

On Congruences of Linear Spaces of Order One

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SUMMARY. - *After presenting the main notions and results about congruences of k -planes, we dwell upon congruences of lines, mainly of order one. We survey the classification results in the projective spaces of dimension 3 and 4, which are almost complete, and the (partial) results and some conjectures in higher dimension. Finally we present some new results, in particular a degree bound for varieties with one apparent double point, a new class of examples with focal locus of high degree, and some general results about the classification of first order congruences of lines in \mathbb{P}^4 with reducible focal surface.*

1. Introduction

Families of linear spaces in the projective space are a classical subject of study, both in algebraic and in differential geometry. Among them, congruences of linear spaces have been particularly studied. Let us define them. A congruence B of k -spaces in \mathbb{P}^n is a (flat) family of k -spaces of dimension $n - k$, or, in other words, a subvariety of the

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Grassmannian $\mathbb{G}(k, n)$ of dimension $n - k$. Therefore, the number of k -spaces of a congruence B passing through a general point of \mathbb{P}^n is finite (possibly zero): it is called the order of B .

The systematic study of congruences was initiated in the second half of the nineteenth century by Kummer ([25]), who gave the first classification theorem of congruences of lines of order one and two in \mathbb{P}^3 . His work was since developed by many algebraic geometers, as for instance Schumacher, Bordiga, C. Segre, Fano, Semple, Roth. In recent times congruences of lines in \mathbb{P}^3 were studied by Goldstein ([21]), who tackled the classification problem from the point of view of the focal locus. More in general Ran, in [32], studied the surfaces of order one in all Grassmannians, giving a modern and more complete formulation of the classical result of Kummer. Smooth congruences of lines in \mathbb{P}^3 , seen as surfaces in a smooth 4-dimensional quadric, in analogy with smooth surfaces in \mathbb{P}^4 , were studied by Arrondo, Sols, Gross, Turrini, Bertolini, Verra.

Congruences of lines of order one in higher-dimensional spaces were considered by Castelnuovo, Palatini, and, in a systematic way in \mathbb{P}^4 and \mathbb{P}^5 , by Marletta ([28]) and his student Sgroi. They tried to give a classification theorem according to the number and the dimension of the components of the focal locus. Their results have been analysed and completed, in modern language, by De Poi, in a series of papers.

As for congruences of linear spaces, we mention the foundational articles of C. Segre [33], and, in recent times, the article of Ciliberto-Sernesi [9], which well explains the differential-geometric techniques involved in the study of the focal and fundamental loci associated to a congruence.

The point of view of algebraic geometry is mixed with that of differential geometry in the long article of Griffiths-Harris [23]. As for the differential-geometric point of view, we quote the classical book of Finikov [19], whereas in modern times the topic has been revived, among others, by Akivis-Goldberg in several papers and books.

Interest towards the above topics is motivated by the several connections they have with some important open problems. We mention just a few of them: the conjectures of Zak on k -normality and

general projections ([35]), the classification of the varieties with one apparent double point ([8]), the connections with systems of PDE of conservation laws, introduced by Agafonov-Ferapontov ([1], [16]).

The aim of this paper is to take stock of the situation regarding congruences of order one. In §2, we recall the basic notions of fundamental and focal point and of multidegree, and we state some general results which are valid in the case of order one. In §3 we present several examples and classes of examples to illustrate the ideas motivating the definitions. In §4 we collect some of the results in dimension 3: the classification theorem for order one by Kummer, the results for congruences of low degree and low order, and for congruences with a fundamental curve. The techniques used in dimension 3 do not always extend to higher spaces. In §5 we consider congruences of lines in any space \mathbb{P}^n . We introduce the notion of parasitical scheme and of fundamental d -locus and prove some general results and bounds on the dimension and the degree of the irreducible components of the fundamental locus. As an application, we obtain a degree bound for the degree of the varieties with one apparent double point. In §6, we consider the case of the congruences of lines of order one with irreducible and reduced focal locus, and discuss the sharpness of the bounds of previous section in this particular situation. We give also a new class of examples of congruences of order one with focal locus of “high” degree in \mathbb{P}^n , for $n \geq 5$. We conjecture that the focal locus is irreducible. We then concentrate in §7 on congruences of lines in \mathbb{P}^4 , where we survey the known results, and state some new results and conjectures. In particular, for a first order congruence B of lines in \mathbb{P}^4 , we prove first that, if B is given by the secant lines to a surface F_1 that meet another surface F_2 also, then either F_1 is a plane or F_2 is a cubic scroll. If instead B is given by the lines meeting three surfaces, then at least one of these surfaces is a plane.

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2. Generalities about families of k -planes in \mathbb{P}^n

We will work with schemes and varieties over the complex field \mathbb{C} , with standard notation and conventions as in [24]. In particular, a variety is always irreducible.

Let $\mathbb{G}(k, n)$ denote the Grassmann variety of k -planes in \mathbb{P}^n : we recall that it is the Hilbert scheme of these spaces.

DEFINITION 2.1. *Let us consider a flat family of k -planes in \mathbb{P}^n obtained by a desingularization of a subvariety B' of dimension $n - k$ of the Grassmannian $\mathbb{G}(k, n)$; this family (or better its basis B or also B') is called congruence of k -planes in \mathbb{P}^n .*

Then, there is a diagram:

$$\begin{array}{ccc} \Lambda \subset B \times \mathbb{P}^n & \xrightarrow{f} & \mathbb{P}^n \\ p \downarrow & & \\ B & & \end{array} \quad (1)$$

where Λ is the incidence correspondence, and p and f are the restrictions to Λ of the two natural projections. We will use the notations: $\Lambda_b := p^{-1}(b)$, and $f(\Lambda_b) =: \Lambda(b)$ is a k -plane in \mathbb{P}^n .

The map p is obviously surjective with all fibres of dimension k , so $\dim \Lambda = n$ and it is expected that f is a surjective morphism. If instead $f(\Lambda) \neq \mathbb{P}^n$, we say that B is a *degenerate* congruence, or a congruence of order zero. In this case, the image is a subvariety of \mathbb{P}^n “covered by many lines”. We are not interested here in analysing this situation, so from now on we will always assume that B is non-degenerate. Therefore f will be a finite morphism and we will denote its degree by a . It is called the *order* of the congruence B , it represents the number of lines of B passing through a general point in \mathbb{P}^n .

DEFINITION 2.2. *A point P in \mathbb{P}^n is a fundamental point of B if the fibre of f over P is infinite. The set of the fundamental points of B is called the fundamental locus. It will be denoted by Φ .*

DEFINITION 2.3. *The ramification divisor of f inside Λ is called the focal scheme of the congruence B . Its schematic image in \mathbb{P}^n is the focal locus of B , it will be denoted by F . Every point of F is called a focus of B .*

It is clear that $\dim \Phi \leq n - 2$ and $\dim F \leq n - 1$. The following facts about the fundamental and the focal loci are classical:

THEOREM 2.4. *Let B be a congruence of k -spaces in \mathbb{P}^n . Then:*

1. *every fundamental point of B is a focus;*
2. *on every space $\Lambda(b)$ of the family, the focal locus either is a hypersurface of degree $n - k$ in $\Lambda(b)$, or coincides with the whole k -plane $\Lambda(b)$;*
3. *if \tilde{F} is an irreducible component of dimension $n - 1$ of F , then every space of the family is tangent to \tilde{F} at its focal but not fundamental points.*

REMARK 2.5. *In the previous theorem, case (2), we should specify that the focal points of the hypersurface of degree $n - k$ are focal points for the k -plane $\Lambda(b)$, in the sense that such points P are those for which $f|_{\Lambda_b}$ drops rank in (b, P) .*

Obviously, we can have also points in $\Lambda(b) \cap F$, which are foci for other k -planes: for example, the dual of a plane curve C , which is not a line, is a congruence, whose focal locus is C , and each tangent line $\mathbb{T}_P C$ intersects C in other points outside P , if $\deg(C) > 2$.

DEFINITION 2.6. *If the k -plane $\Lambda(b)$ is contained in the focal locus, then it is called focal k -plane.*

REMARK 2.7. *If $k > 1$, after restricting the family B to the non-focal k -planes, one can define the second order foci, as the focal points of the family of the focal hypersurfaces of degree $n - k$, of Theorem 2.4 (2), contained in the k -planes of B . This is a family of non-linear varieties, so more attention is required in the definition. It is now clear that we can similarly define higher order foci.*

As we stated in Theorem 2.4 (1), given any congruence B , the fundamental locus is contained in the focal locus, since the fibre of

f at every point of Φ has dimension greater than the general one. The following theorem, of which one implication was not classically known, takes care of the opposite inclusion. It is contained in [13] in the case of congruences of lines. We give here the proof, which is similar, for congruences of k -spaces.

THEOREM 2.8. *Let B be a non-degenerate congruence of k -spaces in \mathbb{P}^n . The focal locus F coincides with the fundamental locus Φ and has codimension > 1 if and only if B has order one.*

Proof. Assume $a = 1$, then f is generically $(1 : 1)$ so the Zariski Main Theorem implies that the two loci coincide. Conversely, assume that the two loci coincide, and consider the restriction of f to $\Lambda \setminus f^{-1}(F)$: it is an unramified covering of $\mathbb{P}^n \setminus F$. By dimensional reasons, $\mathbb{P}^n \setminus F$ is simply connected, and $\Lambda \setminus f^{-1}(F)$ is connected. Therefore $F|_{\Lambda \setminus f^{-1}(F)}$ is a homeomorphism, hence a birational map and B is a first order congruence. \square

We conclude this section introducing the multidegree of a congruence of k -spaces in \mathbb{P}^n . We will use the notation of the Griffiths-Harris book [22] for the Schubert cycles in $\mathbb{G}(k, n)$. Given a complete flag $\{\mathbb{P}^0 \subset \mathbb{P}^1 \subset \dots \subset \mathbb{P}^n\}$ and a sequence of integers c_0, \dots, c_k such that $n - k \geq c_0 \geq \dots \geq c_k \geq 0$, we define: σ_{c_0, \dots, c_k} as the class of analytic cohomology of

$$\{\ell \in \mathbb{G}(k, n) \mid \dim(\ell \cap \mathbb{P}^{n-k+i-c_i}) \geq i, i = 0, \dots, k\},$$

where the dimension of the cycle σ_{c_0, \dots, c_k} is $(k+1)(n-k) - \sum_i c_i$.

Let $[B]$ denote the class of analytic cohomology of a congruence B : since $n - k = \dim B$, we can write:

$$[B] = \sum_{\sum_i c_i = k(n-k)} a_{c_0, \dots, c_k} \sigma_{c_0, \dots, c_k}. \quad (2)$$

We say that *the congruence B has multidegree, $(\dots, a_{c_0, \dots, c_k}, \dots)$* if equation (2) holds.

In the case of the congruences of lines, the expression can be written as follows:

$$[B] = \sum_{i=0}^{\nu} a_i \sigma_{(n-1-i)i}, \quad (3)$$

where we put $\nu := \lfloor \frac{n-1}{2} \rfloor$. So, the congruence of lines B has multidegree (a_0, \dots, a_ν) if equation (3) holds.

REMARK 2.9. We observe that the first coefficient, corresponding to $(c_0, \dots, c_k) = (n - k, \dots, n - k, 0)$ (or a_0 , in the case of the congruences of lines), is the number of k -planes passing through a general point $P \in \mathbb{P}^n$, and therefore it is the order a of the congruence. Clearly, we can give in a similar way the geometric meaning of all the elements of the multidegree; for example, in the case of the congruence of lines, a_j is the number of lines intersecting a general j -plane and contained in a general $(n - j)$ -plane in \mathbb{P}^n .

3. Examples

Let us see now some significant classes of congruences B of k -planes in \mathbb{P}^n .

3.1. $k = n - 1$: curves in $\check{\mathbb{P}}^n$

Let us denote by $\check{\mathbb{P}}^n = \mathbb{G}(n - 1, n)$ the projective dual space of \mathbb{P}^n . Then, the congruences of $(n - 1)$ -planes, *i.e.* with $k = n - 1$, are just the curves \check{C} in $\check{\mathbb{P}}^n$. In this case, the multidegree is just the order, $a_{1, \dots, 1, 0}$, which is in this case the degree of the curve $\check{C} \subset \check{\mathbb{P}}^n$.

3.2. $n = 3$ and $k = 1$: secant lines to curves in \mathbb{P}^3

If we take a smooth skew curve C in \mathbb{P}^3 , then the set of its secant lines, $\text{Sec}(C)$, is a family of dimension two of lines in \mathbb{P}^3 , hence a congruence.

In this case, the congruence has multidegree (a_0, a_1) : the order a_0 is just the number of *apparent double points* of C , that is, the number of double points of the general projection of C to a plane. Thanks to the Clebsch formula, this number is given by

$$a_0 = \binom{d-1}{2} - g, \tag{4}$$

where $d = \text{deg}(C)$ and g is the genus of C .

a_1 is called, also in general for a congruence B in \mathbb{P}^3 , the *order*, and it is the number of lines of B contained in a general plane. In the case $B = \text{Sec}(C)$, it is immediate to see that

$$a_1 = \binom{d}{2}.$$

We note that the curve C is contained in the focal locus, since the two points of secancy are foci. But in general there are also other components of dimension two: the surface of the trisecant lines to C , and the surface of the *stationary secants*, see [3]. A stationary secant is a secant line $\langle P, Q \rangle$, $P, Q \in C$, such that the two tangent lines $\mathbb{T}_P C$ and $\mathbb{T}_Q C$ meet. A stationary secant is a focal line, since it can be shown that the tangent plane $\mathbb{T}_{\langle P, Q \rangle} B$ is contained in $\mathbb{G}(1, 3)$.

3.3. $n = 4$ and $k = 1$: trisecant lines to surfaces in \mathbb{P}^4

Consider now a non-degenerate surface S in \mathbb{P}^4 and let B be the family of its trisecant lines. It is not hard to prove that either B is empty (if we suppose that a line contained in S is not trisecant) or has dimension exactly three.

The smooth surfaces without trisecant lines are classified in [6], and are the elliptic quintic scrolls and the surfaces contained in quadric hypersurfaces.

So, if not empty, the family B has dimension three. S is a component of the fundamental locus. The three secancy points of a general trisecant line are its three foci.

But, in general, we can also have a focal locus of dimension three, if through all points in S pass focal lines: in fact, it is easy to see that the family of 4-secant lines either is empty or has dimension at least two. But clearly a 4-secant line is a focal line, so “in general”—*i.e.* if the family of the 4-secants is not empty—the closure of the union of 4-secant lines is a component of the focal locus of dimension three.

The multidegree is (a_0, a_1) , a_0 the order and a_1 the *class*. If the surface S is smooth, a_0 is the number of its apparent triple points. The following formula holds:

$$a_0 = \binom{d-1}{3} - \pi(d-3) + 2\chi - 2,$$

where $d = \text{deg}(S)$, π is its sectional genus, and χ is its Euler-Poincaré characteristic (see [6]).

The class a_1 is the degree of the surface of the lines of B contained in a hyperplane; it is given by the formula

$$a_1 = h(d - 2) - \binom{d}{3}$$

where h is the number of apparent double points of the general hyperplane section of S , which is given, in the smooth case, by Formula (4). The preceding formula is due to Cayley, and can be found in [26]. Explicitly, if S is smooth, we have

$$a_1 = (d - 2) \left(\frac{(d - 1)(d - 3)}{3} - \pi \right).$$

3.4. Linear congruences

The congruences that come out from linear sections of the Grassmannian $\mathbb{G}(k, n) \subset \mathbb{P}^N$, $N = \binom{n}{k} - 1$, are classically called *linear congruences*.

Since the Schubert cycle that corresponds to a hyperplane section of the Grassmannian is σ_1 , from Pieri's formula it follows that, in general, a linear congruence of k -planes is rationally equivalent to

$$\sigma_1^{n-k} = \sigma_{n-k, \dots, n-k, 0} + (n - k)\sigma_{n-k, \dots, n-k, n-k-1, 1} + \dots$$

In particular, its order is one. For $k = 1$, if we set $h := \lfloor \frac{n-1}{2} \rfloor$, we get

$$\sigma_1^{n-1} = \sum_{i=0}^h \left(\binom{n-2}{i} - \binom{n-2}{i-2} \right) \sigma_{n-1-i, i}$$

—with the convention that $\binom{\ell}{m} = 0$ if $m < 0$ (see [12]).

Therefore, a linear congruence of lines has multidegree

$$\left(1, n-1, \dots, \left(\binom{n-2}{i} - \binom{n-2}{i-2} \right), \dots, \left(\binom{n-2}{\nu} - \binom{n-2}{\nu-2} \right) \right).$$

It can be shown (see [7]) that the focal locus of this congruence of lines is the degeneracy locus F of a general morphism

$$\phi: \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)} \rightarrow \Omega_{\mathbb{P}^n}(2) \tag{5}$$

of (coherent) sheaves on \mathbb{P}^n and that F is smooth if $\dim(F) \leq 3$. If F is the focal locus of a general linear congruence in \mathbb{P}^n , then

1. if n is even, F is a rational variety;
2. if n is odd, F is a scroll over (an open set of) a hypersurface of degree $(n+1)/2$ contained in a \mathbb{P}^{n-2} .

Besides,

$$\deg(F) = \frac{n^2 - 3n + 4}{2}.$$

In low dimension and for a general linear section, we have that (see [7] for details) if $n = 3$, F is the union of two skew lines, if $n = 4$, F is a smooth projected Veronese surface and if $n = 5$, F is a (rational) threefold of degree seven, which is a scroll over a cubic surface in \mathbb{P}^3 . It is also known as Palatini scroll (see [29]).

3.5. Matrices of type $(n-1) \times n$ with linear entries

We construct now an interesting family of first order congruences of lines (see [12]), with the characteristic to have as focal locus an irreducible variety of “high” degree. The first example is the following:

EXAMPLE 3.1. *Let us consider the rational normal curve C in \mathbb{P}^3 ; it is well known that its ideal is generated by the minors of order two of the following matrix:*

$$A := \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

where x_0, \dots, x_3 are the projective coordinates of \mathbb{P}^3 . It is easy to see that a secant line ℓ of C has equations $h_1 = h_2 = 0$ with:

$$h_1 := \sum_{i=0}^2 \lambda_i x_i \qquad h_2 := \sum_{i=0}^2 \lambda_i x_{i+1}$$

where $(\lambda_0 : \lambda_1 : \lambda_2) \in \mathbb{P}^2$, i.e. (h_1, h_2) is a linear combination of the columns of A .

Then, the secant line ℓ has, as Plücker coordinates, the minors of order two of the matrix

$$B := \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & 0 \\ 0 & \lambda_0 & \lambda_1 & \lambda_2 \end{pmatrix}$$

therefore the map

$$\phi: \mathbb{P}^2 \rightarrow \mathbb{G}(1, 3)$$

associates to the point $(\lambda_0 : \lambda_1 : \lambda_2)$ the Plücker coordinates of the line ℓ , i.e.

$$\phi(\lambda_0 : \lambda_1 : \lambda_2) := (\lambda_0^2 : \lambda_0\lambda_1 : \lambda_0\lambda_2 : \lambda_1^2 - \lambda_0\lambda_2 : \lambda_1\lambda_2 : \lambda_2^2).$$

So, the family of the secant lines of the rational normal curve is a Veronese surface.

To generalise this example, consider a general morphism $\phi \in \text{Hom}(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}, \mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1))$, whose minors vanish in the expected codimension two. In this case, $F := V(\phi)$ —the degeneracy locus of ϕ —is a locally Cohen-Macaulay subscheme, whose ideal has minimal free resolution:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}(-n) \xrightarrow{\phi(-n)} \mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1-n) \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_F \rightarrow 0.$$

Then—for example—from the Hilbert polynomial we get

$$\text{deg}(F) = \binom{n}{2} \tag{6}$$

$$\pi(F) = 1 + \frac{2n-7}{3} \binom{n}{2} \tag{7}$$

where $\pi(F)$ is the sectional genus of F . It is easy to prove also that F is rational, and, if $n \leq 5$, it is smooth. Besides, the adjunction map $\varphi_{|K_F+H|}$ exhibits F as the blow-up of \mathbb{P}^{n-2} in a scheme Z of degree $\binom{n+1}{2}$ and sectional genus $\frac{n}{6}(2n-5)(n+1) - 1$. In particular, if $n = 4$, F is a rational sextic which is the blow-up of the plane in 10 points, i.e. a Bordiga surface.

It turns out that the $(n-1)$ -secant lines of the variety F defined as above form a first order congruence of lines B in \mathbb{P}^n . The congruence B is smooth for general ϕ (see [12]).

3.6. Congruences of lines with focal locus of low degree

As last example of congruences of lines, we mention here a family of sextic threefolds in \mathbb{P}^5 , introduced in [17] which is particularly interesting from many points of view. For the aim of this article, it turns out that this threefold is such that the family of its 4-secant lines is a first order congruence B .

This congruence is associated to a *completely exceptional Monge-Ampère system* of PDE's via the construction of Agafonov-Ferapontov (see [1]).

B results to be an irreducible component of a (special) *quadratic congruence*. To define it, let L be a 3-dimensional linear space in \mathbb{P}^5 , and C be a twisted cubic curve in L . As we have seen in Subsection 3.5, the secant lines of C give rise to a Veronese surface V contained in $\mathbb{G}(1, L)$. There exist 2-planes $\pi \subset \check{\mathbb{G}}(1, 5)$, whose Gauss image is V (see [27]). We fix in \mathbb{P}^{14} a linear space Γ of dimension 11 whose projective dual is such a π . We first consider $\mathbb{G}(1, 5) \cap \Gamma \cap H_L$, where H_L is the unique tangent hyperplane to $\mathbb{G}(1, 5)$ along $\mathbb{G}(1, L)$. $\mathbb{G}(1, L)$ results to be an irreducible component of the intersection with multiplicity of intersection 2, hence the other component B' has multidegree $(1, 3, 0)$. The focal locus of B' is non-reduced with support L . It is proved in [17] that there exists a family of dimension 12 of quadric hypersurfaces in \mathbb{P}^{14} containing $\mathbb{G}(1, L) \cup B'$. Fixed such a quadric Q , the following equality defines a congruence B : $\mathbb{G}(1, 5) \cap \Gamma \cap Q = B' \cup \mathbb{G}(1, L) \cup B$.

B results to be a congruence of multidegree $(1, 3, 3)$ and a smooth Fano fourfold of genus 9. Its lines are the 4-secants of the focal locus, which is a threefold X of degree 6 with sectional genus 1, singular along C . The lines of B through a general point of X form a planar pencil. From the multidegree, one sees that X is not contained in any cubic, whereas its hyperplane section is, hence X is an example of non-2-normal threefold in \mathbb{P}^5 . The interest of this example is therefore related to Zak's conjectures on k -normality.

The study of the congruences associated to the completely exceptional Monge-Ampère equations has been performed in any dimension $n \geq 5$ in [18]. The main difference between the case of \mathbb{P}^5 and the higher dimensional cases is that, although the definitions of $L \cong \mathbb{P}^{n-2}$, Γ and H_L can be given easily on the analogy with \mathbb{P}^5 , Γ

always contains the $2(n - 2)$ -dimensional subgrassmannian $\mathbb{G}(1, L)$; therefore, $\mathbb{G}(1, n) \cap \Gamma \cap H_L$ is not a congruence if $n > 6$ for dimensional reasons. In practise, one has to consider the residual of $\mathbb{G}(1, L)$ in $\mathbb{G}(1, n) \cap \Gamma$, and then proceed to the construction. A second important difference is that, for general n , instead of quadrics, one has to use hypersurfaces of degree μ , with $\mu = \lfloor \frac{n+3}{4} \rfloor$. The cohomology class and the multidegree of B are computed in [18]: $[B] = \sigma_1^{n-1} + \sigma_{22}\sigma_1^{n-5}$, so in particular $a_0 = 1$, $a_1 = n - 2$ and $a_2 = \binom{n-2}{2}$. We will come back on this example in §6.

3.7. Congruences of planes in \mathbb{P}^4

Congruences of planes in \mathbb{P}^4 have been used in [9] to give an alternative proof of Torelli theorem. In this case, on a general plane of the family the focal locus is a conic. The case of the congruences with degenerate focal conics has since been studied by Pedreira and Solá-Conde in [30]. To accomplish the classification, the behaviour of the second order foci has to be carefully analysed.

4. Congruences of lines in \mathbb{P}^3

Next theorem was originally proved by Kummer (although case (3) was forgot), then reproved, in modern times, by several authors, including Ran, Arrondo, De Poi.

THEOREM 4.1. *Let B be a congruence of lines in \mathbb{P}^3 of order one. Then there are the following possibilities:*

1. B is the star of lines of centre a point;
2. B is the family of the secant lines of a rational normal cubic curve;
3. B is a union of pencils of lines, with centres on a line L ; congruences of this type of bidegree $(1, d)$ exist for all $d \geq 1$;
4. given a rational curve C of degree m , having a $(m - 1)$ -secant line L , B is the family of the lines meeting C and L .

REMARK 4.2. *As we will see in §5, Theorem 4.1 contains also the classification of the congruences of lines in \mathbb{P}^n , for any n , whose focal locus F has dimension one.*

REMARK 4.3. *Congruences of case (3) have non-reduced focal locus. They are constructed as follows: fix a regular map $\phi: \check{L} \rightarrow L$, of degree d , where \check{L} denotes the dual of L , i.e. the pencil of planes containing L . Then a congruence B of bidegree $(1, d)$ is obtained taking the union of the pencils of lines of centre $\phi(\pi)$ in the plane π , as π varies in \check{L} .*

The classification of the congruences of lines in \mathbb{P}^3 of degree bigger than 1 becomes immediately very complicated. They have been studied mainly under smoothness assumptions, interpreting them as smooth surfaces of \mathbb{P}^5 contained in a smooth quadric, due to the interesting analogies with smooth surfaces in \mathbb{P}^4 .

Goldstein ([21]) gave the classification of the smooth congruences which are scrolls, proving in particular that their degree is bounded by 6. Goldstein also gave a scheme of classification of non-necessarily smooth congruences, according to the dimension of the components of the focal locus ([20]). Arrondo-Sols ([5]) classified the smooth congruences of degree ≤ 9 , describing also the corresponding components of the Hilbert scheme: they proved that there are 25 families with $a_0 \leq a_1$. The congruences with $a_0 \geq a_1$ can be then obtained by duality, exchanging the role of α -planes and β -planes. For degree at most 8, the classification had already been performed by Fano, whereas Verra had considered the case of degree 9. Arrondo-Sols also proved a sort of analogue of the theorem of Ellingsrud-Peskin for surfaces in \mathbb{P}^4 , saying that there are only finitely many families of smooth surfaces in the Grassmannian $\mathbb{G}(1, 3)$ that are not of general type. The classification was continued by Gross, who classified smooth congruences of degree 10. Verra and Gross completed also the classification of the smooth congruences of order 2 and 3, whose degree results to be bounded by 10.

In a different vein, Arrondo and Gross found the complete description of the smooth congruences with a curve of fundamental points. The result is the following ([4]):

THEOREM 4.4. *Let B be a smooth congruence in \mathbb{P}^3 having a curve*

C of fundamental points. Then one of the following holds:

1. C is a line;
2. C is either a twisted cubic or an elliptic quartic and B is the congruence of the secant lines of C ;
3. B is a scroll of degree bigger than 2; there are three examples, of bidegree respectively $(1, 2)$, $(2, 2)$, $(3, 3)$, with C a conic in the first two cases and a smooth plane cubic in the third one;
4. C is a smooth plane cubic and B is a conic bundle over C of bidegree $(3, 6)$.

This theorem has been since extended to the complete classification of the smooth congruences having a fundamental curve in any space \mathbb{P}^n by Arrondo-Bertolini-Turrini (see [2]).

We point out that the methods used in the quoted papers are mainly liaison, degeneracy of maps of vector bundles, embeddings of surfaces in the Grassmannian $\mathbb{G}(1, 3)$ via rank two bundles. These tools, which are very useful in \mathbb{P}^3 , are not easily extended in spaces of higher dimension.

5. General Facts and Degree Bounds

In this section B denotes a non-degenerate congruence of lines. We have seen in §2 that the irreducible components of the fundamental locus Φ have all dimension $\leq n - 2$, that the fundamental locus is contained in the focal locus F , and that $\Phi = F$ if and only if the order of B is one.

If X is an irreducible component of dimension $n - 1$ of the focal locus, then either a general line of B intersects X in finitely many focal points, or X is the union of a subfamily of B formed by focal lines. This second possibility happens for instance with the trisecant lines of a curve C in \mathbb{P}^3 , as focal lines of the congruence of the secant lines (see §3).

Let us consider now an irreducible component X of the fundamental locus. We distinguish two possibilities: either a general line of the family B intersects X (in one point or more), or it does not.

For instance, let B be the congruence of the trisecant lines of a surface S in \mathbb{P}^4 (see §3). If S contains a plane curve C of degree ≥ 3 , then all the lines of the plane generated by C are trisecant to S . Therefore $\langle C \rangle$ is an irreducible component of the focal locus of B , but of course a general line of B does not meet it.

The components of this type are called *parasitical*. More precisely:

DEFINITION 5.1 ([14]). *Let B be a non-degenerate congruence of lines in \mathbb{P}^n . Let X be an irreducible component of dimension d of the fundamental locus of B , with $2 \leq d \leq n-2$. X is called i -parasitical if*

1. *through every point of X there pass infinitely many focal lines of B contained in X ;*
2. *X is a component of Φ with geometrical multiplicity i ;*
3. *a general line of B does not meet X .*

The union of the non-parasitical components of the fundamental locus Φ is the *pure fundamental locus*. We denote by Φ_d the union of its irreducible components of dimension d : this is called the *fundamental d -locus*. Its natural structure of scheme, inherited from the focal locus, comes from a Fitting ideal (see [14]), the one of minors of order $d+1$ of the differential of the map f (see §2). From this the following important fact can be deduced:

PROPOSITION 5.2. *If ℓ is a general line of B and X is an irreducible component of Φ_d , then the focal points of $X \cap \ell$ have multiplicity $\geq (n-2) - d + 1$ in the focal locus.*

The only example of congruence having a fundamental 0-locus is the star of all lines passing through a point P : clearly P is the unique focus and fundamental point for each line of the family, with multiplicity $n-1$ (see [11]). For the congruences whose focal locus has positive dimension, the following holds:

THEOREM 5.3 ([13]). *Let B be a congruence of lines in \mathbb{P}^n . Assume that the focal locus F has dimension $d > 0$. Then $\frac{n-1}{2} \leq d \leq n-1$. If moreover $d = \frac{n-1}{2}$, F_{red} is irreducible and a general line of B meets F_{red} in one point only, then F_{red} is a d -plane.*

Note that if $\dim F \leq n - 2$, then certainly the order of B is 1. As a Corollary of previous theorem, we get that, if $\dim F = 1$, then $n \leq 3$. The congruences of this type are all classified in [13]. If $n = 3$, the list is contained in the one of the congruences of first order, given in §4.

We turn now to the case of the congruences, necessarily of order one, such that $\Phi = F$ and its dimension is $\leq n - 2$. It is possible to give some bounds on the degree of F .

In order to do so, a very useful tool is the hypersurface V_Π (see [14]). It is defined as follows: let Π be a general $(n - 2)$ -plane, then V_Π is the union of the lines of B meeting Π . Clearly V_Π is obtained from a hyperplane section of B in the Plücker embedding. If B has multidegree (a_0, \dots, a_ν) , then as an easy application of Pieri's formula, one computes the degree of the hypersurface V_Π , which results to be equal to $a_0 + a_1$.

Note that, since B is a congruence of order one, if ℓ is a line of B not contained in V_Π , every point P of $V_\Pi \cap \ell$ is a focus for B , because at least two lines of the congruence pass through P .

From this observation, it can be deduced that, if one considers a second general $(n - 2)$ -plane Π' and the corresponding hypersurface $V_{\Pi'}$, then the intersection $V_\Pi \cap V_{\Pi'}$ is the union of the focal locus of B and a scroll Σ , defined as the union of the lines of B meeting both Π and Π' . In some particular cases, it can happen that some irreducible components of Φ are contained in Σ . Using Schubert calculus, the degree of Σ can be computed and it results to be equal to $a_0 + a_1$ if $n \leq 3$ and to $a_0 + 2a_1 + a_2$ if $n \geq 4$.

PROPOSITION 5.4 ([14]). *Let B be a congruence of lines of order one in \mathbb{P}^n , $n \geq 4$. Let F_i , $i = 1, \dots, h$ be the irreducible components of dimension $n - 2$ of the pure fundamental locus; we denote by m_i the degree of F_i and by k_i the algebraic multiplicity of $(F_i)_{\text{red}}$ on V_Π . Finally, we put $s_i := \text{length}((F_i)_{\text{red}} \cap \Lambda(b))$, where $b \in B$ is general. Then the following formulas hold:*

$$\sum_{i=1}^h s_i k_i \leq 1 + a_1; \tag{8}$$

$$(1 + a_1)^2 = \sum_{i=1}^h k_i^2 m_i + (1 + 2a_1 + a_2) + x, \tag{9}$$

where x is the contribution of the parasitical components of F of dimension $n - 2$. Moreover, equality holds in (8) if the pure fundamental locus has pure dimension $n - 2$.

REMARK 5.5. We observe that k_i coincides with the number of lines of B meeting Π and passing through a general point of F_i , or, equivalently, with the number of lines of B contained in a general hyperplane H and passing through a general point of $H \cap F_i$.

Proof. Indeed, if we take a line $\Lambda(b)$ of the congruence not contained in $F \cap V_{\Pi}$, then, intersecting $\Lambda(b)$ with V_{Π} , we obtain a 0-dimensional scheme of length $1 + a_1$, which is the degree of V_{Π} . For all index i , this scheme contains $F_i \cap \Lambda(b)$, which has support in (at most) s_i points, each of them of length k_i , or, in the classical language, there are s_i foci of multiplicity k_i . So formula (8) is proved. To prove (9), note that the degree of $V_{\Pi} \cap V_{\Pi'}$ is $(1 + a_1)^2$, and its components are: the (pure) fundamental locus, with multiplicity $\sum_{i=1}^h k_i^2 m_i$, the parasitical $(n - 2)$ -schemes, and the scroll Σ , which has degree $1 + 2a_1 + a_2$. \square

As a corollary, we get the following theorem, when F is reduced and irreducible.

THEOREM 5.6. *Let B be a congruence of lines of order one, such that F is irreducible and reduced of dimension $n - 2$. Let $\deg F = m$. Then $n - 1 < m < (n - 1)^2$.*

Proof. If we substitute formula (8), that in this case is an equality, in formula (9), we obtain

$$(n - 1)^2 k^2 - m k^2 - (1 + 2a_1 + a_2) = x \geq 0, \quad (10)$$

and since $(1 + 2a_1 + a_2) > 0$, we deduce $m < (n - 1)^2$.

The other inequality $n - 1 < m$ follows simply by degree reasons, since the congruence is given by the $(n - 1)$ -secant lines of F . \square

REMARK 5.7. *Theorem 5.6 is clearly sharp in \mathbb{P}^3 . The upper limit is not sharp in \mathbb{P}^4 , and probably also in \mathbb{P}^n , for $n > 4$, as we will see in the next section.*

Bounds on the degree can be obtained more in general for the components of maximal dimension of the fundamental locus, extending in this way Theorem 5.6. For example, in [8] it is proved that an irreducible smooth threefold of \mathbb{P}^7 with one apparent double point has degree at most 8, and the classification is given. Translated in the language of congruences, this means that if B is a first order congruence of lines in \mathbb{P}^7 whose focal locus F_{red} is smooth and irreducible of dimension three, then $\deg(F) \leq 8$. Here we show the following

THEOREM 5.8. *Let $X \subset \mathbb{P}^{2n+1}$ be a n -dimensional variety with one apparent double point. Then, $\deg(X) < 2^{n+1}$.*

Idea of the proof. By induction, intersecting $n + 1$ hypersurfaces V_{Π} 's, Formulas (8) and (9) become

$$2k = 1 + a_1 \tag{11}$$

and

$$(1 + a_1)^{n+1} \geq k^{n+1} \deg(X) + 1 + (n + 1)a_1 + \dots \tag{12}$$

From (11) and (12) we deduce

$$2^{n+1}k^{n+1} - \deg(X)k^{n+1} \geq 1 + (n + 1)a_1 + \dots > 0,$$

from which the assertion. □

We close this section with the following simple property of congruences of order one.

PROPOSITION 5.9. *Every congruence of order one B is a rational variety.*

Proof. Fixed a general hyperplane H in \mathbb{P}^n , associating to a point $P \in H$ the unique line of B passing through H we get a birational map from H to B . □

REMARK 5.10. *By the same reason, if X is an irreducible component of the focal locus which is met only once by a general line of the congruence, X is rational.*

6. Congruences of order one with irreducible and reduced focal locus

By Theorem 4.1, the only congruence of order one in \mathbb{P}^3 whose focal locus is irreducible and reduced is the congruence of the secant lines of a skew cubic curve.

In \mathbb{P}^4 , congruences of this type are all classified in [14]. Precisely:

THEOREM 6.1. *Let S be an irreducible surface in \mathbb{P}^4 whose trisecant lines generate an irreducible congruence of order one. Then S is one of the following:*

1. *a smooth projected Veronese surface;*
2. *a projection from a point P of a Del Pezzo surface S' of \mathbb{P}^5 , with P not lying in the plane of an irreducible conic of S' ;*
3. *a projection of a smooth rational normal scroll of \mathbb{P}^6 , from a line not intersecting it;*
4. *a (possibly singular) Bordiga surface.*

So the possible degrees of the focal locus are 4, 5, 6, whereas the upper bound given by Theorem 5.6 is $(n - 1)^2 - 1 = 8$.

In \mathbb{P}^n , for $n \geq 5$, a complete classification is still missing. Only the cases in which the focal locus is supposed to be not only irreducible but also smooth are all described.

THEOREM 6.2 ([12]). *Let $X \subset \mathbb{P}^n$ be an irreducible and smooth subvariety of codimension two. Assume that $n \geq 3$ and that the $(n - 1)$ -secant lines of X generate an irreducible congruence of order one. Then $n \leq 5$ and one of the following happens:*

1. *$n = 3$ and X is a twisted cubic;*
2. *$n = 4$ and X is either a projected Veronese surface or a Bordiga surface;*
3. *$n = 5$ and X is a threefold of degree 7, 9 or 10. In the first case, B is a linear congruence and X is a Palatini scroll (see Subsection 3.4); in the second case, X is a non-rational scroll,*

union of the lines parametrised by a K3 surface linear section of $\mathbb{G}(1, 5)$; in the third case X is an aCM threefold defined by the maximal minors of a 4×5 matrix of linear forms (see Subsection 3.5).

In the literature, only two classes of examples are described of congruences of order one with irreducible focal locus (for all $n \geq 3$). They are precisely: linear congruences and congruences of $(n - 2)$ -secant lines of arithmetically Cohen-Macaulay varieties, defined by the minors of maximal order of a $n \times (n - 1)$ matrix of linear forms (see §3). The degree of the focal locus is respectively $\frac{n^2 - 3n + 4}{2}$ and $\binom{n}{2}$.

REMARK 6.3. *As regards the congruences of Subsection 3.6, in \mathbb{P}^5 their focal locus is irreducible, but we do not know if the same holds in general. As we noted, the lines of B through a general focal point P form a pencil in a 2-plane α_P . It results that there are no parasitical components of dimension $n - 2$ and the irreducible components of F are all reduced of dimension $n - 2$. Moreover, for each component F_i of F , its multiplicity k_i in V_Π , introduced in Proposition 5.4, is equal to one: actually, through a general point P of F_i , there is only one line of B intersecting Π , that is the line joining P with $\alpha_P \cap \Pi$.*

We conjecture that F is irreducible. In any case, applying Proposition 5.4, we get that the focal locus has degree $(n - 1)^2 - (1 + 2a_1 + a_2) = \frac{n^2 - 3n + 2}{2}$, i.e. one less than the degree obtained for linear congruences. For more details see [18].

We give now a new class of examples in \mathbb{P}^n , for all $n \geq 5$, having $\Phi = F$ of dimension $n - 2$ and degree $(n - 1)^2 - (n - 2)$. The first case, in \mathbb{P}^5 , is described in an article of M. SgROI ([34], 1927), and does not seem to have been rediscovered in recent times.

EXAMPLE 6.4. *Let $X' \subset \mathbb{P}^n$, $n \geq 5$, be a degenerate subvariety of degree $n - 2$ and codimension 2, contained in a hyperplane H . We assume that X' is a projection of a rational normal scroll $\mathbb{P}^1 \times \mathbb{P}^{n-3}$ of \mathbb{P}^n , singular along a $(n - 3)$ space Π , with multiplicity $n - 3$. X' is obtained by projecting $\mathbb{P}^1 \times \mathbb{P}^{n-3}$ from a \mathbb{P}^{n-5} contained in the linear span of a $\mathbb{P}^1 \times \mathbb{P}^{n-4} \subset \mathbb{P}^1 \times \mathbb{P}^{n-3}$. The ideal sheaf of X' , $\mathcal{I}_{X'}$, has the following minimal free resolution*

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-(n - 2)) \rightarrow \mathcal{O}_{\mathbb{P}^n}(-(n - 1)) \oplus \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow \mathcal{I}_{X'} \rightarrow 0. \quad (13)$$

We define X as the subscheme of codimension 2 which is linked to X' in a linkage of type $(n-1, n-1)$, obtained with two hypersurfaces having both Π as subvariety of multiplicity $n-2$.

By mapping cone (see [31]), the ideal of X has the following minimal free resolution:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-(2n-3)) \oplus \mathcal{O}_{\mathbb{P}^n}(-n) \xrightarrow{A} 3\mathcal{O}_{\mathbb{P}^n}(-(n-1)) \rightarrow \mathcal{I}_X \rightarrow 0, \quad (14)$$

where A is a 2×3 matrix of homogeneous polynomials of the form:

$$A = \begin{pmatrix} L_1 & F_1 \\ L_2 & F_2 \\ L_3 & F_3 \end{pmatrix}, \quad (15)$$

with $\deg(L_i) = 1$ and $\deg(F_i) = n-2$. X has degree $(n-1)^2 - (n-2)$.

THEOREM 6.5. *There exists a congruence of lines of order one formed by $(n-1)$ -secant lines to X .*

Proof. By construction, there is a pencil of hypersurfaces of degree $n-1$ having $X \cup X'$ as base locus. Let P be a general point of \mathbb{P}^n and let T be the hypersurface of the pencil passing through P . Then $T \cap H = X' \cup K$, where $K \cong \mathbb{P}^{n-2}$ and $\Pi \subset K$. Let $H' \cong \mathbb{P}^{n-1}$ be the linear span of P and K , then $H' \cap T = K \cup X''$, where X'' has again degree $n-2$ and, by the assumption made on the singularities of T , it has Π as subspace of multiplicity $n-3$. Therefore also X'' is a projection of the Segre variety $\mathbb{P}^1 \times \mathbb{P}^{n-3}$. We note that $P \in X'' \setminus K$. Let $\ell \subset X''$ be the unique line of the first ruling ($\ell \cong \mathbb{P}^1 \times \{Q\}$, $Q \in \mathbb{P}^{n-3}$) passing through P . Note that $\ell \cap \Pi = \emptyset$. Let T' be any other hypersurface of the pencil: ℓ intersects T' at $n-1$ points, and, since $\ell \subset T$, it intersects every hypersurface of the pencil at the same $n-1$ points, therefore it is $(n-1)$ -secant to $X \cup X'$. But $\ell \cap X' = \emptyset$: indeed $\ell \cap X' \subset H' \cap H = K$, and also $\ell \cap X' \subset \ell \cap T'$, therefore $\ell \cap X' \subset \ell \cap (K \cap T')$. But $K \cap T' = (n-2)\Pi$, so ℓ can meet X' only on Π , which is impossible, therefore ℓ is $(n-1)$ -secant to X . \square

CONJECTURE 6.6. *For $n = 5$, X results to be irreducible. We conjecture that it is irreducible for all $n \geq 5$.*

7. Congruences of lines of order one in \mathbb{P}^4

In this section we collect the known results on the classification of the congruences of lines of order one in \mathbb{P}^4 , considering all the various possibilities for the dimension and the structure of the irreducible components of the focal locus (with its natural structure of scheme). We will also give some contributions towards the completion of the classification.

Let B be a congruence of lines of order one in \mathbb{P}^4 and F its focal locus. We suppose that B is not the star of lines of centre a point. In the light of the results of §5, we can distinguish first of all the following three cases:

1. F is reduced;
2. F is not reduced but is generically reduced;
3. F is generically non reduced.

In case 1. F can have one, two or three irreducible components all of dimension 2, in case 2. there are a component F_1 of dimension two and a component F_2 of dimension one, which is necessarily non reduced, and possibly embedded in F_1 , in case 3. F is either an irreducible surface with a multiple structure, or a union of two irreducible surfaces, one reduced and the other non reduced. The situation is summarised in Table 1.

As indicated in the table, the congruences with non reduced focal locus are classified in [11] and [15]. Since the description of the possible cases is rather involved, we will not report it here.

The only open cases are those with F reduced but reducible. Some partial results were obtained already by Marletta in [28] and are contained in [10]. We prove here the main points.

7.1. Congruences of order one with two reduced focal surfaces

Assume that the focal locus of the congruence B is the union of two surfaces F_1 and F_2 , and B is formed by the secant lines of F_1 meeting also F_2 . For $i = 1, 2$, we denote by m_i the degree of F_i and by k_i the multiplicity of F_i in the hypersurface V_Π (see §5), where Π

Focal locus F	Components of F	Congruence B	Remarks
F reduced	F irreducible	trisecant lines to F	classified (see Th. 6.1)
	two irreducible components F_1, F_2	lines secant F_1 meeting F_2	either F_2 is a plane or F_1 is a rational cubic scroll (see Th. 7.1)
	three irreducible components F_1, F_2, F_3	lines meeting F_1, F_2 and F_3	one of the components is a plane (see Th. 7.2)
$F = F_1 \cup F_2$, $\dim F_1 = 2$, $\dim F_2 = 1$, $(F_2)_{\text{red}} = C$	$C \not\subset F_1$	lines meeting F_1 and C	classified ([11])
	$C \subset F_1$	lines meeting F_1 and C	classified ([11])
F generically non reduced	F_{red} irreducible	lines meeting F_{red}	F_{red} is a plane, classified ([15])
		lines secant F_{red}	F_{red} is a cubic scroll, classified ([15])
	$F = F_1 \cup F_2$, F_1 non reduced	lines meeting $(F_1)_{\text{red}}$ and F_2	one component is a plane and the other a rational surface of sectional genus 0, classified ([15])

Table 1: Congruences of order one in \mathbb{P}^4

is a general plane. For simplicity, we also assume that the general hyperplane section of F_1 is smooth, and denote by h its number of apparent double points. Then:

THEOREM 7.1. *We keep the notations just introduced.*

1. *If the surfaces F_1 and F_2 intersect properly, then F_1 is a rational normal scroll (possibly a cubic cone) and F_2 is a plane. Conversely, given a rational normal scroll F_1 and a plane F_2 intersecting properly, the congruence of the secant lines of F_1 meeting F_2 has order one, multidegree $(1, 4)$ and three 1 – parasitical planes, i.e. the three planes generated by the conics contained in F_1 passing through two of the points of $F_1 \cap F_2$.*
2. *If $F_1 \cap F_2$ is a curve, then either F_2 is a plane or F_1 is a rational*

normal cone, possibly a cubic cone.

Proof. The secant lines of F_1 form a subvariety of $\mathbb{G}(1, 4)$, whose class in the Chow ring can be easily computed using standard Schubert calculus, and is $\binom{m_1}{2}\sigma_{11} + h\sigma_{20}$. Likewise, the class of the lines meeting F_2 is $m_2\sigma_{10}$. If the two surfaces intersect properly, the class of B is the intersection of the two classes just computed, which is:

$$[B] = m_2(h\sigma_{30} + (\binom{m_1}{2} + h)\sigma_{21}).$$

Since B has order one, we obtain $m_2 = h = 1$, so F_2 must be a plane and F_1 have degree 3, because the only curves in \mathbb{P}^3 with one apparent double point are the skew cubics. Conversely, assume that two such surfaces in general position are given. The secant lines of F_1 , passing through a fixed point, form a planar pencil, hence precisely one of those lines intersects also F_2 . Therefore the congruence has order one.

If we consider two of the three points of $F_1 \cap F_2$, the plane of the conic of F_1 containing them intersects F_2 along the line generated by them, so it is parasitical for B . Formulas (8) and (9) now become $2k_1 + k_2 = a_1 + 1$ and $3k_1^2 + k_2^2 = a_1^2 - x$, where x is the contribution of the parasitical planes, which in our case is 3. From Remark 5.5, it follows that $k_1 = 2$ and $k_2 = 1$, hence $a_1 = 4$.

Assume now that $F_1 \cap F_2 =: C$ is a curve and that F_2 is not a plane. The lines joining a point of C with another point of F_1 form a family of dimension 3, hence do not belong to B . If $P \in \mathbb{P}^4$ is a general point, the secant lines of F_1 passing through P form a cone of degree h , hence hm_2 of these lines (counting multiplicities) meet also F_2 . Only one of these lines belongs to B , which has order one, hence the other $u := hm_2 - 1$ lines must intersect F_2 at a point of C .

On the other hand, since F_2 is not a plane, through a general point $Q \in F_2$ cannot pass infinitely many secant lines of F_1 meeting also C , since these lines would be lines of B , so varying the point Q , we would obtain all the lines of B , which would have a fundamental curve.

Therefore, through Q there pass $h(m_2 - 1)$ secant lines of F_1 meeting again F_2 , that must coincide with the u secant lines of F_1

passing through Q and meeting also C . This is due to the fact that, if one of these $h(m_2 - 1)$ lines would meet F_2 outside C , then it would be a focal line, since it contains (at least) four focal points. So, B would have a focal hypersurface.

So, we get $u = hm_2 - 1 = h(m_2 - 1)$, therefore $h = 1$, that implies $m_1 = 3$. \square

7.2. Congruences of order one with three reduced focal surfaces

Assume that the focal locus of the congruence B is the union of three surfaces F_1 , F_2 and F_3 , and B is formed by the lines meeting each of them. For $i = 1, 2, 3$, we denote by m_i the degree of F_i and by k_i the multiplicity of F_i in the hypersurface V_Π (see §5), where Π is a general plane. Then:

THEOREM 7.2. *We keep the notations just introduced.*

1. *At least one of the surfaces F_1 , F_2 and F_3 is a plane.*
2. *If the surfaces F_1 , F_2 and F_3 intersect properly, then they are all planes and B is a linear congruence. B has one parasitical plane, i.e. the plane generated by the points $F_1 \cap F_2$, $F_1 \cap F_3$ and $F_2 \cap F_3$.*

Proof. If the three surfaces intersect two by two in a scheme of dimension zero, in the Chow ring of $\mathbb{G}(1, 4)$ we have:

$$[B] = m_1 m_2 m_3 (2\sigma_{21} + \sigma_{30}),$$

Therefore, in order to obtain a congruence of order one, we must have $m_i = 1$, $i = 1, 2, 3$, i.e. the surfaces are all planes. It is well known and easy to see that, given three planes in general position, the lines meeting all of them form a linear congruence with one parasitical plane.

From now on we will denote by C_{i+j-2} the scheme (of dimension less than or equal to one) intersection of F_i and F_j , $1 \leq i < j \leq 3$. The degree of C_{i+j-2} will be indicated by c_{i+j-2} . We can suppose that at least C_1 is a curve.

Assume by contradiction that none of the surfaces F_1 , F_2 and F_3 is a plane.

We will denote by u the number of lines passing through a general point P of \mathbb{P}^4 which meet the—possibly empty—scheme Z , union of the components of $C_2 \cup C_3$ of dimension one, and meet also F_1 and F_2 .

Let χ_P denote the cone of the lines passing through P , meeting both F_1 and F_2 but not passing through the points of C_1 . χ_P has dimension two and degree $m_1m_2 - c_1$: actually we have to cut out the points of C_1 , because the lines meeting C_1 and F_3 form a congruence which is distinct from B (which is irreducible).

$(m_1m_2 - c_1)m_3$ lines of the cone χ_P meet also F_3 . Indeed, $(m_1m_2 - c_1)m_3$ is the order of the—in general—reducible congruence B' of all the lines meeting F_1 , F_2 , and F_3 . $B \subset B'$ and B is the closure of the lines meeting F_1 , F_2 , and F_3 in three distinct points. If the order of B' is one, we deduce $m_3 = 1$. If instead it has greater order, we deduce that there is another component B'' given by the lines joining Z ($Z \neq \emptyset$ in this case) and F_1 ; this congruence has order u , so we get the formula

$$(m_1m_2 - c_1)m_3 = u + 1. \tag{16}$$

Let us calculate u in another way. Fix a general point $Q \in F_3$; the number of lines of B'' through Q is either finite, in which case this number is u , or it is infinite. In this last case, by the generality of Q , F_3 would be a component of the fundamental locus of B'' ; either it is a parasitic plane and then F_3 is a plane, or it is a 2-fundamental locus, which it cannot be, since every line would be focal, containing more than three foci.

Therefore, the number of lines of B'' through Q is finite; this number is also given by the degree of the intersection of the join of Q with F_3 , with the cone χ_Q (keeping the above notation).

Then, we have proved that

$$u = (m_1m_2 - c_1)(m_3 - 1). \tag{17}$$

From equations (16) and (17) we deduce that $m_1m_2 = c_1 + 1$. This means that, under the projection from a general point $P \in \mathbb{P}^4$,

$\pi_P: \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$, the projections of F_1 and F_2 intersect out of $\pi_P(C_1)$ in a line ℓ .

$(\pi_P)|_{F_1 \cup F_2}^{-1}(\ell)$ is the union of two lines $\ell_1 \subset F_1$ and $\ell_2 \subset F_2$ (non necessarily distinct), and we get two such lines for every choice of $P \in \mathbb{P}^4$. We deduce that the trisecant line of B passing through P must be contained in the intersection of the planes $\langle P, \ell_1 \rangle$ and $\langle P, \ell_2 \rangle$. Then, ℓ_1 and ℓ_2 meet in (at least) one point, which, by definition, must be contained in $C_1 = F_1 \cap F_2$, but this is impossible, since a general line of the congruence B does not intersect C_1 . \square

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