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Classical and non-Classical Positive Solutions of a Prescribed Curvature Equation with Singularities

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Dedicated to Fabio Rossi

Summary. - *We investigate the existence of positive solutions of a prescribed curvature equation with a nonlinearity having one or two singularities. Our approach relies on the method of lower- and upper-solutions, truncation arguments and energy estimates.*

1. Introduction

In this paper, we are interested in the existence of positive solutions of the curvature problem

$$-(\varphi(u'))' = \lambda f(t,u), \quad u(0)=0,\ u(1)=0, \tag{1}$$

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where

$$\varphi(v) = \frac{v}{\sqrt{1+v^2}}.$$

Problem (1) is the one-dimensional counterpart of the elliptic Dirichlet problem

$$-\mathrm{div}\left(\nabla u / \sqrt{1+\|\nabla u\|^2}\right) = \lambda f(x,u) \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega. \tag{2}$$

The existence of positive solutions of problems (1) and (2) has been discussed in the last two decades by several authors (see [1]–[5],[8]–[22]) in connection with various qualitative assumptions on f.

In our recent paper [2], two types of solutions have been considered for problem (1) and referred to as classical or non-classical, respectively.

A *classical* solution of (1) is a function $u : [0,1] \to \mathbb{R}$, with $u \in W^{2,1}(0,1)$, which satisfies the equation in (1) a.e. in $[0,1]$ and the boundary conditions $u(0) = u(1) = 0$. A *non-classical* solution of (1) is a function $u : [0,1] \to \mathbb{R}$ such that $u \in W^{2,1}_{loc}(0,1)$, $|u'(0)| = +\infty$ or $|u'(1)| = +\infty$ and $u' \in C^0([0,1], [-\infty, +\infty])$, which satisfies the equation in (1) a.e. in $[0,1]$ and the boundary conditions $u(0) = u(1) = 0$. Such solutions are said to be *positive* if $u(t) > 0$ on $]0,1[$ and $u'(0) > 0 > u'(1)$.

In [2] existence and multiplicity of positive solutions of (1) have been established under various types of assumptions on the behaviour of the function f at zero and at infinity. In the present work we discuss cases where f exhibits singularities at 0 or at some point $R > 0$; our model nonlinearities are u^{-p}, $(R-u)^{-q}$ and $u^{-p}(R-u)^{-q}$, with $p, q > 0$.

Unlike the fact that a large amount of work has been done for a class of quasilinear elliptic equations in the presence of a singularity at zero, few results have been obtained for the curvature problem (cf. [5]). Singularities on the right have been considered only recently for semilinear elliptic problems; to the best of our knowledge they have never been considered before for the curvature problem.

This paper is organized as follows. Section 2 deals with a nonlinearity that is allowed to be singular at the origin. Within this setting, we prove the existence of at least one positive solution for small values of the parameter $\lambda > 0$. In Section 3 we analyze the nature of the solutions for small values of $\lambda > 0$. Namely, we prove that if f is non-singular or has a weak singularity at the origin (see assumption (7)) and λ is small enough, any small positive solution is classical. On the other hand, if the singularity at the origin is too strong (see assumption (8)), then any positive solution is non-classical. In Section 4, we work out the case of a singularity on the right, while Section 5 deals with possibly two singularities. In this last case, under a quite general assumption, we show that the presence of the second singularity gives rise to a second positive solution. Finally, we present a non-existence result for large values of λ in Section 6, while in the Appendix we show some numerical illustrations of our results.

Throughout this paper the following conditions will be considered. Let $f : [0,1] \times I \to \mathbb{R}$, with I an interval, be a given function.

We say that f is a L^1-*Carathéodory function* if, for a.e. $t \in [0,1]$, $f(t,\cdot) : I \to \mathbb{R}$ is continuous; for every $u \in I$, $f(\cdot,u) : [0,1] \to \mathbb{R}$ is measurable; for every compact set $K \subset I$ there is $h \in L^1(0,1)$ such that, for a.e. $t \in [0,1]$ and every $u \in K$, $|f(t,u)| \leq h(t)$.

We say that f is *locally* L^1-*Lipschitz* with respect to the second variable if, for every compact set $K \subset I$, there is $\ell \in L^1(0,1)$ such that, for every u_1, $u_2 \in K$ and a.e. $t \in [0,1]$,

$$|f(t,u_1) - f(t,u_2)| \leq \ell(t)\,|u_1 - u_2|.$$

2. The singularity at the origin

THEOREM 2.1. *Let* $f : [0,1]\times]0,R[\to [0,+\infty[$, *with* $R \in]0,\infty]$, *be a* L^1-*Carathéodory and locally* L^1-*Lipschitz function. Assume*

(h_1) $\liminf_{u\to 0} f(t,u) > 0$, *uniformly a.e. on* $[0,1]$, *i.e. there exist* $\eta > 0$ *and* $\delta > 0$ *such that, for a.e.* $t \in [0,1]$ *and for every* $u \in]0,\delta]$, *we have* $f(t,u) \geq \eta$.

Then there exists $\lambda_0 > 0$ *such that for any* $\lambda \in]0,\lambda_0]$, *problem* (1) *has at least one positive solution.*

Proof. Step 1 – The modified problem. Let $\bar{R} \in]0,R[$. For each

$n \in \mathbb{N}$, $n > 1$, define

$$f_n(t,u) = \begin{cases} f(t,\bar{R}/n) & \text{if } u \le \bar{R}/n, \\ f(t,u) & \text{if } \bar{R}/n < u \le \bar{R}, \\ f(t,\bar{R}) & \text{if } \bar{R} < u, \end{cases} \tag{3}$$

$$\varphi_n(v) = \begin{cases} \varphi(-n) + \varphi'(-n)(v+n) & \text{if } v \le -n, \\ \varphi(v) & \text{if } -n < v \le n, \\ \varphi(n) + \varphi'(n)(v-n) & \text{if } n < v, \end{cases} \tag{4}$$

and consider the modified problem

$$-(\varphi_n(u'))' = \lambda f_n(t,u), \quad u(0) = 0,\ u(1) = 0. \tag{5}$$

Step 2 – Construction of an upper solution β of (5)*; for any $r \in]0,\bar{R}[$, there exist $\lambda_0 > 0$, $n_0 \in \mathbb{N}$ and $\beta \in W^{2,1}(0,1)$ such that $0 < \beta(t) \le r$ in $[0,1]$ and for any $\lambda \in]0,\lambda_0]$ and any $n \ge n_0$, β is an upper solution of* (5)*.* Let us fix $0 < \hat{r} < r < \bar{R}$. From the L^1-Carathéodory conditions, there exists $h \in L^1(0,1)$ such that $|f(t,u)| \le h(t)$ for a.e. $t \in [0,1]$ and every $u \in [\hat{r}, r]$. Let then $H \in W^{2,1}(0,1)$ be the solution of

$$-H'' = h(t), \quad H(0) = 0,\ H(1) = 0$$

and take $\beta = \hat{r} + \varkappa H$, where $\varkappa > 0$ is small enough so that

$$\beta = \hat{r} + \varkappa H < r.$$

We then compute for $\lambda > 0$ small enough and $n \in \mathbb{N}$ large enough

$$-\beta''(t) = \varkappa h(t) \ge \lambda(1 + \beta'^2(t))^{3/2} f(t,\beta(t)) = \lambda \frac{f_n(t,\beta(t))}{\varphi_n'(\beta'(t))}.$$

Step 3 – Construction of a lower solution $\alpha \le \beta$ of (5)*; for any $\lambda \in]0,\lambda_0]$, there exist $n_1 \in \mathbb{N}$ and $\alpha \in W^{2,1}(0,1)$ such that $0 < \alpha(t) \le \beta(t)$ in $]0,1[$ and, for any $n \ge n_1$, α is a lower solution of* (5)*.* From assumption (h_1), we can find $r_0 \in]0,\hat{r}]$ so that for all large n, a.e. $t \in [0,1]$ and every $u \in]0,r_0]$

$$f_n(t,u) \ge r_0 \frac{\pi^2}{\lambda}.$$

The function $\alpha(t) = r_0 \sin(\pi t) \leq \beta(t)$ is such that

$$-\alpha''(t) = \pi^2 \alpha(t) \leq \pi^2 r_0 \leq \lambda f_n(t, \alpha(t)) \leq \lambda \frac{f_n(t, \alpha(t))}{\varphi_n'(\alpha'(t))}.$$

Step 4 – Existence of a solution u_n of the modified problem (5). Since α and β are lower and upper solutions of (5) for all large n, with $\alpha(t) \leq \beta(t)$ on $]0,1[$, and the equation in (5) can be written as

$$-u'' = \lambda \frac{f_n(t,u)}{\varphi_n'(u')},$$

where the right-hand side is bounded by a L^1-function, a standard result (see [6, Theorem II-4.6]) yields the existence of a solution u_n of (5) satisfying

$$\alpha(t) \leq u_n(t) \leq \beta(t) \quad \text{ in } [0,1]. \tag{6}$$

Step 5 – Existence of a solution u of (1). Let $a \in]0, 1/2[$. From (6) and Step 3 we know that, for all $t \in [a, 1-a]$, we have $\alpha(a) \leq u_n(t) \leq r$. Hence using the L^1-Carathéodory conditions, there exists $h \in L^1(0,1)$ such that for n large enough and a.e. $t \in [a, 1-a]$, we have

$$0 \leq f_n(t, u_n(t)) = f(t, u_n(t)) \leq h(t).$$

Also from the concavity of u_n, we deduce that, for all $t \in [a, 1-a]$,

$$\frac{r}{a} \geq u_n'(a) \geq u_n'(t) \geq u_n'(1-a) \geq -\frac{r}{a}.$$

Hence for n large enough and all $t \in [a, 1-a]$, we get

$$\varphi_n'(u_n'(t)) = \varphi'(u_n'(t)) \geq \varphi'\left(\frac{r}{a}\right).$$

It follows that

$$0 \leq -u_n''(t) = \lambda \frac{f(t, u_n(t))}{\varphi'(u_n'(t))} \leq \lambda \frac{h(t)}{\varphi'(\frac{r}{a})}.$$

From Arzelà-Ascoli Theorem, we infer that a subsequence $(u_n)_n$ converges in $C^1([a, 1-a])$ to a function $u \in C^1([a, 1-a])$. By the C^1_{loc}-estimates previously obtained, u satisfies

$$-u'' = \lambda \frac{f(t,u)}{\varphi'(u')}.$$

Using a diagonalization argument, u can be extended onto $]0,1[$, so that $u \in W^{2,1}_{loc}(0,1)$ satisfies the equation in (1) in $]0,1[$. Moreover, as u is positive and concave, there exist

$$\lim_{t\to 0} u(t) \in [0,+\infty[\quad \text{and} \quad \lim_{t\to 1} u(t) \in [0,+\infty[.$$

Step 6 – $\lim_{t\to 0} u(t) = 0$ *or* $\lim_{t\to 0} u'(t) = +\infty$. Notice that u_n is a solution of the Cauchy problem

$$-u'' = \lambda \frac{f_n(t,u)}{\varphi_n'(u')}, \quad u(\tfrac{1}{2}) = u_n(\tfrac{1}{2}),\ u'(\tfrac{1}{2}) = u_n'(\tfrac{1}{2}),$$

and u is a solution of the limit problem

$$-u'' = \lambda \frac{f(t,u)}{\varphi'(u')}, \quad u(\tfrac{1}{2}) = \lim_{n\to\infty} u_n(\tfrac{1}{2}),\ u'(\tfrac{1}{2}) = \lim_{n\to\infty} u_n'(\tfrac{1}{2}).$$

If it were $\lim_{t\to 0} u(t) > 0$ and $\lim_{t\to 0} u'(t) \in \mathbb{R}$, by continuity with respect to parameters and initial conditions, which follows from the L^1-Carathéodory and locally L^1-Lipschitz conditions, we should get

$$\lim_{t\to 0} u(t) = \lim_{n\to\infty} u_n(0) = 0,$$

which is a contradiction.
Step 7 – $\lim_{t\to 1} u(t) = 0$ *or* $\lim_{t\to 1} u'(t) = -\infty$. The argument is similar to the previous one.
Conclusion – If $\lim_{t\to 0} u(t) = \lim_{t\to 1} u(t) = 0$, then the continuous extension of u on $[0,1]$ is a solution of (1), which may be classical or non-classical. Otherwise, the extension of u obtained by setting $u(0) = u(1) = 0$ is a non-classical solution of (1). □

Remark 2.2. From Step 2 in the proof of Theorem 2.1, we see that, for each $\lambda > 0$ small enough, a solution u_λ of (1) exists such that $\|u_\lambda\|_\infty \to 0$ as $\lambda \to 0$.

Remark 2.3. Suppose that, in addition to the assumptions of Theorem 2.1, the following condition holds

$$\lim_{u\to R} f(t,u) = 0, \qquad \text{uniformly a.e. on } [0,1].$$

Then, for any $\lambda \in]0,+\infty[$, problem (1) has at least one positive solution.

To prove this claim we only need to verify that an upper solution β of (5) can be constructed for any given $\lambda > 0$. We fix λ and, in case $R \neq +\infty$, we extend f by setting $f(t,u) = 0$ for a.e. $t \in [0,1]$ and all $u \geq R$. We define the upper solution β by setting $\beta(t) = M + \varepsilon t(1-t)$, where $M \in]0,R[$ and $\varepsilon \in]0,\frac{1}{2}[$ are such that $[M, M+\varepsilon] \subset]0,R[$ and $f(t,u) \leq \frac{\varepsilon}{\lambda}$ for a.e. $t \in [0,1]$ and all $u \in [M, M+\varepsilon]$. We have then

$$-\beta''(t) = 2\varepsilon \geq (1+\varepsilon^2)^{3/2}\varepsilon \geq \lambda(1+\beta'(t)^2)^{3/2} f(t,\beta(t))$$

a.e. in $[0,1]$.

3. Classical and non-classical solutions

THEOREM 3.1. *Assume $f : [0,1]\times]0,R[\to [0,+\infty[$, with $R \in]0,+\infty[$, is a L^1-Carathéodory function and v is a positive solution of (1) for some $\lambda > 0$. Assume further there exists a function $g :]0,R] \to [0,+\infty[$ having a bounded antiderivative such that, for a.e. $t \in [0,1]$ and all $u \in]0, \|v\|_\infty]$,*

$$f(t,u) \leq g(u). \tag{7}$$

Then, if $\lambda > 0$ is small enough, v is a classical solution.

Proof. Let G be an antiderivative of g and define

$$E(t) = 1 - \frac{1}{\sqrt{1+v'^2(t)}} + \lambda G(v(t)).$$

We have

$$\begin{aligned} E'(t) &= v'(t)[(\varphi(v'(t)))' + \lambda G'(v(t))] \\ &\geq v'(t)[(\varphi(v'(t)))' + \lambda f(t,v(t))] = 0, \end{aligned}$$

for a.e. $t \in [0,1]$ such that $v'(t) \geq 0$. From the concavity of v, we know there exists $t_0 \in]0,1[$ such that $v'(t) \geq 0$ on $[0,t_0]$ and $v'(t_0) = 0$. Assume by contradiction $v'(0) = +\infty$. Since $\lim_{t\to 0} v(t) = v_0 \geq 0$, we have

$$\lambda G(R) \geq \lambda G(v(t_0)) = E(t_0) \geq \lim_{t\to 0} E(t) = 1 + \lambda G(v_0) \geq 1$$

which is impossible for small values of λ.

A similar argument shows that $v'(1) \in \mathbb{R}$. $\square$

Remark 3.2. Theorem 3.1 applies in particular if

$$g(u) = \frac{K}{u^p},$$

where $K > 0$ and $p \in]0, 1[$.

Theorem 3.3. *Assume* $f : [0, 1]\times]0, R[\to [0, +\infty[$, *with* $R \in]0, +\infty]$, *is a* L^1*-Carathéodory function and* v *is a positive solution of* (1) *for some* $\lambda > 0$. *Assume further there exist* $\varepsilon > 0$ *and a function* $h :]0, \varepsilon] \to [0, +\infty[$ *having an unbounded antiderivative such that, for a.e.* $t \in [0, 1]$ *and all* $u \in]0, \varepsilon]$,

$$f(t, u) \geq h(u). \tag{8}$$

Then v *is a non-classical solution. More precisely, we have*

$$\lim_{t\to 0} v(t) > 0 \quad \text{and} \quad \lim_{t\to 1} v(t) > 0.$$

Proof. Assume, by contradiction, that v is a positive solution of (1) satisfying $\lim_{t\to 0} v(t) = 0$. Define

$$E(t) = 1 - \frac{1}{\sqrt{1 + v'^2(t)}} + \lambda H(v(t)),$$

where H is an antiderivative of h. We have

$$\begin{aligned} E'(t) &= v'(t)[(\varphi(v'(t)))' + \lambda h(v(t))] \\ &\leq v'(t)[(\varphi(v'(t)))' + \lambda f(t, v(t))] = 0, \end{aligned}$$

for a.e. $t \in [0, 1]$ such that $v'(t) \geq 0$ and $v(t) \leq \varepsilon$. Let $t_0 \in]0, 1[$ be such that $v'(t) > 0$ and $v(t) \leq \varepsilon$ on $[0, t_0]$. Hence, we get the contradiction

$$E(t_0) \leq \lim_{t\to 0} E(t) = 1 + \lambda \lim_{t\to 0} H(v(t)) = -\infty.$$

A similar argument shows that assuming that $\lim_{t\to 1} v(t) = 0$ leads to a contradiction as well. $\square$

Remark 3.4. Theorem 3.1 applies in particular if

$$h(u) = \frac{K}{u},$$

where $K > 0$.

4. Singularity on the right

THEOREM 4.1. *Let* $f : [0,1] \times [0,R[\to [0,+\infty[$*, with* $R \in]0,+\infty[$*, be a* L^1*-Carathéodory and locally* L^1*-Lipschitz function. Assume*
(h_2) *there is* $p > 1$ *such that*

$$\lim_{u\to R} f(t,u)(R-u)^p = +\infty, \quad \text{uniformly a.e. on } [0,1].$$

Then there exists $\lambda_0 > 0$ *such that for each* $\lambda \in]0,\lambda_0]$ *problem* (1) *has at least one positive solution.*

Proof. Step 1 – The modified problem. For each $n \in \mathbb{N}$, $n > 1$, we define

$$f_n(t,u) = \begin{cases} f(t,0) & \text{if } u \le 0, \\ f(t,u) & \text{if } 0 < u \le \frac{n-1}{n}R, \\ f(t,\frac{n-1}{n}R) & \text{if } \frac{n-1}{n}R < u, \end{cases}$$

and consider the modified problem

$$-(\varphi_n(u'))' = \lambda f_n(t,u), \quad u(0) = 0,\ u(1) = 0, \tag{9}$$

where φ_n is defined from (4).

Step 2 – Construction of an upper solution β *of* (9)*; for any* $r \in]0,R[$*, there exist* $\lambda_0 > 0$*,* $n_0 \in \mathbb{N}$ *and* $\beta \in W^{2,1}(0,1)$ *such that* $0 < \beta(t) \le r$ *in* $[0,1]$ *and, for any* $\lambda \in]0,\lambda_0]$ *and any* $n \ge n_0$*,* β *is an upper solution of* (9)*.* To prove this claim, we repeat the argument used in Step 2 of the proof of Theorem 2.1.

Step 3 – Construction of a lower solution α *of* (9)*; there exist* $n_1 \in \mathbb{N}$ *and* $\alpha \in W^{2,1}(0,1)$ *such that for any* $\lambda \in]0,\lambda_0]$ *and any* $n \ge n_1$*,* α *is a lower solution of* (9)*,* $0 < \alpha(t) < R$ *in* $]0,1[$ *and* $\max(\alpha - \beta) > 0$*.* Let $\lambda > 0$ be fixed. We first choose $\bar{r}$ such that

$$r < \bar{r} < R, \quad (R-\bar{r})^{p-1} < \frac{\lambda}{8R^2},$$

and for a.e. $t \in [0,1]$ and all $u \in [\bar{r},R[$

$$f(t,u) \ge \frac{1}{(R-u)^p} \ge \frac{1}{(R-\bar{r})^p}.$$

Next we write

$$M_0 = \frac{\lambda}{2(R-\bar r)^p},$$
$$k_0 = \frac{\bar r}{M_0} = \frac{2\bar r}{\lambda}(R-\bar r)^p < \frac{1}{8},$$
$$\rho_0 = \frac{1-\sqrt{1-8k_0}}{4} = \frac{2k_0}{1+\sqrt{1-8k_0}}.$$

The function α defined by

$$\alpha(t) = \begin{cases} \bar r + 2M_0\rho_0(t-\frac{1}{2}+\rho_0) & \text{if } 0 \le t < \frac{1}{2}-\rho_0, \\ \bar r + M_0(t-\frac{1}{2}+\rho_0)(\frac{1}{2}+\rho_0-t) & \text{if } \frac{1}{2}-\rho_0 \le t < \frac{1}{2}+\rho_0, \\ \bar r + 2M_0\rho_0(\frac{1}{2}+\rho_0-t) & \text{if } \frac{1}{2}+\rho_0 \le t \le 1, \end{cases}$$

is such that

$$\max(\alpha-\beta) > 0 \quad \text{and} \quad \max\alpha = \alpha(\tfrac{1}{2}) = \bar r + M_0\rho_0^2 < R.$$

To prove the last inequality, we compute

$$M_0\rho_0^2 < 4M_0k_0^2 = 4\frac{\bar r^2}{M_0} \le 8R^2\frac{(R-\bar r)^p}{\lambda} \le R-\bar r.$$

Further, we check that α is a lower solution of (9) for n large enough:

(i) $\alpha(0) = \alpha(1) = \bar r + 2M_0\rho_0(-\frac{1}{2}+\rho_0) = 0$,

(ii) $-\alpha''(t) = 2M_0 = \frac{\lambda}{(R-\bar r)^p} \le \lambda f(t,\alpha(t)) \le \lambda f_n(t,\alpha(t))\varphi_n'(\alpha'(t))$, if $\frac{1}{2}-\rho_0 \le t < \frac{1}{2}+\rho_0$,

(iii) $-\alpha''(t) = 0 \le \lambda f(t,\alpha(t)) \le \lambda f_n(t,\alpha(t))\varphi_n'(\alpha'(t))$, if $0 \le t < \frac{1}{2}-\rho_0$ or $\frac{1}{2}+\rho_0 \le t \le 1$.

Step 4 – Existence of a solution u_n of (9) *for n large enough.* Notice that (9) can be written as

$$-u'' = \lambda\frac{f_n(t,u)}{\varphi_n'(u')}, \quad u(0)=0,\ u(1)=0,$$

where the right-hand side of the equation is bounded by a L^1– function. Using the lower and upper solutions obtained in Step 2 and

Step 3, and applying Theorem 4.1 in [7], we obtain a solution u_n of (9) and points t'_n and $t''_n \in [0,1]$ such that

$$\text{either } u_n(t'_n) > \beta(t'_n) \ \text{ or } \ u_n(t'_n) = \beta(t'_n) \text{ and } u'_n(t'_n) = \beta'(t'_n)$$

and

$$\text{either } u_n(t''_n) < \alpha(t''_n) \ \text{ or } \ u_n(t''_n) = \alpha(t''_n) \text{ and } u'_n(t''_n) = \alpha'(t''_n). \tag{10}$$

Step 5 – The functions u_n are bounded away from R. We choose now $\hat{r}$ such that

$$\bar{r} < \hat{r} < R, \quad (R-\hat{r})^{p-1} < \frac{\lambda}{32R^2}, \quad (R-\hat{r})^p < \frac{\lambda}{32(2M_0\rho_0 - \bar{r})}.$$

Let us prove that $u_n(t) \le \hat{r} + (\frac{R-\hat{r}}{2})$. Assume by contradiction there exists $\hat{s}_n$ such that $\max u_n = u_n(\hat{s}_n) > \hat{r} + (\frac{R-\hat{r}}{2})$. Define then $s'_n < \hat{s}_n < s''_n$ such that $u_n(s'_n) = u_n(s''_n) = \hat{r}$.

Claim 1: $\max u_n \le 2M_0\rho_0$. Since u_n is concave and (10) holds, we compute for $t \ge t''_n$

$$u_n(t) \le u_n(t''_n)\tfrac{t}{t''_n} \le \alpha(t''_n)\tfrac{t}{t''_n} \le \alpha'(0)t \le \alpha'(0) = 2M_0\rho_0.$$

In a similar way, for $t \le t''_n$ we get

$$u_n(t) \le -\alpha'(1)(1-t) \le -\alpha'(1) = 2M_0\rho_0,$$

and the claim follows.

Claim 2: $\hat{s}_n - s'_n < \frac{1}{4}$. Notice that for any $t \in [s'_n, \hat{s}_n]$ one has

$$-(\varphi_n(u'_n(t)))' = \lambda f_n(t, u_n(t)) \ge \frac{\lambda}{(R-\hat{r})^p}.$$

It follows that

$$u'_n(t) \ge \varphi_n(u'_n(t)) \ge \frac{\lambda}{(R-\hat{r})^p}(\hat{s}_n - t)$$

and

$$2M_0\rho_0 - \hat{r} \ge u_n(\hat{s}_n) - \hat{r} \ge \frac{\lambda}{2(R-\hat{r})^p}(\hat{s}_n - s'_n)^2.$$

The claim follows then since

$$(\hat{s}_n - s'_n)^2 \le \tfrac{2}{\lambda}(2M_0\rho_0 - \bar{r})(R-\hat{r})^p < \tfrac{1}{16}.$$

Claim 3: $s''_n - \hat{s}_n < \frac{1}{4}$. This follows from the argument in Claim 2.

Claim 4: $s'_n < \frac{1}{4}$. Define the energy

$$E_n(t) = \Phi_n(u'_n(t)) + \frac{\lambda}{(R-\hat{r})^p} u_n(t),$$

where

$$\Phi_n(v) = \int_0^{\varphi_n(v)} \varphi_n^{-1}(s)\, ds = v\varphi_n(v) - \int_0^v \varphi_n(s)\, ds.$$

Since for $t \in [s'_n, \hat{s}_n]$

$$\begin{aligned} E'_n(t) &= u'_n(t)\left[(\varphi_n(u'_n(t)))' + \frac{\lambda}{(R-\hat{r})^p}\right] \\ &\le u'_n(t)[(\varphi_n(u'_n(t)))' + \lambda f_n(t, u_n(t))] = 0, \end{aligned}$$

we can write

$$\begin{aligned} E_n(s'_n) &= \Phi_n(u'_n(s'_n)) + \frac{\lambda}{(R-\hat{r})^p}\hat{r} \\ &\ge E_n(\hat{s}_n) = \frac{\lambda}{(R-\hat{r})^p} u_n(\hat{s}_n). \end{aligned}$$

It follows that

$$\begin{aligned} {u'}_n^2(s'_n) &\ge u'_n(s'_n)\varphi_n(u'_n(s'_n)) \\ &\ge \Phi_n(u'_n(t)) \\ &\ge \frac{\lambda}{(R-\hat{r})^p}(u_n(\hat{s}_n) - \hat{r}) \\ &\ge \frac{\lambda}{2(R-\hat{r})^{p-1}}. \end{aligned}$$

As

$$\hat{r} = u_n(s'_n) = \int_0^{s'_n} u'_n(s)\, ds \ge s'_n u'_n(s'_n),$$

the claim follows

$${s'_n}^2 \le \frac{\hat{r}^2}{{u'_n}^2(s'_n)} \le \frac{2\hat{r}^2}{\lambda}(R-\hat{r})^{p-1} \le \frac{2R^2}{\lambda}(R-\hat{r})^{p-1} < \frac{1}{16}.$$

Claim 5: $1 - s''_n < \frac{1}{4}$. To prove this claim, we repeat the argument in Claim 4.

Conclusion – We come now to a contradiction since the previous claims imply that $1 = s'_n + (\hat{s}_n - s'_n) + (s''_n - \hat{s}_n) + (1 - s''_n) < \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$.

Step 6 – Existence of a solution of (1). From Step 5, we know that

$$u_n(t) \leq \hat{R} = \hat{r} + (\tfrac{R-\hat{r}}{2}) < R.$$

Hence for n large enough and a.e. $t \in [0,1]$, $f(t, u_n(t)) = f_n(t, u_n(t))$. Let $a \in]0, 1/2[$. The concavity of u_n implies that, for all $t \in [a, 1-a]$,

$$\tfrac{\hat{R}}{a} \geq u'_n(a) \geq u'_n(t) \geq u'_n(1-a) \geq -\tfrac{\hat{R}}{a},$$

so that for n large enough and all $t \in [a, 1-a]$ $\varphi'(u'_n(t)) = \varphi'_n(u'_n(t))$. It follows that

$$0 \leq -u''_n(t) = \lambda \frac{f(t, u_n(t))}{\varphi'(u'_n(t))} \leq \lambda \frac{h(t)}{\varphi'(\frac{\hat{R}}{a})},$$

where $h \in L^1(0,1)$ is such that, for a.e. $t \in [0,1]$ and every $u \in [0, \hat{R}]$, $f(t,u) \leq h(t)$. From Arzelà-Ascoli Theorem, a subsequence of $(u_n)_n$ converges in $C^1([a, 1-a])$ to a function $u \in C^1([a, 1-a])$ which satisfies

$$-u'' = \lambda \frac{f(t,u)}{\varphi'(u')}.$$

Using a diagonalization argument, u can be extended onto $]0,1[$. We observe further that, as $\max u_n \geq \min \beta$, the same holds true for u and hence it is non-trivial. As u is concave and positive, we have

$$\lim_{t \to 0} u(t) \in [0, +\infty[\quad \text{and} \quad \lim_{t \to 1} u(t) \in [0, +\infty[.$$

Arguing as in Step 6 and Step 7 of the proof of Theorem 2.1, we obtain

$$\lim_{t \to 0} u(t) = 0 \quad \text{or} \quad \lim_{t \to 0} u'(t) = +\infty$$

and

$$\lim_{t \to 1} u(t) = 0 \quad \text{or} \quad \lim_{t \to 1} u'(t) = -\infty.$$

Hence, we conclude as in the proof of Theorem 2.1. □

Remark 4.2. Suppose that, in addition to the assumptions of Theorem 4.1, the following condition holds

$$\lim_{u\to 0}\frac{f(t,u)}{u}=0, \qquad \text{uniformly a.e. on } [0,1].$$

Then, for any $\lambda \in]0,+\infty[$, problem (1) has at least one positive solution.

To prove this claim we only need to verify that an upper solution β of (5) can be constructed for any given $\lambda > 0$. We fix λ and define the upper solution β by setting $\beta(t) = \varepsilon t(1-t)$, where $\varepsilon \in]0, \frac{1}{2}[$ is such that $f(t,u) \leq \frac{u}{\lambda}$ for a.e. $t \in [0,1]$ and all $u \in]0,\varepsilon]$. We have then

$$-\beta''(t) = 2\varepsilon \geq (1+\varepsilon^2)^{3/2}\varepsilon \geq \lambda(1+\beta'(t)^2)^{3/2} f(t,\beta(t))$$

a.e. in $[0,1]$.

5. Two singularities

Theorem 5.1. *Let $f : [0,1]\times]0,R[\to [0,+\infty[$, with $R \in]0,+\infty[$, be a L^1-Carathéodory and locally L^1-Lipschitz function. Assume that (h_1) and (h_2) hold. Then there exists $\lambda_0 > 0$ such that for each $\lambda \in]0,\lambda_0]$ problem* (1) *has at least two positive solutions.*

Proof. The modified problem. For each $n \in \mathbb{N}$, $n > 1$, we define

$$f_n(t,u) = \begin{cases} f(t,R/n) & \text{if } u \leq R/n,\\ f(t,u) & \text{if } R/n < u \leq \frac{n-1}{n}R,\\ f(t,\frac{n-1}{n}R) & \text{if } \frac{n-1}{n}R < u, \end{cases}$$

and consider the modified problem

$$-(\varphi_n(u'))' = \lambda f_n(t,u), \quad u(0)=0,\ u(1)=0. \tag{11}$$

where φ_n is defined from (4).

From Step 2 and Step 3 in the proof of Theorem 4.1 there exists $\lambda_0 > 0$ such that, for all $\lambda \in]0,\lambda_0]$, there are upper solutions β_1, β_2 and a lower solution α_2 of (11) such that, for all large n, $\min(\beta_2 - \beta_1) > 0$, $\min\beta_1 > 0$, and $\max(\alpha_2 - \beta_2) > 0$.

From Step 3 in the proof of Theorem 2.1 there exists a lower solution α_1 of (11) such that, for all large n, $\alpha_1(t) > 0$ on $]0,1[$ and $\max \alpha_1 < \min \beta_1$.

Accordingly, for each $\lambda \in]0, \lambda_0]$ there are sequences $(u_n)_n$ and $(v_n)_n$ of solutions of (11) satisfying, for some $R' < R$, $\alpha_1(t) \le u_n(t) \le \beta_1(t)$ on $[0,1]$ and $\min \beta_2 \le \max v_n \le R'$. Arguing as in Step 5 of the proof of Theorem 2.1 and as in Step 6 of the proof of Theorem 4.1 we obtain positive solutions u and v of (1) as limits of subsequences of $(u_n)_n$ and $(v_n)_n$, respectively. Since $\alpha_1(t) \le u(t) \le \beta_1(t)$ on $[0,1]$ and $\min \beta_2 \le \max v$, we have $u \ne v$. □

6. Non-existence for large values of λ

THEOREM 6.1. *Let $f : [0,1] \times]0,R[\to [0,+\infty[$, with $R \in]0,+\infty]$, be a L^1-Carathéodory function. Assume that for some $a \in L^1(0,1)$, with $a(t) \ge 0$ a.e. in $[0,1]$ and $a(t) > 0$ on a set of positive measure, we have*

$$\frac{f(t,u)}{u} \ge a(t), \qquad \text{for a.e. } t \in [0,1] \text{ and for every } u \in]0,R[.$$

Let Λ_1 be the principal eigenvalue of the problem

$$-u'' = \Lambda a(t) u, \quad u(0) = 0, \ u(1) = 0. \tag{12}$$

Then for each $\lambda > \Lambda_1$ problem (1) has no positive solution.

Proof. Let us prove that the existence of a positive solution v of problem (1), for some $\lambda > \Lambda_1$, yields the existence of a positive solution $u \in W^{2,1}(0,1)$ of

$$-u'' = \lambda a(t) u, \quad u(0) = 0, \ u(1) = 0, \tag{13}$$

which is impossible as $\lambda \ne \Lambda_1$.

Step 1 – Construction of an upper solution β of (13). Let β be the continuous extension on $[0,1]$ of the restriction to $]0,1[$ of v. Since

$$-\beta''(t) = \lambda (1 + (\beta'(t))^2)^{3/2} f(t, \beta(t)) \ge \lambda a(t) \beta(t), \quad \text{a.e. in } [0,1],$$

$\beta(0) \ge 0$ and $\beta(1) \ge 0$, we have that β is an upper solution of (13) according to Definition II-4.1 in [6].

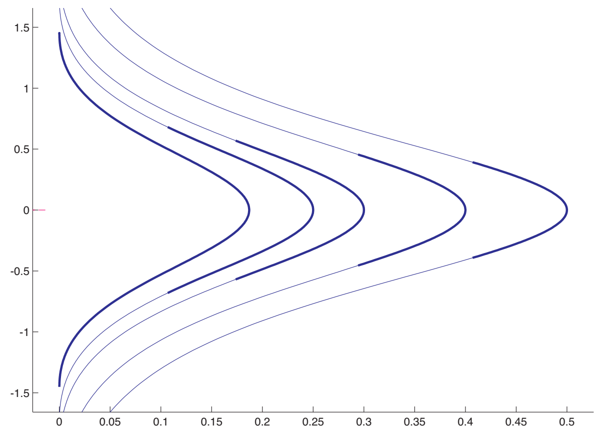

Figure 1: Phase plane for $f(u) = 1/\sqrt{u}$ and $\lambda = 1/2$.

Step 2 – Construction of a lower solution α of (13). Let α be an eigenfunction of (12) corresponding to Λ_1 such that $0 < \alpha(t) \le \beta(t)$ on $]0,1[$. Then α is a lower solution of (13), as

$$-\alpha''(t) = \Lambda_1 a(t)\alpha(t) \le \lambda a(t)\alpha(t).$$

Conclusion – As $\alpha(t) \le \beta(t)$ for all $t \in [0,1]$, Theorem II-4.6 in [6] guarantees that (13) has a positive solution $u \in W^{2,1}(0,1)$. This yields the contradiction. □

A. Numerical illustrations

In this appendix, we present some numerics on autonomous model examples. The computations have been performed by using the MATLAB built-in function ode45. On the phase-plane portraits, the bold part of an orbit corresponds to a time interval of length 1.

In Figure 1, we depict a phase-plane example for a weak singularity in zero. One can see that the time to travel to zero from the

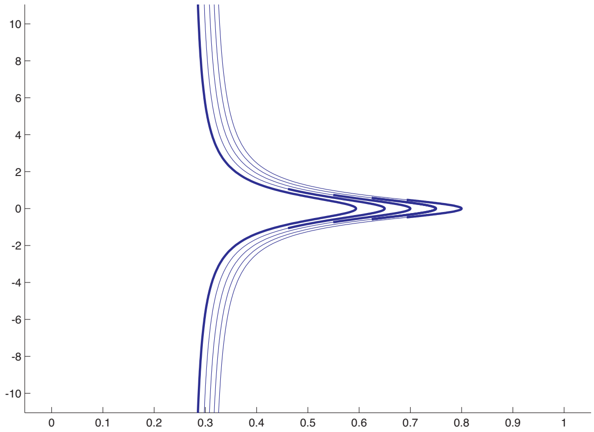

Figure 2: Phase plane for $f(u) = 1/u^2$ and $\lambda = 1/2$.

maximum is increasing. On the left of the bold orbit, the time required to reach zero (in the future or in the past) is less than 1/2 while it is larger than 1/2 for the orbits on the right.

The same phenomenon is depicted for a strong singularity in Figure 2. Observe that in this case, as proved in Theorem 3.3, no orbit reaches zero.

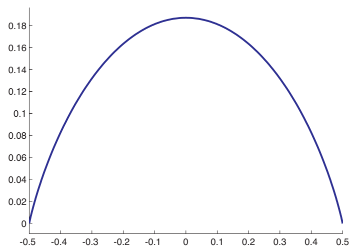

Figure 3: Small classical solution for $f(u) = 1/\sqrt{u}$ and $\lambda = 1/2$.

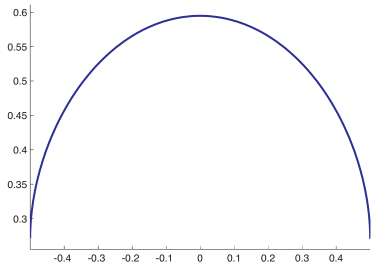

Figure 4: Non-classical solution for $f(u) = 1/u^2$ and $\lambda = 1/2$.

We depict in Figure 3 and Figure 4 the corresponding solutions. In the case of the strong force, the solutions display jumps at the endpoints of the interval.

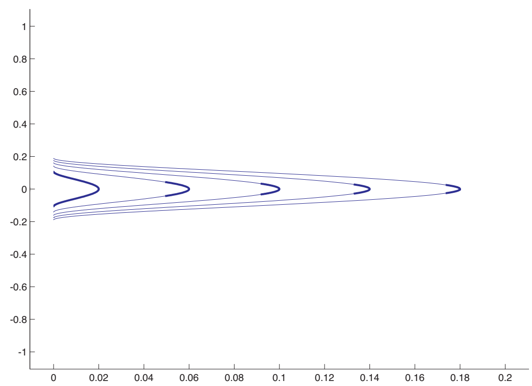

Figure 5: Phase plane for $f(u) = \frac{1}{\sqrt{u}(5-u)^2}$ and $\lambda = 1/2$. Zoom on the small orbits.

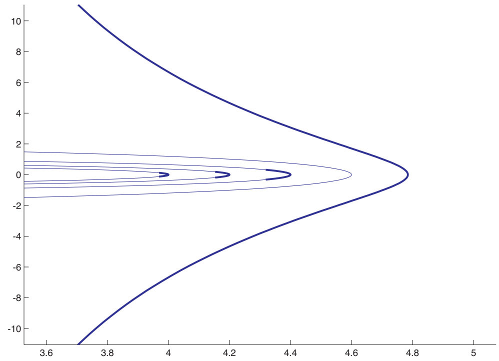

Figure 6: Phase plane for $f(u) = \frac{1}{\sqrt{u}(5-u)^2}$ and $\lambda = 1/2$. Zoom on the large orbits.

The case of two singularities is illustrated by Figure 5 and Figure 6. The time to reach zero is increasing from left to right for small orbits while, for large orbits, the derivative blows up before reaching zero.

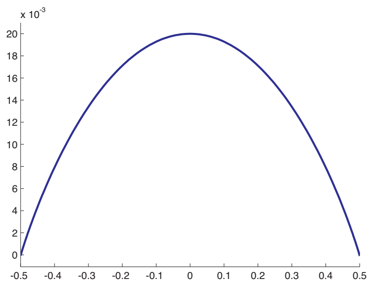

Figure 7: Small classical solution for $f(u) = \frac{1}{\sqrt{u}(5-u)^2}$ and $\lambda = 1/2$.

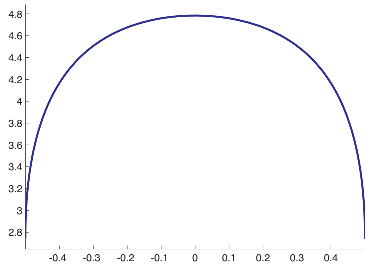

Figure 8: Large non-classical solution for $f(u) = \frac{1}{\sqrt{u}(5-u)^2}$ and $\lambda = 1/2$.

Hence, as proved in Theorem 5.1, we obtain a small classical solution (see Figure 7) and a large solution which turns here to be non-classical (see Figure 8).

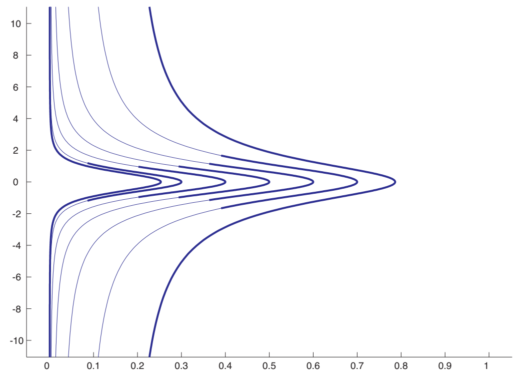

Figure 9: Phase plane for $f(u) = -\frac{1}{u \ln u}$ and $\lambda = 1/2$.

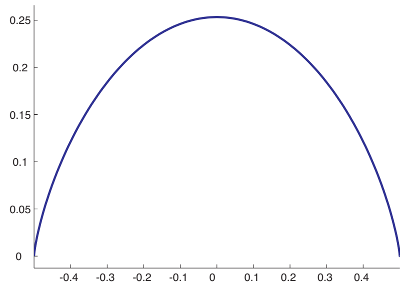

Figure 10: Small non-classical solution for $f(u) = -\frac{1}{u \ln u}$ and $\lambda = 1/2$.

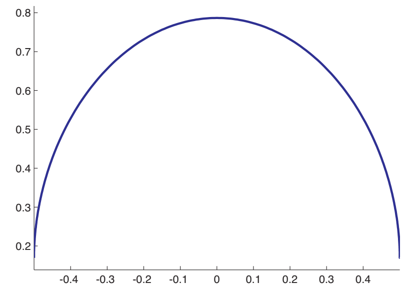

Figure 11: Large non-classical solution for $f(u) = -\frac{1}{u \ln u}$ and $\lambda = 1/2$.

Finally, Figure 9 concerns a case which is not totally covered by our results. Indeed, Theorem 2.1 ensures the existence of a small solution for the model nonlinearity $f(u) = -1/(u \ln u)$; this is non-classical by Theorem 3.3 (see Figure 10). Since the singularity of f at $u = 1$ is not strong, Theorem 5.1 does not apply. However, the numerical computation suggests the existence of a second large non-classical solution (see Figure 11). This example may motivate a further analysis.

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