resolution.*

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