Lifting Finite Groups of Outer Automorphisms of Free Groups, Surface Groups and their Abelianizations

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Dedicated to the memory of Heiner Zieselang.

Summary. - *In the present note, in part written as a survey, we discuss the possibility of lifting finite subgroups, and in particular finite cyclic subgroups, with respect to the canonical projections between automorphism and outer automorphism groups of free groups, surface groups and their abelianizations.*

1. Introduction

For a group G, denote by $Aut\,G$ its automorphism group and by $Out\,G = Aut\,G/Inn\,G$ its outer automorphism group (automorphisms modulo inner automorphisms). For a group homomorphism $\alpha : G \to H$, we say that a subgroup U of H *lifts* to G if there is an injection $\iota : U \to G$ such that $\alpha \circ \iota = id_U$.

Let F_n denote the free group of rank n, and $\pi_1 \mathcal{F}_g$ the fundamental group of a closed orientable surface $\mathcal{F}_g$ of genus g. We consider the natural projections

$$\alpha : Aut\,F_n \to Out\,F_n,$$

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$$\beta : Out\, F_n \to GL(n, \mathbb{Z}),$$

$$\gamma : Aut\, \pi_1 \mathcal{F}_g \to Out\, \pi_1 \mathcal{F}_g,$$

$$\delta : Out_+ \pi_1 \mathcal{F}_g \to Sp(2g, \mathbb{Z})$$

where β and γ are obtained by abelianization of F_n and $\pi_1 \mathcal{F}_g$; by $Sp(2g, \mathbb{Z}) \subset SL(2g, \mathbb{Z})$ we denote the symplectic group (see [6]), and by $Out_+ \pi_1 \mathcal{F}_g$ the subgroup of index two of $Out\, \pi_1 \mathcal{F}_g$ induced by orientation-preserving diffeomorphisms of the surface $\mathcal{F}_g$. It is well-known that the kernels of these four surjections are torsionfree: this is clear for α and γ, for β and δ see e.g. [18]. The main result is the following

THEOREM 1.1. *Let $n > 2$, $g \geq 2$. For each of the projections α, β, γ and δ, there exist finite cyclic subgroups of the target groups which do not lift.*

We note that $Out\, F_2 \cong GL(2, \mathbb{Z})$, and $Out_+ \pi_1 \mathcal{F}_1 \cong SL(2, \mathbb{Z}) = Sp(2, \mathbb{Z})$.

COROLLARY 1.2. *For $n > 2$ and $g \geq 2$, the projections α, β, γ and δ do not have right inverses.*

Compare also [5, Remark 3 in the introduction] and [8, Theorem 2]. We note that all target groups $Out\, F_n$, $GL(n, \mathbb{Z})$, $Out\, \pi_1 \mathcal{F}_g$ and $Sp(2g, \mathbb{Z})$ have torisonfree subgroups of finite index (they are virtually torsionfree), and there is the more subtle question if such a torsionfree subgroup of finite index does lift.

In the following sections, we consider separately the four cases, commenting also on the possibility of lifting other types of finite subgroups, in particular those of maximal order. Some of the proofs use classical results; the most technical case is that of β which we consider last.

We note that, by the positive solution of the Nielsen realization problem, every finite subgroup G of $Out\, \pi_1 \mathcal{F}_g$ can be realized by an action of G as a group of homeomorphisms of the surface $\mathcal{F}_g$; similarly, every finite subgroup G of $Out\, F_n$ can be realized by an action of G on a finite graph with fundamental group F_n (see [20], or the survey [18]). See [3] for a classification of the finite subgroups of $Out_+ \pi_1 \mathcal{F}_2$ and $Out_+ \pi_1 \mathcal{F}_3$, and [19] for the finite subgroups of $Out\, F_3$.

2. The case of $\delta : Out_+\pi_1\mathcal{F}_g \to Sp(2g, \mathbb{Z})$

By the positive solution of the Nielsen realization problem [10], any finite subgroup of $Out_+\pi_1\mathcal{F}_g$ can be realized by a finite subgroup of diffeomorphisms of the surface $\mathcal{F}_g$, and then also, choosing an appropriate structure of $\mathcal{F}_g$ as a Riemann surface, by a finite group of automorphisms of the Riemann surface. We note that the Nielsen realization problem for finite cyclic and solvable groups is a classical result, see [15] for the history of the problem (see also [17] for the solvable case).

Let $a_1, b_1, \ldots, a_g, b_g$ denote a standard symplectic basis of the first homology $\mathbb{Z}^{2g} = (\pi_1\mathcal{F}_g)_{ab}$ of the surface $\mathcal{F}_g$ (see [6, Chapter V.3]). Choose any nontrivial symplectic automorphism of finite order of the subgroup generated by a_1 and b_1 (the possible orders are 2,3,4 and 6), and extend it to a symplectic automorphism of the same order of $\mathbb{Z}^{2g}$ by the identity on the remaining generators $a_2, b_2, \ldots, a_g, b_g$. By [6, Theorem V.3.3], for $g > 2$ this symplectic automorphism is not induced by a periodic automorphism of a Riemann surface of genus g, and hence it does not lift to $Out_+\pi_1\mathcal{F}_g$.

Concerning the case $g = 2$, note that $Sp(4, \mathbb{Z})$ has a subgroup $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$, and hence a cyclic subgroup $\mathbb{Z}_3 \times \mathbb{Z}_4$ of order 12; such a subgroup does not lift to $Out_+\pi_1\mathcal{F}_2$ since, by a result of Wiman, the maximal order of an orientation-preserving diffeomorphism of $\mathcal{F}_g$ is $4g + 2$ (see [16, 4.14.27]).

2.1. Comments

The maximal possible order of finite subgroups of $Out_+\pi_1\mathcal{F}_g$ is well-known:

THEOREM 2.1 ([17]). *For $g > 1$, the order of any finite subgroup of $Out_+\pi_1\mathcal{F}_g$ is bounded above by $84(g-1)$.*

The first proof of this has been given in [17, Satz 5.3], in an equivalent algebraic formulation, as a consequence of a generalized Riemann-Hurwitz formula; of course, (2.1) follows also from the subsequent solution of the Nielsen realization problem [10] and the classical Riemann-Hurwitz formula.

We do not know the maximal order of finite subgroups of the symplectic group $Sp(2g, \mathbb{Z})$. However, $Sp(2g, \mathbb{Z})$ has a finite subgroup U of order $12^g g!$ (permutations of the pairs of generators a_i, b_i and dihedral groups of order 12 for each pair), and by Theorem 1 these subgroups do not lift. By the result of Wiman mentioned above, the maximal order of an orientation-preserving diffeomorphism of $\mathcal{F}_g$ is $4g + 2$; since for almost all values of g the group U has finite cyclic subgroups of larger orders, this gives many cyclic subgroups of $Sp(2g, \mathbb{Z})$ which do not lift to $Out_+ \pi_1 \mathcal{F}_g$.

3. The case of $\gamma : Aut\, \pi_1 \mathcal{F}_g \to Out\, \pi_1 \mathcal{F}_g$.

Since the center of $\pi_1 \mathcal{F}_g$ is trivial, every finite subgroup U of $Out\, \pi_1 \mathcal{F}_g$ determines an extension, unique up to equivalence,

$$1 \to \pi_1 \mathcal{F}_g \to E \to U \to 1$$

which is effective (no element of E acts by conjugation trivially on $\pi_1 \mathcal{F}_g$); note that the extension splits if and only if U lifts to $Aut\, \pi_1 \mathcal{F}_g$. Conversely, any such effective extension defines a subgroup U of $Out\, \pi_1 \mathcal{F}_g$.

By lifting to the universal covering, every cyclic group $\mathbb{Z}_n$ of diffeomorphisms acting freely on a surface $\mathcal{F}_g$ of genus $g > 1$ defines a torsionfree extension

$$1 \to \pi_1 \mathcal{F}_g \to E \to \mathbb{Z}_n \to 1,$$

and hence an inclusion of $\mathbb{Z}_n$ into $Out\, \pi_1 \mathcal{F}_g$ which does not lift to $Aut\, \pi_1 \mathcal{F}_g$. In fact, the group E acts on the universal covering $\mathbb{H}^2$ of $\mathcal{F}_g$ (as an extension of the universal covering group $\pi_1 \mathcal{F}_g$); if it has torsion, by Smith fixed point theory some element of prime power order must have fixed points. Examples are the covering involutions of the orientable 2-fold coverings of the nonorientable surfaces of genus $g > 2$.

Alternatively, start with a torsionfree co-compact group E of isometries of the hyperbolic plane (a surface group) and consider a normal subgroup $\pi_1 \mathcal{F}_g$ with cyclic factor group $\mathbb{Z}_n$; then $\mathbb{Z}_n$ acts freely by isometries on the surface $\mathcal{F}_g$ defined by the normal subgroup, and the induced group $\mathbb{Z}_n$ of outer automorphisms of $\pi_1 \mathcal{F}_g$ does not lift to $Aut\, \pi_1 \mathcal{F}_g$.

3.1. Comments

Each group $Out\,\pi_1\mathcal{F}_g$ has many noncyclic and nonabelian finite subgroups (see [3] for $g = 2$ and 3). On the other hand, the finite subgroups of $Aut\,\pi_1\mathcal{F}_g$ are very special, in fact one has

THEOREM 3.1. *For $g > 1$, the finite subgroups of $Aut\,\pi_1\mathcal{F}_g$ are either cyclic or dihedral.*

Proof. A finite subgroup U of $Aut\,\pi_1\mathcal{F}_g$ defines a split extension

$$1 \to \pi_1\mathcal{F}_g \to E \to U \to 1,$$

so U injects into E. By results of Nielsen (see [17, Section 2]), the extension E acts as a group of homeomorphisms of the boundary S^1 of the unit disk (the sphere at infinity of the hyperbolic plane), and it is easy to see that finite groups of homeomorphisms of S^1 are cyclic or dihedral (see also [17, Lemma 2.1]).

Alternatively, one may apply again the solution of the Nielsen realization problem. By this solution, U can be realized by a group of diffeomorphisms of the surface $\mathcal{F}_g$, and then also by a group of isometries of a suitable hyperbolic surface $\mathcal{F}_g$. Lifting to the universal covering of $\mathcal{F}_g$ (the hyperbolic plane $\mathbb{H}^2$), this realizes E as a group of isometries of $\mathbb{H}^2$, and every finite group of isometries of $\mathbb{H}^2$ is cyclic or dihedral. $\square$

4. The case of $\alpha : Aut\,F_n \to Out\,F_n$

A finite subgroup U of $Out\,F_n$ determines an effective extension

$$1 \to F_n \to E \to U \to 1,$$

and the extension splits if and only if U lifts to $Aut\,F_n$. Conversely, any such extension defines a finite subgroup U of $Out\,F_n$.

Considering extensions

$$1 \to F_n \to F_{n'} \to \mathbb{Z}_m \to 1$$

where also E is a free group $F_{n'}$ (so $(1 - n) = m(1 - n')$), for each n one easily constructs finite cyclic subgroups $\mathbb{Z}_m$ of $Out\,F_n$ which do not lift to $Aut\,F_n$ (e.g. for $n' = 2$).

4.1. Comments

The finite subgroups of maximal order of $Out\ F_n$ and $Aut\ F_n$ are given by the following

THEOREM 4.1 ([14]). *For $n > 2$, the maximal order of a finite subgroup of $Out\ F_n$ and $Aut\ F_n$ is $2^n n!$. For $n > 3$, up to conjugation there is a unique subgroup of maximal order, generated by permutations and inversions of a system of free generators.*

The finite subgroups of $Out\ F_3$ are determined in [19]. For the possible orders of finite cyclic subgroups of $Out\ F_n$ and $Aut\ F_n$, see [1] or [13]. The maximal order of finite abelian subgroups of $Out\ F_n$ and $Aut\ F_n$ is determined in [2] and equal to 2^n, for $n > 3$.

5. The case of $\beta : Out\ F_n \to GL(n, \mathbb{Z})$

Denoting by $e_1, \ldots, e_n$ the standard basis of $\mathbb{Z}^n$, we define an automorphism ϕ of order six of $\mathbb{Z}^n$ by

$$\phi(e_1) = -e_2,\ \phi(e_2) = e_1 + e_2,\ \phi(e_i) = e_i\ \text{ for }\ i \geq 3.$$

We will show that, for $n > 2$, the cyclic subgroup of $GL(n, \mathbb{Z})$ generated by ϕ does not lift to $Out\ F_n$.

Consider an extension

$$1 \to \mathbb{Z}^n \to \bar{E} \to \mathbb{Z}_6 \to 1$$

where a generator of $\mathbb{Z}_6$ induces the automorphism ϕ of $\mathbb{Z}^n$. If $\bar{E}$ is the semidirect product $\mathbb{Z}^n \ltimes \mathbb{Z}_6$ then the abelianization of $\bar{E}$ is $\mathbb{Z}_6 \times \mathbb{Z}^{n-2}$, if the extension does not split the abelianization is $\mathbb{Z}^{n-2}$, $\mathbb{Z}_2 \times \mathbb{Z}^{n-2}$ or $\mathbb{Z}_3 \times \mathbb{Z}^{n-2}$.

Suppose that ϕ can be lifted to an outer automorphism of F_n of order six, represented by an automorphism ψ of F_n. Then ψ defines an extension, unique up to equivalence,

$$1 \to F_n \to E \to \mathbb{Z}_6 \to 1,$$

and the abelianization of E is $\mathbb{Z}_m \times \mathbb{Z}^{n-2}$, $m = 1, 2, 3$ of 6.

The group E is a finite effective extension of the free group F_n. By [9], the finite extension E of the free group F_n is isomorphic to the

fundamental group $\pi_1(\Gamma, \mathcal{G})$ of a finite graph of finite groups $(\Gamma, \mathcal{G})$ (the iterated free product with amalgamation and HNN-extension over the vertex groups, amalgamated over the edge groups of a maximal tree, the HNN-generators corresponding to the edges in the complement of the chosen maximal tree). The *Euler characteristic* of $E \cong \pi_1(\Gamma, \mathcal{G})$ or of the graph of groups $(\Gamma, \mathcal{G})$ is

$$\chi(E) = \chi(\Gamma, \mathcal{G}) = \sum 1/|G_v| - \sum 1/|G_e|$$

where the sum is taken over all vertex groups G_v resp. all edge groups G_e of $(\Gamma, \mathcal{G})$. The Euler characteristic behaves multiplicatively under finite extensions, in particular in our situation we have $\chi(F_n) = 1 - n = 6\,\chi(E)$, or

$$-\chi(E) = \frac{n-1}{6}.$$

Since the kernel F_n of the surjection of E onto $\mathbb{Z}_6$ is torsionfree, the vertex and edge groups of $E = \pi_1(\Gamma, \mathcal{G})$ inject into $\mathbb{Z}_6$ and hence are cyclic groups of orders 1,2,3 or 6; in the following, we shall call an edge or vertex with associated group $\mathbb{Z}_m$ an *m-edge* or an *m-vertex*.

We shall assume that the graph of groups $(\Gamma, \mathcal{G})$ is *reduced*, i.e. has no non-closed edges such that the edge group coincides with one of the two vertex groups (such an edge can be contracted obtaining a graph of groups with fewer edges). Denote by T a maximal tree of the underlying graph Γ; then $\Gamma - T$ has exactly $n - 2$ edges (considering the abelianization of E). Note that any 6-vertex of $(\Gamma, \mathcal{G})$ contributes a direct summand $\mathbb{Z}_6$ or $\mathbb{Z}_2$ to the abelianization of E, and any 2-vertex contributes a summand $\mathbb{Z}_2$. Also, all 6-edges, 3-edges and 2-edges of $(\Gamma, \mathcal{G})$ are closed, and hence T consists only of 1-edges.

Suppose that Γ has more than one vertex. Then the contribution of T to $-\chi(E)$ is ≥ 0, and $-\chi(E) \geq (n-2)/6$ (considering only the contribution of the edges in $\Gamma - T$). Since $-\chi(E) = (n-1)/6$ it follows that $\Gamma - T$ has only 6-edges except maybe for a single 3-edge and, estimating $-\chi(E)$ from below, one easily obtains a contradiction.

Hence Γ has exactly one vertex which has to be a 6-vertex. Then E is a split extension of F_n and $\mathbb{Z}_6$, and the abelianization of E is $\mathbb{Z}_6 \times \mathbb{Z}^{n-2}$. It follows now easily that $(\Gamma, \mathcal{G})$ has no 1-edge, and either one 2-edge and $n - 3$ 6-edges, or two 3-edges and $n - 4$ 6-edges. In both cases, since the unique vertex group $\mathbb{Z}_6$ survives in

the abelianization, there is a nontrivial subgroup of the vertex group which is central in E. But then the extension E was not effective which is a contradiction.

5.1. Comments

a) The maximal order finite subgroups of $Out\ F_n$ are given by (4.1). The situation for finite subgroups of $GL(n, \mathbb{Z})$ is more complicated. It is shown in [7] that, for values on n larger than some constant, the maximum value of finite subgroups of $GL(n, \mathbb{Z})$ is again $2^n n!$, and that the maximal groups are generated by permutations and inversions of the standard generators of $\mathbb{Z}^n$. However, for $n = 2, 4, 6, 7, 8, 9$ and 10 there are subgroups of larger orders, the Weyl groups of the exceptional Lie groups of types G_2, F_4, E_6, E_7 and E_8 (of orders 12, 1152, 51840, 2903040 and 696729600). On the basis of a result of Weisfeiler, Feit gave a complete classification of the maximal order finite subgroups of $GL(n, \mathbb{Z})$ (see [7, 12]; this uses the classification of the finite simple groups).

For the maximal orders of finite cyclic subgroups of $GL(n, \mathbb{Z})$, see [12] or [13]. The maximal orders of finite abelian subgroups of $GL(n, \mathbb{Z})$ are determined in [7] and are larger than those for $Out\ F_n$ (see [2]).

b) It is more difficult to construct cyclic subgroups of prime order p of $GL(n, \mathbb{Z})$ which do not lift to $Out\ F_n$. Any integral representation of $\mathbb{Z}_p$ can be written as a direct sum of indecomposable representations which (in the language of [4, Section 1]) are either trivial, regular, cyclotomic or "exotic" (corresponding to a non-principal ideal in a cyclotomic representation $\mathbb{Z}[\lambda]$ where λ is a primitive p th root of unity and a generator of $\mathbb{Z}_p$ acts by multiplication with λ; so in this case, the representation has a nontrivial ideal class invariant in the ideal class group of $\mathbb{Z}[\lambda]$). Now any subgroup $\mathbb{Z}_p$ of $Out\ F_n$ can be induced by the action of $\mathbb{Z}_p$ on a finite graph with fundamental group F_n [20], and it follows from an argument due to Swan (see [4, Section 1]), or from [11, Theorem 15.5] that the induced representation of $\mathbb{Z}_p$ on the abelianization of the fundamental group is standard (has no exotic indecomposable summand). On the other hand, if an integer representation of $\mathbb{Z}_p$ is standard than it is easy to construct an action of $\mathbb{Z}_p$ on a finite graph (with one global fixed point) which induces

this representations (for each regular summand one takes a bouquet of p circles permuted cyclically by a generator of $\mathbb{Z}_p$, for each cyclotomic summand a graph with two vertices and $p + 1$ connecting edges permuted cyclically). Hence the following holds

THEOREM 5.1. *A cyclic subgroup $\mathbb{Z}_p$ of prime order p of $GL(n, \mathbb{Z})$ lifts to Out F_n if and only if the corresponding integer representation of $\mathbb{Z}_p$ is standard.*

Exotic integer representations of $\mathbb{Z}_p$ do not exist for $p < 23$. On the other hand, the situation for general cyclic subgroups $\mathbb{Z}_m$ of $GL(n, \mathbb{Z})$ appears to be rather complicated.

PROBLEM 5.2: Which cyclic subgroups $\mathbb{Z}_m$ of $GL(n, \mathbb{Z})$ lift to *Out F_n*?

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