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Algebraic Aspects of Commutation of Linear Operators up to a Factor

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Summary. - *When representing projective geometry by means of a vector space, commutativity can be replaced by commutativity up to a factor. This feature was investigated by F. Cecioni under very weak assumptions, but it is hard to generalize the methods of [4] to a wider algebraic context. In this note, we develop the independent treatment of H. Weyl, and extend the approach of [13] to non-commutative rings under suitable assumptions on the endomorphisms. From this point of view, we show that commutativity of operators up to a non-trivial factor is an exceptional phenomenon in comparison to strict commutativity.*

1. Introduction

The problem treated in this note arises from the investigation of commutativity of automorphisms in modular lattices, especially in projective geometries [6],[11]. It can be tackled by passing to the relevant vector spaces, in which the automorphisms can be described by semi-linear mappings and commutativity is described by commutativity up to a factor.

Strict commutativity of semi-linear operators, especially in complex spaces, was investigated in particular by the second author and his collaborators [1]. In the present paper, we treat commutativity of linear operators up to a scale factor. We shall demonstrate that

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commutativity up to a factor other than unity imposes some significant constraints. For instance, the phenomenon cannot occur if the ambient space is $\mathbb{R}^{2n+1}$, or one of the operators is invertible and the other operator has only one non-zero eigenvalue. The same is true in a $\mathbb{Z}^{2n+1}$ module if one operator is again invertible and diagonalizable (see Examples 1).

Such questions had been treated in about 1930 at an early stage in the development of modern algebra and geometry by F. Cecioni [4], and somewhat later by others [5]. These authors addressed the problem under very weak assumptions on the operators involved, but their use of nilpotent matrices and Jordan forms makes it hard to generalize the theory, for example to non-commutative rings. An independent and simpler treatment, which is based on eigenvalues and eigenvectors, is to be found in Hermann Weyl's 1931 book [13], which considers diagonalizable operators only. Further developments are treated in a paper of J. Schwinger [9], and applications in more recent papers such as [12], [10], [8], [7].

A motivating example is the following. Let $(\mathbf{e}_1 \ldots, \mathbf{e}_n)$ be a basis of $\mathbb{C}^n$, and set $\varphi = e^{2\pi i/n}$. Let U be the endomorphism of $\mathbb{C}^n$ for which $\mathbf{e}_j$ is an eigenvector with eigenvalue φ^j, and S one that cyclically permutes the basis. Then

$$U(\mathbf{e}_j) = \varphi^j \mathbf{e}_j, \qquad S(\mathbf{e}_j) = \begin{cases} \mathbf{e}_{j+1}, & j \neq n \\ \mathbf{e}_1, & j = n, \end{cases}$$

and

$$US = \varphi\, SU. \tag{1}$$

Taking determinants of both sides, $\varphi^n = 1$ and so a first restriction is that φ is a root of unity. By passing to the limit $n \to \infty$ in a suitable way, Weyl showed that one recovers the standard representation of the Heisenberg group and the fundamental commutation relations of quantum mechanics.

We shall formalize Weyl's approach so as to identify the minimal assumptions that are required, and this leads us to formulate explicit statements that are easily interpretable. Our treatment refers to modules over a ring that is not necessarily commutative, and we provide examples in the skew-field of quaternions. Given an endomorphism U of a module X over a ring R, conditions for the existence of automorphisms of X commuting with U up to a factor $\varphi \neq 1$ are examined.

Theorem 1 concerns *necessary conditions*: they hold under remarkably general assumptions, and exclude commutativity up to a factor $\varphi \neq 1$ (meaning the equation (1)) in a number of cases. In particular, Theorem 1 strongly restricts the point spectrum Λ and the set of the eigenelements of U, as well as the possible values of φ. We present several examples where commutativity up to $\varphi \neq 1$ cannot occur. In comparison with the Weyl's investigation, one of the most relevant assumptions that emerges is finiteness of Λ rather than finiteness of an overall dimension. Thus, some of our statements apply in infinite dimensions, and generalize to rings in place of fields.

Theorem 2 provides *necessary and sufficient conditions* under restrictive assumptions which nonetheless allow us to formulate the problem in terms of a module X over a ring R. We require Λ to lie in the center R' of R, and this enables us to group the eigenelements of U into "eigenmodules" $X_\lambda \subseteq X$. We further require X be the direct sum of the X_λ. The treatment is developed via tensor products, and the condition on U that we obtain may be expressed by means of tensor products involving the motivating example above. Corollary 1 concerns the easier case of vector spaces. Some final examples are given with endomorphisms that do commutate up to a factor $\varphi \neq 1$.

2. Necessary conditions for the commutation up to a non-trivial factor

Throughout this paper, R denotes a ring with unity and X a module over R. We let $L(X)$ denote the algebra of the endomorphisms of X, and $GL(X)$ the algebra of the invertible endomorphisms of X.

Definition 1. *For any pair of elements $S, U \in L(X)$ we say that S* and U commute up to a factor φ *if there exists $\varphi \in R$ such that*

$$US = \varphi SU.$$

We say that S and U commute up to a non-trivial factor *whenever $\varphi \neq 1$.*

Given $U \in L(X)$, we let

$$\Lambda_U = \{\lambda \in R \mid \exists\, \mathbf{e} \in X \setminus \{\mathbf{0}\} \text{ with } U\mathbf{e} = \lambda\mathbf{e}\}$$

denote the point spectrum of U ($\mathbf{0}$ is of course the null element of R). The following result then holds.

Lemma 1. *Let $U \in L(X)$. Suppose that $S \in GL(X)$ commutes with U up to a factor $\varphi \in R$, so that $US = \varphi SU$. Let $\Lambda = \Lambda_U$. Then*

a_1) if $\lambda \in \Lambda$ and $j \in \mathbb{N}$ then $\varphi^j\lambda \in \Lambda$; indeed,

a_2) any eigenelement $\mathbf{e}$ of U with eigenvalue λ is mapped by S^j to an eigenelement $S^j\mathbf{e}$ with eigenvalue $\varphi^j\lambda$.

Suppose also that the inverse φ^{-1} of φ exists in R (actually, it suffices to assume the existence of a left inverse). Then:

b_1) if $\lambda \in \Lambda$ and $j \in \mathbb{Z}$ then $\varphi^j\lambda \in \Lambda$;

b_2) The statement a_2) above also holds for $j \in \mathbb{Z}$.

Proof. By the very definition of Λ, we have that $\lambda \in \Lambda$ if and only if $U\mathbf{e} = \lambda\mathbf{e}$ for some $\mathbf{e} \neq \mathbf{0}$. It follows that $SU\mathbf{e} = \lambda S\mathbf{e}$, with $\mathbf{e} \neq \mathbf{0}$. From $US = \varphi SU$, we have $U(S\mathbf{e}) = \varphi(\lambda S\mathbf{e})$. As S is invertible, $S\mathbf{e} \neq \mathbf{0}$. We can therefore conclude that $\lambda \in \Lambda \Rightarrow \varphi\lambda \in \Lambda$. By induction on j, we deduce a_1) and a_2).

In order to prove b_1), let us now assume that a left inverse φ^{-1} of φ exists. From $US = \varphi SU$ and the invertibility of S, it follows immediately that $US^{-1} = \varphi^{-1}S^{-1}U$. Using the same argument as before with S^{-1} instead of S and φ^{-1} instead of φ, we deduce that $\lambda \in \Lambda \Rightarrow \varphi^{-1}\lambda \in \Lambda$. Combining this result with the preceding points a_1) and a_2), we obtain b_1) and b_2). ◻

We next now apply some elementary group theory to the multiplicative group R^* of invertible elements of R. Let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ denote the set of positive integers.

Proposition 1. *Let $\varphi \in R$. The following conditions are equivalent:*

i) φ is a root of unity, i.e. there exists $j \in \mathbb{N}^*$ such that $\varphi^j = 1$;

ii) The set $\varphi^{\mathbb{N}^*} = \{\varphi^j \mid j \in \mathbb{N}^*\}$ is finite.

Whenever these conditions are satisfied, φ is invertible. Indeed, $\varphi^{-1} = \varphi^{m-1}$, where m can be defined in the following two equivalent ways:

iii) φ *is a primitive root of unity of order* m *in the sense that* m *equals the* least *positive integer* j *such that* $\varphi^j = 1$;

iv) $m = \mathrm{card}(\varphi^{\mathbb{N}^*}) = \mathrm{card}(\varphi^{\mathbb{Z}})$, *so that* $\varphi^{\mathbb{Z}} = \varphi^M$ *where* $M = \{1, 2, \dots, m\}$.

This elementary proposition is based upon the fact that the mapping $j \mapsto \varphi^j$ determines a homomorphism of $\mathbb{Z}$ onto a cyclic group isomorphic to $\mathbb{Z}/(m)$.

We may now state the main theorem of this section.

Theorem 1. *Let* X *be a module over a ring* R *without zero divisors. Let* $U \in L(X)$ *be an endomorphism whose point spectrum* $\Lambda = \Lambda_U$ *is finite. Set* $\Lambda^* = \Lambda \setminus \{0\}$ *and* $\ell = \mathrm{card}\,\Lambda^*$; *we suppose also that* $\Lambda^* \neq \emptyset$. *Let* $S \in GL(X)$ *and* $\varphi \in R$ *be given such that* $US = \varphi SU$. *Then*

i) φ *is a root of unity whose order* m *divides* ℓ *(hence commutativity up to a non-trivial factor may only occur if* $\ell \neq 1$*);*

ii) setting $\ell = mc$ *and* $M = \{1, 2, \dots, m\}$, *the non-zero spectrum* Λ^* *may be expressed (non-uniquely) as*

$$\Lambda^* = \varphi^M \cdot P = \{\varphi^j \rho_\gamma \mid 1 \leqslant j \leqslant m,\ 1 \leqslant \gamma \leqslant c\}, \qquad (2)$$

where $P = \{\rho_1, \dots, \rho_c\}$ *is a suitable subset of* R *of cardinality* c.

iii) For each $\lambda \in \Lambda^*$, *let* X_λ *denote the subset of* X *consisting of eigenelements of* U *with eigenvalue* λ. *Then, for each fixed* γ, *the subsets* $X_{\varphi^j \rho_\gamma}$ *with* $j = 1, 2, .., m$ *are isomorphic:* $X_{\varphi^j \rho_\gamma} \cong X_{\rho_\gamma}$.

Suppose in addition that R *is a (commutative) field, which we denote here by* K, *so that* X *becomes a vector space over* K. *Let* n *be the dimension of the subspace generated by the eigenvectors of* U. *Then the eigenspaces* $X_{\varphi^j \rho_\gamma}$ *with* $j = 1, 2, \dots, m$ *are eigenspaces with the same dimension* d_γ, *so that* $\dim X_{\varphi^j \rho_\gamma} = \dim X_{\rho_\gamma} = d_\gamma$ *for each fixed* γ. *Thus,* m *divides* n, *and commutativity up to a non-trivial factor cannot occur if* ℓ *and* n *are coprime.*

Remark. It will also be convenient to write (2) in the form

$$\Lambda^* = \varphi^M \cdot P = \bigcup_{\gamma \in \Gamma} \Lambda_\gamma, \tag{3}$$

where $\Gamma = \{1, 2, \ldots, c\}$ and

$$\Lambda_\gamma = \varphi^M \cdot \rho_\gamma = \{\varphi^j \rho_\gamma \mid j \in M\}. \tag{4}$$

Proof. Let us first prove point 1. From the equation $US = \varphi SU$, the fact that S is invertible and U non-null, we deduce that $\varphi \neq 0$. Given $\lambda \in \Lambda^*$, we know from Lemma 1 that $\varphi^{\mathbb{N}}\lambda \subseteq \Lambda^*$, whence $\mathrm{card}(\varphi^{\mathbb{N}}\lambda) < \infty$. Proposition 1 now tells us that there exists a smallest positive integer m such that $\varphi^m = 1$.

Let $\sim$ be the relation on Λ^* defined by

$$\lambda \sim \lambda' \iff \lambda' \in \varphi^{\mathbb{Z}}\lambda.$$

This is an equivalence relation: reflexivity and transitivity are obvious, while symmetry follows from the additivity property of the group $\mathbb{Z}$. The finite non-empty set Λ^* is partitioned into say c equivalence classes, with $c \in \mathbb{N}^*$. We express the equivalence class of Λ^* containing an element λ by

$$[\lambda] = \varphi^{\mathbb{Z}}\lambda.$$

Let us show that each of these classes has the same number m of elements. In effect, we have a chain of equalities

$$\mathrm{card}([\lambda]) = \mathrm{card}(\varphi^{\mathbb{Z}}\lambda) = \mathrm{card}(\varphi^{\mathbb{Z}}) = \mathrm{card}(\varphi^M) = m.$$

The first equality follows from the definition of $[\lambda]$. In order to examine the second, consider the mapping $R \to R$ defined by $x \mapsto x\lambda$; it is injective since R has no zero divisors and

$$x_1\lambda = x_2\lambda \iff (x_1 - x_2)\lambda = 0 \implies x_1 = x_2.$$

The same mapping is bijective when restricted to $\varphi^{\mathbb{Z}}$, and $\mathrm{card}(\varphi^{\mathbb{Z}}\lambda) = \mathrm{card}(\varphi^{\mathbb{Z}})$. As the number c of classes is finite, we conclude that $\ell = mc$, and so c divides ℓ.

Let us turn to the proof of 2. Having set $\Gamma = \{1, 2, \ldots, c\}$, let $(\Lambda_\gamma)_{\gamma\in\Gamma}$ be the equivalence classes of eigenvalues, and let $(\rho_\gamma)_{\gamma\in\Gamma}$ be any system of representatives of these classes. According to the remarks above and Proposition 1, we have

$$\Lambda_\gamma = \varphi^{\mathbb{Z}}\rho_\gamma = \varphi^M\rho_\gamma = \{\varphi^j\rho_\gamma \mid j \in M\},$$

whence

$$\Lambda^* = \bigcup_{\gamma\in\Gamma}\Lambda_\gamma = \bigcup_{\gamma\in\Gamma}\{\varphi^M\rho_\gamma\} = \varphi^M\bigcup_{\gamma\in\Gamma}\{\rho_\gamma\} = \varphi^M P,$$

where $P = \{\rho_\gamma \mid \gamma \in \Gamma\}$. This justifies (2) and (4).

The proof of 3 proceeds as follows. In the following discussion, we fix $\gamma \in \Gamma$, meaning $1 \leqslant \gamma \leqslant c$. From Theorem 1, we know that any eigenelement $\mathbf{e}$ with eigenvalue $\varphi^j\rho_\gamma$ is mapped by S into an eigenelement $S\mathbf{e}$ with eigenvalue $\varphi^{j+1}\rho_\gamma$. Let us temporarily abbreviate $X_{\varphi^j\rho_\gamma}$ to $X_{j,\gamma}$, so that in this notation, $S(X_{j,\gamma}) \subseteq X_{j+1,\gamma}$. Conversely, any eigenelement $\mathbf{e}$ with eigenvalue $\varphi^{j+1}\rho_\gamma$ may be obtained by applying S to the eigenelement $S^{-1}\mathbf{e}$, which is (according to statement 2 of Theorem 1) an eigenelement with eigenvalue $\varphi^{-1}\varphi^{j+1}\rho_\gamma$. Thus, $S^{-1}(X_{j+1,\gamma}) \subseteq X_{j,\gamma}$. It follows that

$$S(X_{j,\gamma}) = X_{j+1,\gamma},$$

and S defines the required isomorphism of $X_{j,\gamma}$ onto $X_{j+1,\gamma}$.

For the last assertion, suppose that $R = K$ is a field. This immediately implies that $X_{j,\gamma}$ and $X_{j+1,\gamma}$ have the same dimension, which we denote by m. Consider the vector space generated by the eigenvectors of U, and suppose that n is its (finite) dimension. It follows that n is a finite multiple of m: as ℓ also is a finite multiple of m (see point 1), we conclude that n and ℓ cannot be coprime. □

To state the next result, let X be a vector space, or more generally a module over a ring R without zero divisors and which is torsion free (so $\lambda\mathbf{e} = \mathbf{0}$ implies either $\lambda = 0$ or $\mathbf{e} = \mathbf{0}$). Let $L(X)$ be the algebra of endomorphisms of X, and let O denote the null endomorphism.

Lemma 2. *With the above assumptions, suppose that $S \in L(X)$ and $U \in L(X)$ commute up to a factor, so that $US = \varphi SU$ with $\varphi \in R$ (Definition 1). The following statements hold:*

a) $SU \neq O \;\Leftrightarrow\; US = \varphi SU$ *for a unique* $\varphi \in R$*;*

b) $SU = O \;\Leftrightarrow\; US = \psi SU$ *for all* $\psi \in R$*.*

Proof. As for *a)*, we first observe that $SU \neq O$ implies the uniqueness of the factor φ. For given

$$US = \varphi SU = \varphi' SU, \qquad \varphi, \varphi' \in R,$$

we infer that $(\varphi - \varphi')SU = O$, whence $\varphi = \varphi'$. The converse follows by reversing the operations.

Let us now prove *b)*. Assume that $SU = O$, so $\psi SU = O$ for all ψ and

$$US = \varphi SU = O = \psi SU, \qquad \forall\, \psi \in R. \tag{5}$$

Conversely, we already know that (5) implies that $SU = O$. ☐

We conclude this section with a few examples.

2.1. Examples 1

As Theorem 1 provides *necessary* conditions for the occurrence of commutativity up to a factor $\varphi \neq 1$, it enables us to find instances where the phenomenon cannot occur, or at least to restrict the possible values of the factor φ.

i) Such restrictions depend on the very nature of the coefficients of the ring R. Commutativity up to a factor $\varphi \neq 1$ cannot occur at all if the ring R has no non-trivial roots of unity, as is the case when R is the field $\mathbb{Z}/(p)$ of cardinality p. The restriction $\varphi = \pm 1$ applies for the field $\mathbb{R}$ and the ring $\mathbb{Z}$, and more generally for the quotient ring $\mathbb{Z}/(r)$ for any integer $r > 0$.

ii) The impossibility of commutativity up to a factor $\varphi \neq 1$ may depend on the module X, independently of the diagonalizable operator U under consideration. According to the last statement of Theorem 1, this situation occurs when X is an odd-dimensional real vector space, and also when $R = \mathbb{Z}^{2n+1}$. Moreover, Theorem 2 implies that if X is a vector space of prime dimension and U is diagonalizable, then no non-trivial commutation is possible.

iii) The impossibility of commutativity up to a factor $\varphi \neq 1$ may depend on the particular operator U. This occurs with any module X, for instance an infinite-dimensional vector space, whenever U has only one non-zero eigenvalue. It also occurs for when X is one of $\mathbb{R}^2$, $\mathbb{C}^2$, $\mathbb{Q}^2$, $\mathbb{Z}^2$ ($\mathbb{Q}$ denotes the skew-field of quaternions) if U is a diagonal matrix with entries $1, 2$. Or, we could take X to be one of $\mathbb{R}^4$, $\mathbb{C}^4$, $\mathbb{Q}^4$, $\mathbb{Z}^4$ and U to be a diagonal matrix with entries $1, 1, 1, -1$.

3. A necessary and sufficient condition for the commutation up to a non-trivial factor

Let U continue to denote an endomorphism of a module X over a ring R with unity. In this section we make the further assumption that the point spectrum $\Lambda = \Lambda_U$ lies the center $R^{'}$ of R. For any $\lambda \in \Lambda$, the set

$$X_\lambda = \{\mathbf{e} \in X \mid U\mathbf{e} = \lambda\mathbf{e}\} \tag{6}$$

of eigenelements of U with eigenvalue λ is then a sub*module* of X, and we call it an *eigenmodule*. If we further assume that the sets X_λ generate X, then the latter is a *direct* sum of the submodules X_λ in view of the assumption $\Lambda \subseteq R'$. In these circumstances, we can write

$$X = \bigoplus_{\lambda \in \Lambda} X_\lambda, \qquad X_* = \bigoplus_{\lambda \in \Lambda^*} X_\lambda. \tag{7}$$

We shall shortly formulate a necessary and sufficient condition for the existence of operators S that commute with U up to a non-trivial factor (Theorem 2). This simplifies considerably in the case of vector spaces (Corollary 1).

In order to state the results compactly, we first introduce the following notation. We denote by

$$mX_\lambda = X_\lambda \otimes mR$$

the external direct sum of m copies of X_λ (see [2]). Given a primitive mth root of unity φ, we shall also work with the diagonal matrix

$$E_m = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \varphi^{m-1} \end{pmatrix} \tag{8}$$

of size $m \times m$. Given a submodule such as X_λ, we denote by $I(X_\lambda)$ and $O(X_0)$ the identity and null matrices representing endomorphisms of X_λ. With these preliminaries we state our main result.

THEOREM 2. *Let $U \in L(X)$, and suppose that its point spectrum $\Lambda = \Lambda_U$ is finite. Set $\Lambda^* = \Lambda \setminus \{0\}$ and $\ell = \operatorname{card} \Lambda^*$, and assume that $\Lambda^* \neq \emptyset$. Assume further that $\Lambda \subseteq R'$. Suppose that X is generated by the submodules (6), so that (7) applies. Then there exists an operator $S \in GL(X)$ commuting with U up to a non-trivial factor if and only if the following conditions all hold:*

1. *there exists a primitive root φ of unity of order $m > 1$ that divides ℓ, and an index set $\Gamma = \{1, 2, \ldots, c\}$ such that Λ^* is partitioned into a family $\left(\Lambda_\gamma = \varphi^M \cdot \rho_\gamma\right)_{\gamma \in \Gamma}$ of subsets as in (4),*

2. *the eigenmodules associated to the elements of each subset are isomorphic, so that*

$$X = X_0 \ \oplus \ \bigoplus_{\gamma \in \Gamma} \bigoplus_{j \in M} X_{\varphi^j \rho_\gamma} \cong \ X_0 \ \oplus \ \dot{\bigoplus_{\gamma \in \Gamma}} mX_{\rho_\gamma}. \tag{9}$$

3. *the matrix of U relative to (9) has the form*

$$O(X_0) \quad \oplus \quad \dot{\bigoplus_{\gamma \in \Gamma}} E_m \otimes \rho_\gamma I(X_{\rho_\gamma}). \tag{10}$$

Remark. In spite of the difficulties that can arise when considering the tensor product of modules over a non-commutative ring, there is no problem here because the left factor is a module over the center of the ring (see e.g. N. Bourbaki [3]).

Proof. Let us first prove necessity, that is 1 implies 2. We assume that an operator $S \in GL(X)$ exists, commuting with U up to a non-trivial factor φ. We have the decompositions (7) in which we use

- the description of Λ_* in (3),

- the isomorphism $X_{\varphi^j \rho_\gamma} \cong X_{\rho_\gamma}$ for each $\gamma \in \Gamma$,

both consequences of from Theorem 1. We express the result as a tensor product

$$X_* = \bigoplus_{\gamma\in\Gamma} \bigoplus_{j\in M} X_{\varphi^j \rho_\gamma} = \dot{\bigoplus_{\gamma\in\Gamma}} m X_{\rho_\gamma}.$$

The corresponding non-null component of the operator U can be expressed as

$$\begin{aligned} U_* &= \bigoplus_{\gamma\in\Gamma} \bigoplus_{j\in M} \varphi^j \rho_\gamma I(X_{j,\gamma}) \\ &= \dot{\bigoplus_{\gamma\in\Gamma}} \rho_\gamma \dot{\bigoplus_{j\in M}} \varphi^j I(X_{\rho_\gamma}) \\ &\cong \dot{\bigoplus_{\gamma\in\Gamma}} E_m \otimes \rho_\gamma I(X_{\rho_\gamma}). \end{aligned}$$

This ends the proof of point 1.

Let us now prove sufficiency. We thereby adopt points 1,2,3 as hypotheses. Relative to the representation (9) of X, the operator U is expressed as

$$U \; = \; O(X_0) \; \oplus \; \dot{\bigoplus} \rho_\gamma E_m \otimes I(X_{\rho_\gamma}).$$

Define, for each $j \in M$, the matrix

$$P_m = \begin{pmatrix} 0 & 1 \\ I_{m-1} & 0 \end{pmatrix} \tag{11}$$

that is easily seen to satisfy the equation

$$E_m P_m = \varphi P_m E_m.$$

In this notation, the permutation operator defined in the Introduction is none other than P_n.

In this more general situation, let us define the operator S by summing such permutation operators over the non-zero eigenmodules:

$$S = \bigoplus_{\gamma\in\Gamma} P_m \otimes I(X_{\rho_\gamma}).$$

Ignoring to write explicitly the null component of U, we have

$$\begin{aligned}
US &= \left(\dot{\bigoplus} \rho_\gamma E_m \otimes I(X_{\rho_\gamma})\right)\left(\bigoplus_{\gamma\in\Gamma} P_m \otimes I(X_{\rho_\gamma})\right) \\
&= E_m P_m \otimes \bigoplus_{\gamma\in\Gamma} \rho_\gamma I(X_{\rho_\gamma}) \\
&= \varphi\, P_m E_m \otimes \bigoplus_{\gamma\in\Gamma} \rho_\gamma I(X_{\rho_\gamma}) \\
&= \varphi \left(\bigoplus_{\gamma\in\Gamma} P_m \otimes I(X_{\rho_\gamma})\right)\left(\bigoplus_{\gamma\in\Gamma} E_m \otimes \rho_\gamma I(X_{\rho_\gamma})\right) \\
&= \varphi SU.
\end{aligned}$$

The matrix S thus constructed belongs to $GL(\mathrm{X})$, as it is invertible.

□

As a consequence of the preceding theorem, we can prove the following result in the case of vector spaces.

COROLLARY 1. *Let X be any finite-dimensional vector space over a (possibly skew-) field F. Let $U \in L(X)$. As usual, set $\Lambda = \Lambda_U$, and assume that $\Lambda^* = \Lambda \setminus \{0\}$ is* non-empty. *Assume further that $\Lambda \subseteq K'$. Suppose that the eigenvectors of U generate X, so that (7) applies. Then the following conditions are equivalent:*

1. *There exists an operator $S \in GL(X)$ which commutes with U up to a non-trivial factor.*

2. *There is a primitive root φ of unity of order $m > 1$ such that, relative to a basis of eigenvectors, U can be expressed in the equivalent ways*

$$U = E_m \otimes V = \dot{\bigoplus_{\delta\in\Delta}} \sigma_\delta E_m. \tag{12}$$

Here E_m is the $m \times m$ matrix over K defined by (8), V is a suitable finite-diagonalizable matrix over K, and $(\sigma_\delta)_{\delta\in\Delta}$ is a finite family of elements of K.

Whenever the conditions 1 *and* 2 *are satisfied, m divides* $\dim X$.

Proof. The statement of the Corollary is more handy than that of Theorem 2, but its proof is complicated by the manipulation of finite sets of numbers. (In the following formulas, we also include the eigenvalue 0.)

We start by observing that the existence of a basis of eigenvectors of U in X is a particular case of the assumption that X be generated by the X_λ, as in Theorem 2.

We consider the formula (12), and write the last term expression as a sum of tensor products involving identity matrices I_δ:

$$U = E_m \otimes V = \dot{\bigoplus_{\delta\in\Delta}} \sigma_\delta E_m = \dot{\bigoplus_{\delta\in\Delta}} E_m \otimes \sigma_\delta I_\delta.$$

Using the argument that follows, we can deduce that

$$V = \dot{\bigoplus_{\delta\in\Delta}} \sigma_\delta I_\delta. \tag{13}$$

Indeed, $E_m = \dot{\bigoplus\limits_{j\in M}} \varphi^j I_j$ and we may write $E_m \otimes A = \dot{\bigoplus\limits_{j}} \varphi^j A_j$ for any operator A. Hence,

$$\begin{aligned} E_m \otimes A = E_m \otimes B \quad &\Leftrightarrow \quad \dot{\bigoplus_{j}} \varphi^j A_j = \dot{\bigoplus_{j}} \varphi^j B_j \\ &\Leftrightarrow \quad A_j = B_j \quad \text{for all } j \in M \\ &\Leftrightarrow \quad A = B. \end{aligned}$$

To derive Corollary 1 from Theorem 2, we merely compare the expression (13) with that originating from (10), namely

$$V = \dot{\bigoplus_{\gamma\in\Gamma}} \rho_\gamma I(X_{\rho_\gamma}),$$

in the vector space context. In (13), V is effectively expressed by a diagonal matrix whose entries are the σ_δ with $\delta \in \Delta$. The resulting family is partitioned into subsets

$$(\sigma_\delta)_{\delta\in\Delta_\gamma} = \Big((\rho_\gamma)_\delta\Big)_{\delta\in\Delta_\gamma},$$

parametrized by Γ, with each subset collecting $\ell_\gamma = \mathrm{card}(\Delta_\gamma)$ equal elements ρ_γ. $\square$

This formula shows that the two expressions for U in (12) are equal up to isomorphism. Moreover, they specialize the expressions for U in Theorem 2 to the case of vector spaces, so that the implication from 1) to 2) of the Corollary follows easily.

The implication from 2) to 1) also follows because any choice of U or V in (12) can be put in the form described in Theorem 2 by partitioning the family $(\sigma_\delta)_{\delta\in\Delta}$ into subsets as explained. This concludes the proof.

3.1. Examples 2

Let us give some examples using the *sufficiency* for the occurrence of commutativity up to a factor $\varphi \neq 1$ from Corollary 1. In particular, the last example shows that for a given U there may exist operators S which commute up to different factors.

Below, E_m is the matrix (8), I_r is the identity matrix on r copies of R, $\mathbb{Q}$ the skew-field of quaternions, and $i^2 = -1$. In examples (1),(2),(3), R is one of $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, and in example (4), R is $\mathbb{C}$ or $\mathbb{Q}$. We point out that example (1) is well known, especially to physicists.

$$\begin{aligned}
(1)\quad X &= R\oplus R \\
U &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= E_2 \otimes I_1 \\
&= 1\cdot E_2
\end{aligned}
\qquad\qquad
\begin{aligned}
(2)\quad X &= \textstyle\bigoplus_1^3 R \\
U &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= E_2 \otimes I_1 + O \\
&= 1\cdot E_2 \oplus O
\end{aligned}$$

(3)
$$\begin{aligned} X &= \bigoplus_1^4 R \\ U &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} \\ &= E_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \\ &= 1 \cdot E_2 \oplus aE_2 \end{aligned}$$

(4)
$$\begin{aligned} X &= \bigoplus_1^4 R \\ U &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \\ &= E_4 \otimes I_1 \\ &= 1 \cdot E_4 \end{aligned}$$

The *sufficiency* statement from Corollary 1 ensures that commutativity up to a factor $\varphi \neq 1$ *does* occur. In (1),(2),(3) with $a \neq i, -i$ we have $m = 2$ and $\varphi = -1$; in (4) we have $m = 4$ and $\varphi = -1$ or $-i$.

In each of the four examples in turn, an operator S commuting with U up to a factor φ, as in the proof of Theorem 2, is given by

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that example (4) with $\varphi = -1$ is a particular case of example (3) with $a = i$, except for the order of the basis vectors.

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