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A Note on the CR Cohomology of Levi-Flat Minimal Orbits in Complex Flag Manifolds

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Summary. - *We prove a relation between the $\bar{\partial}_M$ cohomology of a minimal orbit M of a real form $\mathbf{G}_0$ of a complex semisimple Lie group $\mathbf{G}$ in a flag manifold $\mathbf{G}/\mathbf{Q}$ and the Dolbeault cohomology of the Matsuki dual open orbit X of the complexification $\mathbf{K}$ of a maximal compact subgroup $\mathbf{K}_0$ of $\mathbf{G}_0$, under the assumption that M is Levi-flat.*

1. Introduction

Many authors have studied the $\bar{\partial}_M$ cohomology of CR manifolds (see e. g. [6, 7, 12] and references therein). In particular, since Andreotti and Fredricks [2] proved that every real analytic CR manifold M can be embedded in a complex manifold X, it is natural to try to find relations between the $\bar{\partial}_M$ cohomology of M and the Dolbeault cohomology of X.

In this paper we examine this problem for a specific class of homogeneous CR manifolds, namely minimal orbits in complex flag manifolds that are Levi-flat.

Given a (generalized) flag manifold $Y = \mathbf{G}/\mathbf{Q}$, with $\mathbf{G}$ a complex semisimple Lie group and $\mathbf{Q}$ a parabolic subgroup of $\mathbf{G}$, a real form

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$\mathbf{G}_0$ of $\mathbf{G}$ acts on Y with finitely many orbits. Among these there is exactly one orbit that is compact, the minimal orbit $M = \mathbf{G}_0 \cdot o$. Let $\mathbf{K}_0$ be a maximal compact subgroup of $\mathbf{G}_0$, and $\mathbf{K}$ its complexification. Then $X = K \cdot o$ is an open dense complex submanifold of $\mathbf{G}/\mathbf{Q}$ and contains M as an embedded submanifold. It is known that M is a deformation retract of X, so $H^p(M, \mathbb{C}) = H^p(X, \mathbb{C})$ ([4], [8]).

Let $\boldsymbol{E}$ be a $\mathbf{K}$-homogeneous complex vector bundle on X and $\boldsymbol{E}|_M$ its restriction to M. Under the additional assumption that M is Levi-flat we prove that the restriction map from the Dolbeault cohomology $H^{p,q}(X, \boldsymbol{E})$ to the $\bar{\partial}_M$ cohomology $H^{p,q}(M, \boldsymbol{E}|_M)$ is continuous, injective and has a dense range. More precisely we show that

$$H^{p,q}(M, \boldsymbol{E}|_M) = \mathcal{O}_M(M) \otimes_{\mathcal{O}_X(X)} H^{p,q}(X, \boldsymbol{E})$$

where $\mathcal{O}_M(M)$ (resp. $\mathcal{O}_X(X)$) is the space of CR (resp. holomorphic) functions on M (resp. X), and that the restriction map from $\mathcal{O}_X(X)$ to $\mathcal{O}_M(M)$ is injective, continuous and has a dense range.

2. Preliminaries on minimal orbits in complex flag manifolds

Let $\mathbf{G}$ be a complex connected semisimple Lie group, with Lie algebra $\mathfrak{g}$, and let $\mathbf{Q}$ be a parabolic subgroup of $\mathbf{G}$, with Lie algebra $\mathfrak{q}$. Then $\mathbf{Q}$ is the normalizer of $\mathfrak{q}$, $\mathbf{Q} = \mathbf{N}_{\mathbf{G}}(\mathfrak{q})$ and is connected. The coset space $Y = \mathbf{G}/\mathbf{Q}$ is a compact complex manifold, called a *flag manifold* (it is a complex smooth projective variety). It is not restrictive to assume that $\mathfrak{q}$ does not contain any simple ideal of $\mathfrak{g}$.

Let σ be an anti-holomorphic involution of $\mathbf{G}$; we will also denote by σ its differential at the identity and we will write $\bar{x} = \sigma(x)$. A *real form* of $\mathbf{G}$ is an open subgroup $\mathbf{G}_0$ of $\mathbf{G}^\sigma$. It is a Lie subgroup and its Lie algebra $\mathfrak{g}_0$ satisfies $\mathfrak{g}_0 = \mathfrak{g}^\sigma$ and $\mathfrak{g} = \mathbb{C} \otimes \mathfrak{g}_0$.

Let $\mathbf{K}_0$ be a maximal compact subgroup of $\mathbf{G}_0$, and θ the corresponding Cartan involution: $\mathbf{K}_0 = \mathbf{G}_0^\theta$. Still denoting by θ the complexification of θ, there is exactly one open subgroup $\mathbf{K}$ of $\mathbf{G}^\theta$ such that $\mathbf{K} \cap \mathbf{G}_0 = \mathbf{K}_0$. Let $\mathfrak{k}$ and $\mathfrak{k}_0$ be the corresponding Lie subalgebras.

The groups $\mathbf{G}_0$ and $\mathbf{K}$ act on Y via left multiplication. There is exactly one closed $\mathbf{G}_0$-orbit M ([1, 19]) and it is contained in the unique open $\mathbf{K}$ orbit X ([3, 14]), we denote by $j\colon M \to X$ the inclusion. The open orbit X is dense in Y and is the dual orbit of M, in the terminology of [14].

The manifolds M and X do not depend on the choice of $\mathbf{G}$ and $\mathbf{G}_0$, but only on $\mathfrak{g}_0$ and $\mathfrak{q}$ ([1, 14]). So there is no loss of generality assuming that $\mathbf{G}$ is simply connected, that $\mathbf{G}_0$ is connected and that M and X are the orbits through the point $o = e\mathbf{Q}$. We will write $M = M(\mathfrak{g}_0, \mathfrak{q})$.

The isotropy subgroups at o of the actions of $\mathbf{G}_0$ and $\mathbf{K}$ are $\mathbf{G}_+ = \mathbf{G}_0 \cap \mathbf{Q}$ and $\mathbf{L} = \mathbf{K} \cap \mathbf{Q}$, with Lie algebras $\mathfrak{g}_+ = \mathfrak{g}_0 \cap \mathfrak{q}$ and $\mathfrak{l} = \mathfrak{k} \cap \mathfrak{q}$. Since M is compact, the action on M of maximal compact subgroup $\mathbf{K}_0$ is transitive: $M = \mathbf{K}_0/\mathbf{K}_+$, where $\mathbf{K}_+ = \mathbf{K}_0 \cap \mathbf{Q} = \mathbf{G}_0 \cap \mathbf{L}$ and $\mathrm{Lie}(\mathbf{K}_+) = \mathfrak{k}_+ = \mathfrak{k}_0 \cap \mathfrak{q} = \mathfrak{g}_0 \cap \mathfrak{l}$.

In the language of [1] the pair $(\mathfrak{g}_0, \mathfrak{q})$ is an effective parabolic minimal CR algebra and M is the associated minimal orbit. On M there is a natural CR structure induced by the inclusion into X.

We recall that M is totally real if the partial complex structure is trivial. We give a more complete characterization of totally real minimal orbits:

Theorem 2.1. *The following are equivalent:*

1. *M is totally real.*
2. $\bar{\mathfrak{q}} = \mathfrak{q}$.
3. $\mathfrak{l} = \mathfrak{k}_+^{\mathbb{C}}$.
4. *X is a Stein manifold.*
5. *X is a smooth affine algebraic variety defined over $\mathbb{R}$ and M is the set of its real points.*

Proof. (5) $\Rightarrow$ (4) because every closed complex submanifold of a complex vector space is Stein.

(4) $\Rightarrow$ (3) Since X is Stein, its covering $\tilde{X} = \mathbf{K}/\mathbf{L}^0$ is also Stein ([17]). Furthermore $\mathbf{K}$ is a linear algebraic group that is the complexification of a maximal compact subgroup $\mathbf{K}_0$; the result then follows from Theorem 3 of [15] and Remark 2 thereafter.

(3) $\Rightarrow$ (5) If $\mathfrak{l}$ is the complexification of $\mathfrak{k}_+$, then $\mathbf{L}$ is the complexification of $\mathbf{K}_+$. Hence $X = \mathbf{K}/\mathbf{L}$ is the complexification of

$M = \mathbf{K}_0/\mathbf{K}_+$ in the sense of [9], and (5) follows from Theorem 3 of the same paper.

(1) $\Leftrightarrow$ (2) is easy, and proved in [1].

(5) $\Rightarrow$ (1) is obvious.

(2) $\Rightarrow$ (3) We have that $\bar{\mathfrak{k}} = \mathfrak{k}$, thus $\mathfrak{k} \cap \mathfrak{q} = (\mathfrak{k} \cap \mathfrak{q} \cap \mathfrak{g}_0)^{\mathbb{C}} = \mathfrak{k}_+^{\mathbb{C}}$. □

We denote by $\mathcal{O}_N$ the sheaf of smooth CR functions on a CR manifold N. If N is complex or real, then $\mathcal{O}_N$ is the usual sheaf of holomorphic or smooth (complex valued) functions. For every open set $U \subset N$, the space $\mathcal{O}_N(U)$ is a Fréchet space (with the topology of uniform convergence of all derivatives on compact sets).

Corollary 2.2. *If M is totally real then $j^*\big(\mathcal{O}_X(X)\big)$ is dense in $\mathcal{O}_M(M) = C^\infty(M)$.*

Proof. Let $X \subset \mathbb{C}^N$ be an embedding as in (5) of Theorem 2.1, so that $M = X \cap \mathbb{R}^N$. The restrictions of complex polynomials in $\mathbb{C}^N$ are contained in $\mathcal{O}_X(X)$ and dense in $\mathcal{O}_M(M)$ (see e.g. [18]). □

3. Levi-flat orbits and the fundamental reduction

In this paper we consider Levi-flat minimal orbits. They are orbits $M = M(\mathfrak{g}_0, \mathfrak{q})$, where $\mathfrak{q}' = \mathfrak{q} + \bar{\mathfrak{q}}$ is a subalgebra (necessarily parabolic) of $\mathfrak{g}$. Let $\mathbf{Q}' = \mathbf{N}_{\mathbf{G}}(\mathfrak{q}')$, $Y' = \mathbf{G}/\mathbf{Q}'$, $\mathbf{G}'_+ = \mathbf{G} \cap \mathbf{Q}'$, $M' = M(\mathfrak{g}_0, \mathfrak{q}') = \mathbf{G}_0/\mathbf{G}'_+$, $\mathbf{K}'_+ = \mathbf{K}_0 \cap \mathbf{Q}'$, $\mathbf{L}' = \mathbf{K} \cap \mathbf{Q}'$ and $X' = \mathbf{K}/\mathbf{L}'$.

From Theorem 2.1 we have that M' is totally real and X' is Stein.

The inclusion $\mathbf{Q} \to \mathbf{Q}'$ induces a fibration

$$\pi \colon Y = \mathbf{G}/\mathbf{Q} \longrightarrow \mathbf{G}/\mathbf{Q}' = Y' \tag{1}$$

with complex fiber $F \simeq \mathbf{Q}'/\mathbf{Q}$. This fibration is classically referred to as the Levi foliation, and is a special case of the fundamental reduction of [1]. In fact Levi-flat minimal orbits are characterized by the property that the fibers of the fundamental reduction are totally complex.

We identify F with $\pi^{-1}(e\mathbf{Q})$.

Lemma 3.1. *$\pi^{-1}(M') = M$, $\pi^{-1}(X') = X$ and F is a compact connected complex flag manifold.*

Proof. First we observe that F is connected because $\mathbf{Q}'$ is connected. Let F' be the fiber of the restriction $\pi|_M : M \to M'$. Then F' is totally complex and CR generic in F, hence an open subset of F. Proposition 7.3 and Theorem 7.4 of [1] show that there exists a connected real semisimple Lie group $\mathbf{G}_0''$ acting on F' with an open orbit, and that the Lie algebra of the isotropy is a t-subalgebra (i. e. contains a maximal triangular subalgebra) of $\mathfrak{g}_0''$. Hence a maximal compact subgroup $\mathbf{K}_0''$ of $\mathbf{G}_0''$ has an open orbit, which is also closed. Since F' is open in F and F is connected, $\mathbf{K}_0''$ is transitive on F, and $F' = F$, proving the first two statements.

Furthermore the isotropy subgroup $\mathbf{G}_+''$ for the action of $\mathbf{G}_0''$ on F and the homogeneous complex structure are exactly those of a totally complex minimal orbit, hence by [1, § 10] F is a complex flag manifold. $\square$

The total space M is locally isomorphic to an open subset of $M' \times F$, hence to $U \times \mathbb{R}^k$, where U is open in $\mathbb{C}^n$, for some integers n and k.

For a Levi flat CR manifold N and a nonnegative integer p, let Ω_N^p be the sheaf of p-forms that are CR (see [7] for precise definitions). They are $\mathcal{O}_N$-modules and $\Omega_N^0 \simeq \mathcal{O}_N$.

Let $\mathcal{A}_N^{p,q}$ be the sheaf of (complex valued) smooth (p,q)-forms on N, $\bar{\partial}_N$ the tangential Cauchy-Riemann operator and $\mathcal{Z}_N^{p,q}$, (resp. $\mathcal{B}_N^{p,q}$) the sheaf of closed (resp. exact) (p,q)-forms. As usual we denote by $H^{p,q}(N) = \mathcal{Z}_N^{p,q}(N)/\mathcal{B}_N^{p,q}(N)$ the cohomology groups of the $\bar{\partial}_M$ complex on smooth forms. The Poincaré lemma is valid for $\bar{\partial}_N$ (see [11]), thus the complex:

$$0 \to \Omega_N^p \to \mathcal{A}_N^{p,0} \xrightarrow{\bar{\partial}_N} \dots \xrightarrow{\bar{\partial}_N} \mathcal{A}_N^{p,q} \xrightarrow{\bar{\partial}_N} \dots$$

is a fine resolution of Ω_N^p. This implies that $H^{p,q}(N) \simeq H^q(\Omega_N^p)$.

Let $\boldsymbol{E}_N$ be a homogeneous CR vector bundle on N (i. e. a complex vector bundle with transition functions that are CR) with fiber E, and let $\mathcal{E}_N$ be the sheaf of its CR sections.

We denote by $\boldsymbol{E}_N^p$ the bundle of CR , $\boldsymbol{E}_N$-valued, p-forms, with associated sheaf of CR sections $\mathcal{E}_N^p = \Omega_N^p \otimes_{\mathcal{O}_N} \mathcal{E}_N$.

Let $\mathcal{A}_{N,\boldsymbol{E}_N}^{p,q} = \mathcal{A}_N^{p,q} \otimes_{\mathcal{O}_N} \mathcal{E}_N$, denote by $\bar{\partial}_{\boldsymbol{E}_N}$ the tangential Cauchy-Riemann operator on $\boldsymbol{E}_N$ and let $\mathcal{Z}_{N,\boldsymbol{E}_N}^{p,q}$ (resp. $\mathcal{B}_{N,\boldsymbol{E}_N}^{p,q}$) be the sheaf of $\bar\partial_{\boldsymbol{E}_N}$-closed (resp. exact) smooth forms with values in $\boldsymbol{E}_N$.

Then $H^{p,q}(N,\boldsymbol{E}_N)=\mathcal{Z}^{p,q}_{N,\boldsymbol{E}_N}(N)/\mathcal{B}^{p,q}_{N,\boldsymbol{E}_N}(N)$, but we also have:

$$H^{p,q}(N,\boldsymbol{E}_N)=H^q(\mathcal{E}^p_N)=H^{0,q}(N,\boldsymbol{E}^p_N)=\mathcal{Z}^{0,q}_{N,\boldsymbol{E}^p_N}(N)/\mathcal{B}^{0,q}_{N,\boldsymbol{E}^p_N}(N).$$

For any open set $U\subset N$, the spaces $\mathcal{A}^{p,q}_{N,\boldsymbol{E}_N}(U)$ and $\mathcal{Z}^{p,q}_{N,\boldsymbol{E}_N}(U)$ are Fréchet spaces with the topology of uniform convergence of all derivatives on compact sets. If $\mathcal{B}^{p,q}_{N,\boldsymbol{E}_N}(N)$ is closed in $\mathcal{Z}^{p,q}_{N,\boldsymbol{E}_N}(N)$, then $H^{p,q}(N,\boldsymbol{E}_N)$, with the quotient topology, is also a Fréchet space.

4. Statements and proofs

Let $\boldsymbol{E}_F$ be a $\mathbf{L}'$-homogeneous holomorphic vector bundle on F. The $\mathbf{L}'$ action induces a natural $\mathbf{L}'$ action on $\mathcal{A}^{p,q}_{F,\boldsymbol{E}_F}$, hence on $H^{p,q}(F,\boldsymbol{E}_F)$, because the action of $\mathbf{L}'$ preserves closed and exact forms. Since F is a compact complex manifold, $H^{p,q}(F,\boldsymbol{E}_F)$ is finite dimensional and we can construct the $\mathbf{K}$-homogeneous holomorphic vector bundle on X':

$$\boldsymbol{H}^{p,q}_{X'}(F,\boldsymbol{E}_F)=\mathbf{K}\times_{\mathbf{L}'}H^{p,q}(F,\boldsymbol{E}_F).$$

In a similar way we define a $\mathbf{K}_0$-homogeneous complex vector bundle on M':

$$\boldsymbol{H}^{p,q}_{M'}(F,\boldsymbol{E}_F)=\mathbf{K}_0\times_{\mathbf{K}'_+}H^{p,q}(F,\boldsymbol{E}_F).$$

The following thorem has been proved by Le Potier ([13], see also [5]):

THEOREM 4.1. *Let X, X' and F be as above, $\boldsymbol{E}_X$ a $\mathbf{K}$-homogeneous holomorphic vector bundle on X and $\boldsymbol{E}_X|_F$ its restriction to F. Then there exists a spectral sequence ${}^pE^{s,t}_r$, converging to $H^{p,q}(X,\boldsymbol{E}_X)$, with*

$${}^pE^{s,t}_2=\bigoplus_i H^{i,s-i}\big(X',\boldsymbol{H}^{p-i,t+i}_{X'}(F,\boldsymbol{E}_X|_F)\big).\quad\square$$

For $p=0$ the spectral sequence collapses at $r=2$ and, recalling that X' is a Stein manifold, we obtain:

$$H^{0,q}(X,\boldsymbol{E}_X)=H^{0,0}\big(X',\boldsymbol{H}^{0,q}_{X'}(F,\boldsymbol{E}_X|_F)\big).$$

Recalling that $H^{p,q}(X,\boldsymbol{E}_X)=H^{0,q}(X,\boldsymbol{E}^p_X)$ we finally obtain:

Proposition 4.2. *Let X, X' and F be as above, $\boldsymbol{E}_X$ a $\mathbf{K}$-homogeneous holomorphic vector bundle on X and $\boldsymbol{E}_X|_F$ its restriction to F. Then:*

$$H^{p,q}(X, \boldsymbol{E}_X) = H^{0,0}\big(X', \boldsymbol{H}^{0,q}_{X'}(F, \boldsymbol{E}^p_X|_F)\big)$$

as Fréchet spaces. □

A statement analogous to the last proposition holds for M:

Proposition 4.3. *Let M, M' and F be as above, $\boldsymbol{E}_M$ a $\mathbf{K}_0$-homogeneous CR vector bundle on M and $\boldsymbol{E}_M|_F$ its restriction to F. Then:*

$$H^{p,q}(M, \boldsymbol{E}_M) = H^{0,0}\big(M', \boldsymbol{H}^{0,q}_{M'}(F, \boldsymbol{E}^p_M|_F)\big)$$

as Fréchet spaces.

Proof. Fix p, q, let $\mathcal{Z}_{M'} = \pi_*(\mathcal{Z}^{0,q}_{M,\boldsymbol{E}^p_M})$, $\mathcal{B}_{M'} = \pi_*(\mathcal{B}^{0,q}_{M,\boldsymbol{E}^p_M})$ and $\mathcal{H}_{M'}$ be the sheaf of sections of $\boldsymbol{H}^{0,q}_{M'}(F, \boldsymbol{E}^p_M|_F)$ We already know that $H^{p,q}(M, \boldsymbol{E}_M) \simeq \mathcal{Z}_{M'}(M')/\mathcal{B}_{M'}(M')$.

We now define a map $\phi\colon \mathcal{Z}_{M'} \to \mathcal{H}_{M'}$ as follows.

Let $U \subset M'$, $x \in U$, $x = g\mathbf{K}'_+$, $g \in \mathbf{K}_0$ and $\xi \in \mathcal{Z}_{M'}(U)$. Let $\xi_g = (g^{-1} \cdot \xi)|_F$. This is a closed $\boldsymbol{E}^p_M$-valued $(0,q)$-form on F, that determines a cohomology class $[\xi_g]$ in $H^{0,q}(F, \boldsymbol{E}^p_M|_F)$. Then the class of $(g, [\xi_g])$ in $\boldsymbol{H}^{0,q}_{M'}(F, \boldsymbol{E}^p_M|_F)$ does not depend on the particular choice of g, but only on x, hence it defines a section $s_\xi = \phi(\xi)$ of $\mathcal{H}_{M'}$ on U.

The sheaves $\mathcal{Z}_{M'}$, $\mathcal{B}_{M'}$, $\mathcal{H}_{M'}$, $\ker\phi$ are $\mathcal{O}_{M'}$-modules and ϕ is a morphism of $\mathcal{O}_{M'}$-modules. Since $\mathcal{O}_{M'}$ is fine, to prove that $\phi(M')$ is continuous, surjective and that its kernel is exactly $\mathcal{B}_{M'}(M')$ it sufices to check that this is true locally, and this reduces to a straightforward verification. □

We prove now the main theorem of this paper:

Theorem 4.4. *Let M and X be as above, $\boldsymbol{E}_X$ a $\mathbf{K}$-homogeneous holomorphic vector bundle over X. Then:*

$$H^{p,q}(M, \boldsymbol{E}_X|_M) \simeq \mathcal{O}_M(M) \otimes_{\mathcal{O}_X(X)} H^{p,q}(X, \boldsymbol{E}_X).$$

Proof. Define M' and X' as above, fix integers $p, q \geq 0$ and let $\boldsymbol{H}_{X'} = \boldsymbol{H}^{0,q}_{X'}(F, \boldsymbol{E}^p_X|_F)$, $\boldsymbol{H}_{M'} = \boldsymbol{H}^{0,q}_{M'}(F, \boldsymbol{E}^p_X|_F) = \boldsymbol{H}_{X'}|_{M'}$. By Propositions 4.2 and 4.3, we have that $H^{p,q}(M, \boldsymbol{E}_X|_M) = \Gamma(M', \boldsymbol{H}_{M'})$ and $H^{p,q}(X, \boldsymbol{E}_X) = \Gamma(X', \boldsymbol{H}_{X'})$.

Since $\dim_{\mathbb{R}} M' = \dim_{\mathbb{C}}(X')$, a global section of $\boldsymbol{H}_{X'}$ that is zero on M' must be zero on X', i. e. the restriction map $\Gamma(X', \boldsymbol{H}_{X'}) \to \Gamma(M', \boldsymbol{H}_{M'})$ is injective.

On the other hand, X' is Stein, thus $\boldsymbol{H}_{X'}$ is generated at every point by its global sections. Together with the fact that $\boldsymbol{H}_{M'} = \boldsymbol{H}_{X'}|_{M'}$ this implies that

$$\Gamma(M', \boldsymbol{H}_{M'}) = \mathcal{O}_{M'}(M') \otimes_{\mathcal{O}_{X'}(X')} \Gamma(X', \boldsymbol{H}_{X'}),$$

where in the right hand side we implicitly identify global holomorphic sections on X' with their restrictions to M'.

The theorem follows from the observation that $\mathcal{O}_M(M) \simeq \mathcal{O}_{M'}(M')$ and $\mathcal{O}_X(X) \simeq \mathcal{O}_{X'}(X')$ because the fiber F of π is a compact connected complex manifold. □

This, together with Corollary 2.2, implies the following.

Corollary 4.5. *With the same assumptions, the inclusion $j \colon M \to X$ induces a map:*

$$j^* \colon H^{p,q}(X, \boldsymbol{E}_X) \longrightarrow H^{p,q}(M, \boldsymbol{E}_X|_M)$$

that is continuous, injective and has a dense range. □

5. An example

Let $\mathbf{G} = \mathbf{SL}(4, \mathbb{C})$, and $\mathbf{Q}$ be the parabolic subgroup of upper triangular matrices. We consider the real form $\mathbf{G}_0 = \mathbf{SU}(1, 3)$, identified with the group of linear trasformations of $\mathbb{C}^4$ that leave invariant the Hermitian form associated to the matrix

$$B = \begin{pmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{pmatrix}.$$

Then $\mathbf{G}/\mathbf{Q}$ is the set of complete flags $\{\ell^1 \subset \ell^2 \subset \ell^3 \subset \mathbb{C}^4\}$ and $M = \mathbf{G}_0 \cdot e\mathbf{Q}$ is the submanifold $\{\ell^1 \subset \ell^2 \subset \ell^3 \subset \mathbb{C}^4 \mid \ell^3 = (\ell^1)^\perp\}$.

Let $\mathbf{Q}'$ be the set of block upper triangular matrices of the form

$$\mathbf{Q}' = \left\{ g = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \mid g \in \mathbf{G} \right\},$$

so that M' is the totally real manifold $\{\ell^1 \subset \ell^3 \subset \mathbb{C}^4 \mid \ell^3 = (\ell^1)^\perp\}$ and M fibers over M' with typical fiber F isomorphic to $\mathbb{CP}^1$. The fibration is given by

$$(\ell^1, \ell^2, \ell^3) \longmapsto (\ell^1, \ell^3). \tag{*}$$

Choose $\mathbf{K}$ to be the stabilizer in $\mathbf{G}$ of the subspaces $V = \mathrm{Span}(\mathrm{e}_1 - \mathrm{e}_4)$ and $W = \mathrm{Span}(\mathrm{e}_1 + \mathrm{e}_4, \mathrm{e}_2, \mathrm{e}_3)$ so that $\mathbf{K}$ is isomorphic to $\mathbf{S}(\mathbf{GL}(1,\mathbb{C}) \times \mathbf{GL}(3,\mathbb{C}))$ and $\mathbf{K}_0$ to $\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(3))$. Then X is the set of flags $\{\ell^1 \subset \ell^2 \subset \ell^3 \subset \mathbb{C}^4\}$ in a generic position with respect to the subspaces V and W, and X' is the set of flags $\{\ell^1 \subset \ell^3 \subset \mathbb{C}^4\}$ in a generic position with respect to V and W. The map from X to X' given by (*) is a fibration with typical fiber isomorphic to $\mathbb{CP}^1$ and X' is a Stein manifold.

Finally let $\boldsymbol{E} = X \times \mathbb{C}$ be the trivial line bundle. According to Propositions 4.2 and 4.3 the cohomology of M and X is given by

$$\begin{aligned} H^{p,q}(X) &= H^{p,q}(X, \boldsymbol{E}) = H^{0,0}\big(X', \boldsymbol{H}^{0,q}_{X'}(F, \boldsymbol{E}^p|_F)\big), \\ H^{p,q}(M) &= H^{p,q}(M, \boldsymbol{E}|_M) = H^{0,0}\big(M', \boldsymbol{H}^{0,q}_{M'}(F, \boldsymbol{E}^p|_F)\big). \end{aligned}$$

Recalling that $H^{p,q}(F) \simeq \mathbb{C}$ if $p = q = 0$ or $p = q = 1$ and 0 otherwise, we obtain:

$$\begin{cases} H^{p,q}(X) \simeq \mathcal{O}_X(X) \simeq \mathcal{O}_{X'}(X') & \text{if } p = q = 0 \text{ or } p = q = 1; \\ H^{p,q}(X) = 0 & \text{otherwise;} \end{cases}$$

and analogously:

$$\begin{cases} H^{p,q}(M) \simeq \mathcal{O}_M(M) \simeq \mathcal{O}_{M'}(M') & \text{if } p = q = 0 \text{ or } p = q = 1; \\ H^{p,q}(M) = 0 & \text{otherwise;} \end{cases}$$

and it is clear that

$$H^{p,q}(M) = \mathcal{O}_M(M) \otimes_{\mathcal{O}_X(X)} H^{p,q}(X).$$

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