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Note on Elongations of Summable p-Groups by $p^{\omega+n}$-Projective p-Groups

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Summary. - *We prove that a special ω-elongation of a summable p-group by a $p^{\omega+n}$-projective p-group is itself a summable p-group. This parallels our recent result in (Liet. matem. rink., 2006) related to totally projective p-groups.*

1. Introduction

Let A be an abelian p-group, written additively as is the custom when discussing abelian groups, with a first Ulm subgroup denoted via $p^\omega A = \cap_{i<\omega} p^i A$. Consulting with [10], such a group A is called $p^{\omega+n}$*-projective* for $n \in \mathbb{N}$ if there exists $P \leq A[p^n]$ such that A/P is a direct sum of cyclic groups; thus we can write $P = L + p^\omega A$ for some $L \leq A[p^n]$. Actually, the first popular example of a separable (i.e. $p^\omega A = 0$) $p^{\omega+1}$-projective p-group which is not a direct sum of cyclics is given in the remarkable paper [6]. Moreover, in [10] is also stated the definition of a *totally projective* p-group by using homological machinery (for more details see [7] too). Following [9] (see also [7], vol. II, p. 123, Chapter 84) the group A is said to be *summable* provided that $A[p] = \oplus_{\alpha<\lambda} A_\alpha$, where, for each $\alpha < \lambda = length(A)$, $A_\alpha \setminus \{0\} \subseteq p^\alpha A \setminus p^{\alpha+1} A$. An abelian p-group A is called a Σ-*group*

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when some its high subgroup is a direct sum of cyclics; it then follows that all its high subgroups retain that property. It is well-known that all totally projective and summable p-groups are Σ-groups, while the converse claim is demonstrably wrong. In [3] we have shown that every Σ-group (in particular, every summable group) is a $p^{\omega+1}$-projective p-group if and only if it is totally projective of length at most $\omega + 1$ (i.e. a direct sum of countable p-groups of length not exceeding $\omega + 1$) and that this is not the case for any other $p^{\omega+n}$-projectives when $n \geq 2$ (cf. [4] as well).

Referring to [11], a p-group A is termed *an ω-elongation of a totally projective group by a separable $p^{\omega+n}$-projective group* provided $p^\omega A$ is totally projective and $A/p^\omega A$ is $p^{\omega+n}$-projective, i.e. $p^\omega A$ is totally projective and there is $P \leq A$ so that $p^n P \subseteq p^\omega A \subseteq P$ and A/P is a direct sum of cyclics. In [8], Fuchs jointly with Irwin started the study of certain ω-elongations of totally projective p-groups by separable $p^{\omega+n}$-projective p-groups. Specifically, they considered the following construction: $p^\omega A$ is totally projective and $\exists P \leq A[p^n] : P \cap p^\omega A = 0$ and $A/(P \oplus p^\omega A)$ is a direct sum of cyclic groups. This p-group A was called by us in [4] *a separate strong ω-elongation of a totally projective group by a separable $p^{\omega+n}$-projective group*. In [4] we have investigated such a strong ω-elongation without the restriction on separability of P in A, that is, $P \cap p^\omega A = 0$, which extra restriction gives the name "separate". In other words, in our article $p^\omega A$ is totally projective and there is $P \leq A[p^n]$ so that $A/(P + p^\omega A)$ is a direct sum of cyclics. To simplify the terminology, we termed this special ω-elongation as *a strong ω-elongation of a totally projective p-group by a $p^{\omega+n}$-projective p-group*, thus removing the word "separable". We have also proved there that a p-torsion Σ-group is a strong ω-elongation of a totally projective group by a $p^{\omega+1}$-projective group only when it is totally projective and, as a consequence, that a p-torsion summable group is a strong ω-elongation of a totally projective group by a $p^{\omega+1}$-projective group precisely when it is a totally projective group of length at most Ω, the first uncountable (limit) ordinal, that is by [10] a direct sum of countable groups.

The purpose of the present short paper is to examine the same situation but concerning ω-elongations of summable groups. Nevertheless, because of the truthfulness of criterion 3 listed below, the analogue with the corresponding result for totally projective groups is not valid in general. Thereby, we shall demonstrate in what follows that under some additional circumstances on the existing group P these *(separate) strong ω-elongations of summable groups by $p^{\omega+n}$-projective groups* are themselves summable groups. Note that the required limitations on P are essential and cannot be dropped off.

2. The main result

Before proceed by proving the central theorem, for a facilitating of the exposition, we need the following concepts which are similar to those from [4].

DEFINITION 2.1. *The abelian p-group A is called a strong ω-elongation of a summable group by a $p^{\omega+n}$-projective group when $p^\omega A$ is summable and there exists $P \leq A[p^n]$ with the property that $A/(P + p^\omega A)$ is a direct sum of cyclics.*

If, in addition, $P \cap p^n A \subseteq p^\omega A$, then A is called a bounded strong ω-elongation of a summable p-group by a $p^{\omega+n}$-projective p-group.

DEFINITION 2.2. *The abelian p-group A is said to be a separate strong ω-elongation of a summable group by a $p^{\omega+n}$-projective group when $p^\omega A$ is summable and there exists $P \leq A[p^n]$ with the properties that $P \cap p^\omega A = 0$ and $A/(P \oplus p^\omega A)$ is a direct sum of cyclics.*

If, in addition, $P \cap p^n A = 0$, then A is said to be a bounded separate strong ω-elongation of a summable p-group by a $p^{\omega+n}$-projective p-group.

Notice that definition 2.2 obviously yields definition 2.1. However, it is worth noting that, since both $P \cap p^n A \subseteq p^\omega A$ and $P \cap p^\omega A = 0$ guarantee that $P \cap p^n A = 0$, each (bounded) strong ω-elongation must be a (bounded) separate strong ω-elongation, respectively, provided that P is separable in A; equivalently P has no nonzero elements of infinite height as computed in A.

We also recollect the major instruments that we shall apply in the sequel for verification of the assertion from the main theorem.

Criterion 1 ([6]). *Suppose A is an abelian p-group with a subgroup G whose quotient A/G is a direct sum of cyclic groups. Then A is a direct sum of cyclic groups $\iff$ $G = \cup_{i<\omega} G_i, G_i \subseteq G_{i+1} \leq G$ and, $\forall i \geq 1$, $G_i \cap p^i A = 0$.*

The next necessary and sufficient condition assures a more convenient for us form than the definition of a Σ-group quoted above, which form is needed for further applications.

Criterion 2 ([1]). *The abelian p-group A is a Σ-group $\iff$ $A[p] = \cup_{i<\omega} A_i, A_i \subseteq A_{i+1} \leq A$ and, $\forall i \geq 1$, $A_i \cap p^i A \subseteq p^\omega A$ or, equivalently, $A_i \cap p^i A = (p^\omega A)[p]$.*

As early mentioned, there exist p-primary Σ-groups that are not summable. So, in that aspect, the next necessary and sufficient condition ensures the realistic circumstances under which it is true.

Criterion 3 ([5], [2]). *The abelian p-group A is summable $\iff$ A is a Σ-group and $p^\omega A$ is summable. In particular, if $p^\omega A$ is summable and $A/p^\omega A$ is a direct sum of cyclic groups, then A is summable.*

It is worthwhile noticing that a valuable corollary, argued in [1], is that for lengths no more than $\omega + n$, $n \in \mathbb{N}$, these two classes of groups do coincide. However, for lengths greater than or equal to $\omega \cdot 2$ such a claim fails (see, for instance, [2]).

We continue with two statements about the direct summand property and the behavior of direct sums of the elongations explored. They unambiguously show that these elongations are closed with respect to direct summands and direct sums.

Proposition 2.3. *A direct summand of a (bounded) (separate) strong ω-elongation of a summable p-group by a $p^{\omega+n}$-projective p-group is one also.*

Proof. Write $A = B \oplus C$ and put $K = B \cap (P + p^\omega A)$, where P is defined as above. It is self-evident that $B/K \cong (B + P + p^\omega A)/(P + p^\omega A) \subseteq A/(P + p^\omega A)$ is a direct sum of cyclics as being an isomorphic copy of a subgroup of a direct sum of cyclics.

On the other hand, since $p^\omega A = p^\omega B \oplus p^\omega C$, we detect that $p^\omega B$ is summable being a direct factor of the summable group $p^\omega A$.

Furthermore, owing to the modular law (e.g. [7]), we obtain that $K = B \cap (P + p^\omega B + p^\omega C) = p^\omega B + [B \cap (P + p^\omega C)] = p^\omega B + T$ by putting $T = B \cap (P + p^\omega C)$. Because $p^n P = 0$ we find that $p^n T \subseteq p^n B \cap p^\omega C = 0$, whence $T \leq B[p^n]$. Moreover, we observe that $P \cap p^\omega A = 0$ along with $B \cap C = 0$ insure $p^\omega B \cap T = p^\omega B \cap (P + p^\omega C) = 0$. Besides, in view of the modular law ([7]), $T \cap p^n B = (P + p^\omega C) \cap p^n B \subseteq (P + p^\omega C) \cap p^n A = p^\omega C + (P \cap p^n A) \subseteq p^\omega A$ whenever $P \cap p^n A \subseteq p^\omega A$, hence $T \cap p^n B \subseteq p^\omega A \cap B = p^\omega B$. Consequently, all needed conditions are satisfied, in fact, and we are finished. □

PROPOSITION 2.4. *The direct sum of (bounded) (separate) strong ω-elongations of summable p-groups by $p^{\omega+n}$-projective p-groups is a (bounded) (separate) strong ω-elongation of a summable p-group by a $p^{\omega+n}$-projective p-group.*

Proof. Write $A = \oplus_{i \in I} A_i$, where all components A_i satisfy the assumptions from the text either of definitions 2.1 or 2.2. For an arbitrary but fixed index $i \in I$, there is $P_i \leq A_i[p^n]$ such that $A_i/(P_i + p^\omega A_i)$ is a direct sum of cyclics and, eventually, either $P_i \cap p^\omega A_i = 0$ or $P_i \cap p^n A_i \subseteq p^\omega A_i$. It is apparent that $p^\omega A = \oplus_{i \in I} p^\omega A_i$ is summable (for example [7], p. 123) since each $p^\omega A_i$ is. Set $P = \oplus_{i \in I} P_i$, whence clearly $P \leq A[p^n]$. Therefore, we infer that $A/(P + p^\omega A) = \oplus_{i \in I} A_i / (\oplus_{i \in I} P_i + \oplus_{i \in I} p^\omega A_i) = \oplus_{i \in I} A_i / \oplus_{i \in I} (P_i + p^\omega A_i) \cong \oplus_{i \in I} (A_i/(P_i + p^\omega A_i))$. This isomorphism allows us to derive that this quotient is really a direct sum of cyclic groups. On the other hand, $P \cap p^\omega A = (\oplus_{i \in I} P_i) \cap (\oplus_{i \in I} p^\omega A_i) = \oplus_{i \in I} (P_i \cap p^\omega A_i) = 0$ provided that $P_i \cap p^\omega A_i = 0$, $\forall i \in I$, and $P \cap p^n A = (\oplus_{i \in I} P_i) \cap (\oplus_{i \in I} p^n A_i) = \oplus_{i \in I} (P_i \cap p^n A_i) \subseteq \oplus_{i \in I} p^\omega A_i = p^\omega (\oplus_{i \in I} A_i) = p^\omega A$ provided that $P_i \cap p^n A_i \subseteq p^\omega A_i$, $\forall i \in I$. These conclusions substantiate our claim, and so we are done. □

Now we have accumulated all the information, necessary to prove the following affirmation.

THEOREM 2.5 (MAIN THEOREM). *Any bounded strong ω-elongation of a summable p-group by a $p^{\omega+n}$-projective p-group is a summable p-group.*

Proof. By definition, in conjunction with the classical Kulikov's criterion for direct sums of cyclic groups or with its generalization - criterion 1 (see, for instance, [6] or [7]), we write down $A/(P+p^\omega A) = \cup_{i<\omega}[A_i/(P+p^\omega A)]$, where, for all $i \geq 1$, $A_i \subseteq A_{i+1} \leq A$ and $A_i \cap p^i A \subseteq P+p^\omega A$ for some $P \leq A[p^n]$. Thus $A = \cup_{i<\omega} A_i$. Furthermore, in accordance with the modular law (cf. [7]), we calculate for every $i \in \mathbb{N}$ that $A_i \cap p^i A \subseteq (P+p^\omega A) \cap p^i A = p^\omega A + (P \cap p^i A) = p^\omega A$ because $P \cap p^i A \subseteq p^\omega A$ by taking $i \geq n$ which is possible since the chain $(A_i)_{i<\omega}$ is increasing. Consequently, we deduce with the aid of criterion 2 that A must be a Σ-group and, even more, that $A/p^\omega A$ is a direct sum of cyclics. Henceforth, criterion 3 works to conclude that A is indeed summable, as expected. □

3. Remark

The condition on the p^n-subsocle P of A to have a finite number of finite heights as computed in A, that is $P \cap p^n A \subseteq p^\omega A$, is crucial and cannot be omitted as simple examples show. It is noteworthy that the Main Theorem holds true even for totally projective groups by employing the foregoing idea of proof and [10].

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