Planar Convex Bodies with a Common Directed X-ray

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Summary. - *We study $\mathcal{X}(K)$, the set of convex bodies in the plane with the same directed X-ray as the convex body K. We show that $\mathcal{X}(K)$ is complete in the metrics of the uniform and L^p norms. In fact these metrics turn out to be equivalent even though $\mathcal{X}(K)$ is almost always infinite dimensional. In addition, we characterize the compact subsets of $\mathcal{X}(K)$ and determine necessary and sufficient conditions for $\mathcal{X}(K)$ to be uniformly bounded.*

1. Introduction

Generally, geometric tomography is centered around the question of what X-ray data is sufficient to uniquely determine a convex body. The most significant results of this type when the data are point or directed X-rays were given by Falconer [2], Gardner [3] and Volčič [9]. An overview of these and other results can be found in Gardner [4]. The directed X-ray of a convex body from a known source consists of the lengths of the intersection of the body with all rays emanating from the source. Continuing work in [6] we investigate a

situation where uniqueness does not obtain, that of a single directed X-ray. (If the source is interior to the body, the body is clearly uniquely determined by this data, but we always assume the source to be outside the body.) Throughout K is a convex body (compact, convex set with nonempty interior) in the plane $\mathbb{R}^2$. The boundary of K is denoted ∂K. We study topological and geometric properties of $\mathcal{X}(K)$, the set of convex bodies in the plane which have the same directed X-ray as K. In particular, we show that even though $\mathcal{X}(K)$ is (with some singular exceptions) infinite dimensional, the compact subsets of $\mathcal{X}(K)$ are precisely the closed and bounded sets in both the uniform and L^p topologies. Also, in contrast to the situation with parallel X-rays, we show that $\mathcal{X}(K)$ may be bounded and characterize when this occurs in terms of the directed X-ray. A better understanding of $\mathcal{X}(K)$ may help in developing algorithms for reconstruction and techniques for attacking questions of determination.

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2. Background

Since there will be only one X-ray source, it is convenient to choose polar coordinates with the origin at the source and assume that K is contained in the open upper half-plane $\mathcal{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. A ray $\varphi = \varphi_0$ which meets the interior of K intersects ∂K at two points $(r(\varphi_0), \varphi_0)$ (a near side point) and $(R(\varphi_0), \varphi_0)$ (a far side point) with $0 < r(\varphi_0) < R(\varphi_0)$. Rays $\varphi = \alpha$, $\varphi = \beta$ which meet K, but do not meet the interior of K, are called supporting rays. Since $K \subset \mathcal{H}$, $0 < \alpha < \beta < \pi$. The near and far side points at α are defined respectively by $\min\{r : (r, \alpha) \in K\}$ and $\max\{r : (r, \alpha) \in K\}$. The near and far side points at β are defined analogously. The set of near side points of K determines the near side function $r(\varphi)$ and the set of far side points determines the far side function $R(\varphi)$. The directed X-ray of K is $X(\varphi) = R(\varphi) - r(\varphi)$. K is uniquely determined

by any pair of the three functions r, R, X.

Throughout, the independent variable of a function will be the polar variable $\varphi \in [\alpha, \beta]$ and functions f, F, r, R, X are defined and nonnegative on that interval.

A function f is concave away from the origin (or is a near side function) if it is continuous and given any two points on the graph of f, the chord joining the two points is separated from the origin by the graph of f. f is concave toward from the origin (or is a far side function) if it is continuous and given any two points on the graph of f, the chord joining the two points separates the graph of f from the origin. The X-ray body of K, written K_X, is the body with near side $r = 0$ and far side $R = X$. Note the origin $\mathbf{O} \in K_X$ and K_X is the only element of $\mathcal{X}(K)$ that contains a point in $\partial \mathcal{H}$. (A result of Longinetti [8] shows that K_X is convex.)

It is useful to have analytical methods to determine whether a function is concave toward or away from the origin. Such are provided by the quadratic form Q introduced in [7] and operators derived from it in [6].

DEFINITION 2.1. *Suppose that* $0 < \alpha \leq \varphi_1 < \varphi_2 < \varphi_3 \leq \beta < \pi$ *and* $f : [\alpha, \beta] \to [0, \infty)$. *We define the quadratic form Q by*

$$Qf(\varphi_1, \varphi_2, \varphi_3) = f(\varphi_1)f(\varphi_2)\sin(\varphi_2 - \varphi_1)$$

$$+ f(\varphi_2)f(\varphi_3)\sin(\varphi_3 - \varphi_2) - f(\varphi_1)f(\varphi_3)\sin(\varphi_3 - \varphi_1).$$

We will repeatedly use both geometric and analytic aspects of the following result [7, 6].

LEMMA 2.2. *Let* $\alpha \leq \varphi_1 < \varphi_2 < \varphi_3 \leq \beta$ *and* $f \geq 0$ *on* $[\alpha, \beta]$. *The sum of the areas of the triangles with vertices* $(\mathbf{O}; (f(\varphi_1), \varphi_1);$ $(f(\varphi_2), \varphi_2))$ *and* $(O; (f(\varphi_2), \varphi_2); (f(\varphi_3), \varphi_3))$ *is* $\leq$ *($\geq$) the area of the triangle with vertices* $(\mathbf{O}; (f(\varphi_1), \varphi_1); (f(\varphi_3), \varphi_3))$ *if and only if* $Qf(\varphi_1, \varphi_2, \varphi_3) \leq 0$ *($Qf(\varphi_1, \varphi_2, \varphi_3) \geq 0$). Consequently a continuous, nonnegative f on $[\alpha, \beta]$ is concave away from the origin (toward the origin) if and only if $Qf(\varphi_1, \varphi_2, \varphi_3) \leq 0$ ($Qf(\varphi_1, \varphi_2, \varphi_3) \geq 0$) for all choices of $\varphi_1, \varphi_2, \varphi_3$ as designated.*

Throughout, when we state that $Qf \geq 0$ or $Qf \leq 0$ we mean that this holds for all choices of the arguments with $\alpha \leq \varphi_1 < \varphi_2 < \varphi_3 \leq \beta$.

We will need a formula for $Q(f+g)$ in terms of Qf and Qg. Due to the length of the expression we write f_j for $f(\varphi_j)$ and g_j for $g(\varphi_j)$.

$$Q(f+g) = \frac{(f_1+g_1)\sin(\varphi_2-\varphi_1)+(f_3+g_3)\sin(\varphi_3-\varphi_2)}{f_1\sin(\varphi_2-\varphi_1)+f_3\sin(\varphi_3-\varphi_2)}Q(f)$$

$$+\frac{(f_1+g_1)\sin(\varphi_2-\varphi_1)+(f_3+g_3)\sin(\varphi_3-\varphi_2)}{g_1\sin(\varphi_2-\varphi_1)+g_3\sin(\varphi_3-\varphi_2)}Q(g)$$

$$-\frac{\sin(\varphi_2-\varphi_1)\sin(\varphi_3-\varphi_2)}{(f_1\sin(\varphi_2-\varphi_1)+f_3\sin(\varphi_3-\varphi_2))}\times$$

$$\frac{\sin(\varphi_3-\varphi_1)[f_1g_3-f_3g_1]^2}{(g_1\sin(\varphi_2-\varphi_1)+g_3\sin(\varphi_3-\varphi_2))}. \tag{1}$$

We sketch the elementary and tedious derivation. From Definition 2.1

$$Q(f+g) = (f_1+g_1)(f_2+g_2)\sin(\varphi_2-\varphi_1)+(f_2+g_2)(f_3+g_3)\sin(\varphi_3-\varphi_2)$$

$$-(f_1+g_1)(f_3+g_3)\sin(\varphi_3-\varphi_1).$$

Solving for f_2 and g_2 in the definition of $Q(f)$ and $Q(g)$ gives

$$f_2+g_2 = \frac{Q(f)+f_1f_3\sin(\varphi_3-\varphi_1)}{f_1\sin(\varphi_2-\varphi_1)+f_3\sin(\varphi_3-\varphi_2)}+$$

$$\frac{Q(g)+g_1g_3\sin(\varphi_3-\varphi_1)}{g_1\sin(\varphi_2-\varphi_1)+g_3\sin(\varphi_3-\varphi_2)}.$$

Now replace f_2+g_2 in the formula for $Q(f+g)$ by the above expression. One obtains the first two term on the right hand side of (1) and the additional terms

$$[(f_1+g_1)\sin(\varphi_2-\varphi_1)+(f_3+g_3)\sin(\varphi_3-\varphi_2)]\times$$

$$\left[\frac{f_1f_3\sin(\varphi_3-\varphi_1)}{f_1\sin(\varphi_2-\varphi_1)+f_3\sin(\varphi_3-\varphi_2)}+\right.$$

$$\left. \frac{g_1 g_3 \sin(\varphi_3 - \varphi_1)}{g_1 \sin(\varphi_2 - \varphi_1) + g_3 \sin(\varphi_3 - \varphi_2)} \right] - (f_1 + g_1)(f_3 + g_3) \sin(\varphi_3 - \varphi_1).$$

It remains to show that this expression is equal to the last term in

(1). Rationalizing gives the correct denominator and with a little algebra the numerator becomes $\sin(\varphi_3 - \varphi_1)$ times

$$f_1 f_3 [g_1 \sin(\varphi_2 - \varphi_1) + g_3 \sin(\varphi_3 - \varphi_2)]^2 +$$

$$g_1 g_3 [f_1 \sin(\varphi_2 - \varphi_1) + f_3 \sin(\varphi_3 - \varphi_2)]^2 - (f_1 g_3 + f_3 g_1) \times$$

$$[(f_1 \sin(\varphi_2 - \varphi_1) + f_3 \sin(\varphi_3 - \varphi_2)) \times$$

$$(g_1 \sin(\varphi_2 - \varphi_1) + g_3 \sin(\varphi_3 - \varphi_2))]$$

$$= \sin(\varphi_2 - \varphi_1) \sin(\varphi_3 - \varphi_2)[2 f_1 f_3 g_1 g_3 - f_1^2 g_3^2 - f_3^2 f_1^2]$$

$$= -\sin(\varphi_2 - \varphi_1) \sin(\varphi_3 - \varphi_2)[f_1 g_3 - f_3 g_1]^2.$$

This completes the derivation of (1).

If we let $\varphi_2 \downarrow \varphi_1$ in the definition of Q we obtain

$$\lim_{\varphi_2 \downarrow \varphi_1} \frac{Qf}{\varphi_2 - \varphi_1} \tag{2}$$

$$= f(\varphi_1)^2 - f(\varphi_1) f(\varphi_3) \cos(\varphi_3 - \varphi_1) + f'_+(\varphi_1) f(\varphi_3) \sin(\varphi_3 - \varphi_1).$$

Similarly, letting $\varphi_2 \uparrow \varphi_3$ gives

$$\lim_{\varphi_2 \uparrow \varphi_3} \frac{Qf}{\varphi_3 - \varphi_2} \tag{3}$$

$$= f(\varphi_3)^2 - f(\varphi_1) f(\varphi_3) \cos(\varphi_3 - \varphi_1) + f'_-(\varphi_3) f(\varphi_1) \sin(\varphi_3 - \varphi_1).$$

The sign of these expressions also characterizes whether a continuous f is concave away (≤ 0) or toward (≥ 0) the origin. Here f'_+ (f'_-) denotes the right (left) hand derivative.

Another operator that we will use that characterizes the direction of concavity is the curvature operator defined by

$$\mathcal{K}f(\varphi) = f^2 + 2(f')^2 - f f'' = f^3 \left(\frac{1}{f} + \left(\frac{1}{f} \right)'' \right) \tag{4}$$

$$= \kappa_f(\varphi)(f^2 + (f')^2)^{3/2},$$

where $\kappa_f(\varphi)$ is the signed curvature of the graph of f at φ. This operator is well defined when f is C^2. In order to deal with the general situation we will need the lower and upper curvature operators;

$$\underline{\mathcal{K}}f(\varphi) = f^2 + 2(D_1 f)^2 - f\overline{D}_2 f, \tag{5}$$

and

$$\overline{\mathcal{K}}f(\varphi) = f^2 + 2(D_1 f)^2 - f\underline{D}_2 f, \tag{6}$$

where

$$D_1 f(t) = \frac{f'_+(t) + f'_-(t)}{2},$$

is the average of the left and right drivatives,

$$\underline{D}_2 f(t) = \underline{\lim}_{h \to 0+} \frac{f(t-h) + f(t+h) - 2f(t)}{h^2},$$

$$\overline{D}_2 f(t) = \overline{\lim}_{h \to 0+} \frac{f(t-h) + f(t+h) - 2f(t)}{h^2},$$

are the lower and upper second derivatives respectively. A function $f : [\alpha, \beta] \to [0, \infty)$ is concave away from the origin (toward the origin) if and only if it is continuous and $\overline{\mathcal{K}}f(\varphi) \leq 0$ ($\underline{\mathcal{K}}f(\varphi) \geq 0$) for all $\varphi \in (\alpha, \beta)$. If f is concave away from or toward the origin, then f has a second derivative at almost every point and at such points $\mathcal{K}f(\varphi) = \underline{\mathcal{K}}f(\varphi) = \overline{\mathcal{K}}f(\varphi)$. Moreover, $\overline{\mathcal{K}}f$ exists at every point, allowing the value $-\infty$ which occurs at all nonsmooth points (points at which f does not have a derivative). A similar situation holds for $\underline{\mathcal{K}}f$ when f is concave toward the origin provided $-\infty$ is replaced by $+\infty$. Thus if f is concave away from (toward) the origin $D_1 f = f'$ at points where $\overline{\mathcal{K}}f$ ($\underline{\mathcal{K}}f$) are finite. See [6] for derivations of these properties of the curvature operators.

We conclude this section with two results that will be used in the sequel. The first addresses what the conditions $Qf \leq 0$, $Qf \geq 0$ imply without the assumption that f is continuous.

LEMMA 2.3.   *1. Suppose that $f \geq 0$ and $Qf \leq 0$ on $[\alpha, \beta]$. If $f > 0$ at two points in $[\alpha, \beta]$ or $(\alpha, \beta]$, then $f > 0$ on $[\alpha, \beta]$ and is continuous on (α, β). In any event, for any $\varphi_0 \in [\alpha, \beta]$, $\lim_{\varphi \to \varphi_0} f(\varphi)$ exists and is $\leq f(\varphi_0)$.*

2. *Suppose $F \geq 0$ and $QF \geq 0$ on $[\alpha, \beta]$. If $F > 0$ on (α, β), then F is continuous on (α, β) and $\lim_{\varphi \downarrow \alpha} F(\varphi) \geq F(\alpha)$ and $\lim_{\varphi \uparrow \beta} F(\varphi) \geq F(\beta)$. (One of the limits may be $+\infty$.)*

Proof. We prove 1. (The proof of 2 is similar, although a little care needs to be taken to ensure that the limits are finite in (α, β).) Suppose that $\alpha \leq \varphi_1 < \varphi_2 < \varphi_3 \leq \beta$ with $f(\varphi_j) > 0$ for $j = 1, 2$. If $f(\varphi_3) = 0$, then $Qf(\varphi_1, \varphi_2, \varphi_3) = f(\varphi_1)f(\varphi_2)\sin(\varphi_2 - \varphi_1) > 0$ which is a contradiction. Hence $f > 0$ on $[\varphi_2, \beta]$. The same argument shows that if $\varphi \in [\alpha, \varphi_2)$ and $f(\varphi) = 0$, then $Qf(\varphi, \varphi_2, \beta) > 0$. Hence $f > 0$ on $[\alpha, \beta]$. This shows that if f vanishes at any point, then there are at most two points where f is nonzero. So, if f vanishes at any point, $\lim_{\varphi \to \varphi_0} f(\varphi) = 0 \leq f(\varphi_0)$. Thus we need only consider the case where $f > 0$ on $[\alpha, \beta]$. In fact, repeating the argument above we see that this implies $\inf f > 0$, so f is bounded below by some positive number.

Next we show that that the left and right hand limits exist at every point. From Lemma 2.2 $f(\varphi) \leq \max\{f(\alpha), f(\beta)\}$, so f is bounded. Let $f^* = \underline{\lim}_{\varphi \downarrow \varphi_0} f(\varphi)$ and $f^{**} = \overline{\lim}_{\varphi \downarrow \varphi_0} f(\varphi)$. Let $\varphi_j \downarrow \varphi_0$, $\psi_j \downarrow \varphi_0$ with $\varphi_j < \psi_j$ such that $f(\varphi_j) \to f^*$ and $f(\psi_j) \to f^{**}$. Then for all $j > 1$

$$0 \geq Qf(\varphi_j, \psi_j, \psi_1) = f(\varphi_j)f(\psi_j)\sin(\psi_j - \varphi_j)$$

$$+f(\psi_j)f(\psi_1)\sin(\psi_1 - \psi_j) - f(\psi_1)f(\varphi_j)\sin(\psi_1 - \psi_j).$$

Letting $j \to \infty$ we obtain

$$0 \geq f(\psi_1)(f^{**} - f^*)\sin(\psi_1 - \varphi_0).$$

Since $f(\psi_1) \neq 0$, we have

$$\overline{\lim}_{\varphi \downarrow \varphi_0} f(\varphi) = f^{**} \leq f^* = \underline{\lim}_{\varphi \downarrow \varphi_0} f(\varphi).$$

Thus equality holds throughout and the right hand limit exists. If we replace φ_j by φ_0 and repeat the same argument, we find $\lim_{\varphi \downarrow \varphi_0} f(\varphi) \leq f(\varphi_0)$. The proof for the left hand limit is essentially the same. It remains to show that f is continuous on (α, β).

Fix $\varphi_0 \in (\alpha, \beta)$ and define $f(\varphi_0+) = \lim_{\varphi \downarrow \varphi_0} f(\varphi)$ and $f(\varphi_0-) = \lim_{\varphi \uparrow \varphi_0} f(\varphi)$. Let $\varphi_1 = \varphi_0 - \gamma$, $\varphi_2 = \varphi_0$ and $\varphi_3 = \varphi_0 + \gamma$ where

$\gamma > 0$ is small. Since $Qf \leq 0$, $0 \geq \lim_{\gamma \to 0} \frac{Qf(\varphi_1, \varphi_2, \varphi_3)}{\sin \gamma}$. On the other hand

$$\lim_{\gamma \to 0} \frac{Qf(\varphi_1, \varphi_2, \varphi_3)}{\sin \gamma} = f(\varphi_0)f(\varphi_0+) + f(\varphi_0-)f(\varphi_0) - 2f(\varphi_0-)f(\varphi_0+)$$

$$= f(\varphi_0+)(f(\varphi_0) - f(\varphi_0-) + f(\varphi_0-)(f(\varphi_0) - f(\varphi_0+)) \geq 0,$$

since $f(\varphi_0) \geq f(\varphi_0+)$ and $f(\varphi_0) \geq f(\varphi_0-)$. Since $f(\varphi_0+) > 0$ and $f(\varphi_0-) > 0$, we must have $f(\varphi_0) = f(\varphi_0+) = f(\varphi_0-)$, completing the proof. □

The last result of the section summarizes some topological properties of $\mathcal{X}(K)$ that we will use. See Theorem 5.1 [6].

THEOREM 2.4. *Let $K \subset \mathcal{H}$ be a convex body with near side r, far side R and directed X-ray $X = R - r$. Then for $0 \leq t \leq 1$, the body K_t with near side tr and far side $tr + X$ is convex. Hence $\mathcal{X}(K)$ is star-shaped with respect to the X-ray body K_X. Thus $\mathcal{X}(K)$ is both path connected and simply connected.*

3. Metrics and Compactness.

We first study $\mathcal{X}(K)$ under the topology of uniform convergence.

DEFINITION 3.1. *Let L, $M \in \mathcal{X}(K)$. The distance between L and M is $d_\infty(L, M) = \|r - s\|_\infty = \max_{\varphi \in [\alpha, \beta]} |r(\varphi) - s(\varphi)|$, where r, s are the near side functions of L, M respectively.*

Since the near side of the X-ray body K_X is the zero function, $d_\infty(K_X, L) = \|r\|_\infty = \max_{\varphi \in [\alpha, \beta]} r(\varphi) = \max\{r(\alpha), r(\beta)\}$. A subset S of $\mathcal{X}(K)$ is bounded (more precisely uniformly bounded) if there exists a finite $A > 0$ such that if r is the near side function of a body in S, then $\|r\|_\infty \leq A$.

DEFINITION 3.2. *Let $\mathcal{F}$ be a family of continuous real valued functions defined on an interval I. $\mathcal{F}$ is equicontinuous on I if for each $\epsilon > 0$ there exists $\delta > 0$ such that $g \in \mathcal{F}$, x, $y \in I$ and $|x - y| < \delta$ implies $|g(x) - g(y)| < \epsilon$. $\mathcal{F}$ is equicontinuous at a point $x_0 \in I$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $g \in \mathcal{F}$, $x \in I$ and $|x - x_0| < \delta$ implies $|g(x) - g(x_0)| < \epsilon$.*

We will say that a subset of $\mathcal{X}(K)$ is equicontinuous on an interval (at a point) when the family of near side functions, associated to the bodies in the set, is equicontinuous according to the above definition.

The main result of this section is the following.

THEOREM 3.3. *The space $\mathcal{X}(K)$ is complete in the metric of uniform convergence. Moreover closed, bounded subsets of $\mathcal{X}(K)$ are compact and consequently bounded subsets are equicontinuous.*

(By compact we mean sequentially or countably compact. That is, a subset A of a metric space M is compact if every sequence from A contains a subsequence that converges to a point in A. As is well known, in a metric space this is equivalent to the open covering definition of compactness. See [5] for example .)

The completeness is easy. Indeed, if $\{r_j\}$ is a Cauchy sequence of near side functions of convex bodies in $\mathcal{X}(K)$, then by the completeness of the continuous functions on a compact interval in the uniform topology, $\{r_j\}$ converges uniformly to a continuous function r and $Qr \leq 0$ since each Qr_j is. So r is concave away from the origin. Similarly, $R_j = r_j + X$ converges uniformly to the continuous function $R = r + X$. $QR \geq 0$ since each QR_j is, and hence R is concave toward the origin. Thus the body with near side function r and far side function R is convex with directed X-ray X.

The characterization of the compact sets requires two lemmas.

LEMMA 3.4. *Bounded subsets of $\mathcal{X}(K)$ are equicontinuous on compact subsets of (α, β).*

Proof. Fix α_0, β_0 such that $\alpha < \alpha_0 < \beta_0 < \beta$ and let r be a near side function of a convex body in $\mathcal{X}(K)$. In light of (1), (2) and the fact that $Q(r + X) \geq 0$, evaluating $\lim_{\varphi_2 \downarrow \varphi_1} \frac{Q(r+X)}{\varphi_2 - \varphi_1}$ gives

$$0 \leq \frac{r_3 + X_3}{r_3}[r_1^2 - r_1 r_3 \cos(\varphi_3 - \varphi_1) + r_3 r'_{1+} \sin(\varphi_3 - \varphi_1)]$$

$$+\frac{r_3 + X_3}{X_3}[X_1^2 - X_1 X_3 \cos(\varphi_3 - \varphi_1) + X_3 X'_{1+} \sin(\varphi_3 - \varphi_1)]$$

$$-\frac{(r_1 X_3 - r_3 X_1)^2}{r_3 X_3},$$

where we again use the abbreviated notation $r_j = r(\varphi_j)$ etc. Since $Qr \leq 0$, (2) gives $r_1^2 - r_1 r_3 \cos(\varphi_3 - \varphi_1) + r_3 r'_{1+} \sin(\varphi_3 - \varphi_1) \leq 0$, so

$$|r_1 X_3 - r_3 X_1|$$

$$\leq \sqrt{r_3(r_3 + X_3)}\sqrt{X_1^2 - X_1 X_3 \cos(\varphi_3 - \varphi_1) + X_3 X'_{1+} \sin(\varphi_3 - \varphi_1)}.$$

Now $r_1 X_3 - r_3 X_1 = X_3(r_1 - r_3) + r_3(X_3 - X_1)$ and hence

$$X_3|r_3 - r_1| - r_3|X_3 - X_1| \leq |r_1 X_3 - r_3 X_1|.$$

Combining these inequalities we obtain

$$|r_1 - r_3| \leq \frac{r_3}{X_3}|X_3 - X_1| +$$

$$\frac{\sqrt{r_3(r_3 + X_3)}}{X_3}\sqrt{X_1^2 - X_1 X_3 \cos(\varphi_3 - \varphi_1) + X_3 X'_{1+} \sin(\varphi_3 - \varphi_1)}.$$

If we restrict X to $[\alpha_0, \beta_0]$, then $X_3 \geq a > 0$ for some fixed positive number a and $|X'_{1+}|$ is bounded by [6] Lemma 2.1.5. Hence if $\|r\|_\infty \leq M$, then given $\epsilon > 0$, there exists $\delta > 0$ such that $\varphi_1, \varphi_3 \in [\alpha_0, \beta_0]$ and $\varphi_3 - \varphi_1 < \delta$ implies $|r_1 - r_3| < \epsilon$. So, bounded subsets of $\mathcal{X}(K)$ are equicontinuous on $[\alpha_0, \beta_0]$. □

The Arzela Theorem, [5] p. 102, characterizes the compact subsets of the continuous functions on a compact interval (in the uniform topology) as precisely those sets which are closed, (uniformly) bounded and equicontinuous. We need to look more closely at near side functions of bodies in $\mathcal{X}(K)$ near the endpoints of the interval.

LEMMA 3.5. *Let $\{r_j\}$ be a sequence of near side functions of bodies in $\mathcal{X}(K)$. If for each $\varphi \in [\alpha, \beta]$, $\{r_j(\varphi)\}$ converges to a finite number, then $\{r_j\}$ converges to the near side function, r, of a body in $\mathcal{X}(K)$. Moreover, $\{r_j\}$ is equicontinuous on $[\alpha, \beta]$ and $r_j \to r$ uniformly.*

Proof. Define r on $[\alpha, \beta]$ by $r(\varphi) = \lim r_j(\varphi)$. The sequences $\{r_j(\alpha)\}$ and $\{r_j(\beta)\}$ converge and hence are bounded. Since r_j is concave away from the origin, $\|r_j\|_\infty = \max(r_j(\alpha), r_j(\beta))$. Hence $\{r_j\}$ is bounded. Also, since $Qr_j \leq 0$ and $Q(r_j + X) \geq 0$, the pointwise convergence guarantees that $Qr \leq 0$ and $Q(r + X) \geq 0$. Hence, if r

is continuous on $[\alpha, \beta]$, then r is the near side function of a body in $\mathcal{X}(K)$.

From the previous lemma, $\{r_j\}$ is equicontinuous on compact subsets of (α, β). By the Arzela theorem and a standard diagonalization process, there is a subsequence of $\{r_j\}$ that converges uniformly on all compact subsets of (α, β). Hence r is continuous on (α, β). We need to show that r is continuous at the endpoints of the interval. From Lemma 2.3.1 $r(\alpha+) \leq r(\alpha)$. It remains to show equality. If not, then $r(\alpha+) < r(\alpha)$. Let $R = r + X$ and $R_j = r_j + X$. Each $QR_j \geq 0$ and R_j converges pointwise to $R = r + X \geq X > 0$ on (α, β). Hence we have $R(\alpha+) = X(\alpha) + r(\alpha+) < X(\alpha) + r(\alpha) = R(\alpha)$, which contradicts Lemma 2.3.2. The proof is identical at β.

Now we establish the equicontinuity of $\{r_j\}$ on $[\alpha, \beta]$. Since this sequence is equicontinuous on all compact subsets of (α, β), a simple triangle inequality argument shows that it suffices to establish equicontinuity at α and β. The proofs being identical, we show the sequence is equicontinuous at α. For purposes of contradicton suppose that $\{r_j\}$ is not equicontinuous at α. Then there exists an $\epsilon > 0$ and a sequence $\varphi_j \downarrow \alpha$ such that

$$|r_j(\varphi_j) - r_j(\alpha)| > \epsilon.$$

Hence for j sufficiently large, either $r_j(\varphi_j) > r(\alpha) + \epsilon/2$ or $r(\alpha) > r_j(\varphi_j) + \epsilon/2$. We show that each leads to a contradiction.

Suppose that $r_j(\varphi_j) > r(\alpha) + \epsilon/2$ for j sufficiently large. Then

$$Qr_j(\alpha, \varphi_j, \varphi_1) = r_j(\alpha)r_j(\varphi_j)\sin(\varphi_j - \alpha)$$

$$+r_j(\varphi_1)[r_j(\varphi_j)\sin(\varphi_1 - \varphi_j) - r_j(\alpha)\sin(\varphi_1 - \alpha)]$$

$$= r_j(\alpha)r_j(\varphi_j)\sin(\varphi_j - \alpha) + r_j(\varphi_1)\times$$

$$[(r_j(\varphi_j) - r_j(\alpha))\sin(\varphi_1 - \varphi_j) - r_j(\alpha)(\sin(\varphi_1 - \alpha) - \sin(\varphi_1 - \varphi_j)] > 0,$$

unless $r_j(\alpha) = r_j(\varphi_1) = 0$. Lemma 2.3 and the fact that $r_j(\varphi_j) > r(\alpha) + \epsilon/2 > 0$ show this is impossible and we have a contradiction.

It remains to consider the case $r(\alpha) > r_j(\varphi_j) + \frac{\epsilon}{2}$ for j sufficiently large. Note $R_j = r_j + X$ and from the previous inequality and the continuity of X

$$R_j(\alpha) - R_j(\varphi_j) = r_j(\alpha) - r_j(\varphi_j) + X(\alpha) - X(\varphi_j) > \frac{\epsilon}{4}$$

for j sufficiently large. Proceeding as before with r_j replaced by R_j,

$$0 \leq QR_j(\alpha, \varphi_j, \varphi_1) = R_j(\varphi_j)R_j(\alpha)\sin(\varphi_j - \alpha)$$
$$+R_j(\varphi_j)R_j(\varphi_1)(\sin(\varphi_1 - \varphi_j) - \sin(\varphi_1 - \alpha))$$
$$-R_j(\varphi_1)\sin(\varphi_1 - \alpha)(R_j(\alpha) - R_j(\varphi_j)).$$

Again as $j \to \infty$ the first two summands go to zero, and hence for j sufficiently large

$$0 \leq QR_j(\alpha, \varphi_j, \varphi_1) < -\epsilon R_j(\varphi_1)\sin(\varphi_1 - \alpha)/8.$$

This is a contradiction. Hence $\{r_j\}$ is equicontinuous at α.

To see that the convergence $r_j \to r$ is uniform, note that by equicontinuity and the Arzela theorem every subsequence of $\{r_j\}$ has a uniformly convergent subsequence. Since $\{r_j\}$ converges pointwise to r, each uniformly convergent subsequence must also converge to r. Hence $\{r_j\}$ itself must converge uniformly to r.

□

Now, the proof of the remainder of the main theorem is easy.

Proof of Theorem 3.3. Let $\{r_j\}$ be a bounded sequence of near side functions of bodies in $\mathcal{X}(K)$. By Lemma 3.4 $\{r_j\}$ is equicontinuous on compact subsets of (α, β) and thus contains a subsequence that converges uniformly on compact subsets of (α, β) to a function r that is continuous on (α, β). Choosing further subsequences if necessary, we obtain a subsequence of the original that converges also at α and β. By the previous lemma, this subsequence converges uniformly to a function that is the near side of a convex body in $\mathcal{X}(K)$. Hence closed, bounded subsets of $\mathcal{X}(K)$ are compact.

Further, from the Arzela Theorem the compact subsets are precisely those sets which are closed, bounded and equicontinuous. Hence bounded subsets of $\mathcal{X}(K)$ are equicontinuous completing the proof.

Before discussing the L^p topologies we will need the following.

LEMMA 3.6. *Fix $\epsilon > 0$ and define $C_\epsilon = \inf\left[\frac{\min r(\varphi)}{\|r\|_\infty}\right]$ where the inf is taken over all $L \in \mathcal{X}(K)$ whose near side function satisfies $\|r\|_\infty \geq \epsilon$. Then $C_\epsilon > 0$ and is an increasing function of ϵ. Morevoer, if $\inf_{\varphi \in (\alpha,\beta)} \underline{K}X > 0$, then $C_\epsilon \to 0$ as $\epsilon \to 0$.*

Proof. First we show that for each $\epsilon > 0$, $C_\epsilon > 0$. Suppose not. Then there exists a sequence of near side functions $\{r_j\}$ of bodies in $\mathcal{X}(K)$ such that $\|r_j\|_\infty \geq \epsilon$ and $\frac{\min r_j(\varphi)}{\|r_j\|_\infty} \to 0$ as $j \to \infty$. Consider the functions $s_j = \frac{\epsilon r_j}{\|r_j\|_\infty}$. $\|s_j\|_\infty = \epsilon$ and from Theorem 2.4 each s_j is a nearside function of a body in $\mathcal{X}(K)$. Thus $\{s_j\}$ contains a subsequence that converges uniformly to a near side function s of a body in $\mathcal{X}(K)$. By construction $\|s\|_\infty = \epsilon > 0$ and $\min s = 0$ which contradicts Lemma 2.3.1. Hence $C_\epsilon > 0$. It is clear from the definition that C_ϵ is an increasing function of ϵ. Suppose now that $\inf \underline{K}X > 0$. By Theorem 7.3 [6] for any line (segment) ℓ that intersects both supporting rays, the body with near side $t\ell$ and far side $t\ell + X$ is in $\mathcal{X}(K)$ when $t > 0$ is sufficiently small. In particular, for $\alpha > \psi > \beta - \pi$ consider the line $\ell(\varphi) = \frac{\csc(\psi-\varphi)}{\csc(\psi-\alpha)}$. The ratio $\frac{\ell(\beta)}{\ell(\alpha)} = \frac{\csc(\psi-\beta)}{\csc(\psi-\alpha)}$ goes to 0 as $\psi \to \alpha$. $\square$

The lemma has an immediate corollary that will be useful when discussing boundedness of $\mathcal{X}(K)$.

COROLLARY 3.7. *Let $\{r_j\}$ be a sequence of near side functions of bodies in $\mathcal{X}(K)$. If $\|r_j\|_\infty \to \infty$ as $j \to \infty$, then $\min r_j \to \infty$ as $j \to \infty$.*

The L^p metrics, $1 \leq p < \infty$ on $\mathcal{X}(K)$ are defined in the natural way. If r, s are near side functions of bodies L, M in $\mathcal{X}(K)$, then the distance between L and M is

$$d_p(L, M) = \|r - s\|_p = \left[\int_\alpha^\beta |r(\varphi) - s(\varphi)|^p d\varphi \right]^{1/p}.$$

Since the area under a polar curve r on $[\alpha, \beta]$ is $\frac{1}{2} \int_\alpha^\beta r^2 d\varphi$, $\|r\|_2$ has a geometric interpretation. The case $p = 2$ of Theorem 3.9 below shows that $\mathcal{X}(K)$ is isomorphic to a closed subset of the Hilbert space $L^2[\alpha, \beta]$.

LEMMA 3.8. *Let r be the near side function of a body in $\mathcal{X}(K)$. Then for $1 \leq p < \infty$*

  1. $\|r\|_p \leq (\beta - \alpha)^{1/p} \|r\|_\infty$;

2. $\|r\|_\infty < \epsilon$ or $\|r\|_\infty \leq \|r\|_p/(C_\epsilon(\beta - \alpha)^{1/p})$,

where C_ϵ is the constant defined in Lemma 3.6. Consequently, if $S \subset \mathcal{X}(K)$ is bounded in any of the the metrics d_p, $1 \leq p < \infty$, then S is bounded in the metric d_∞.

Proof. The first inequality is standard. There is nothing to prove in the second if $\|r\|_\infty < \epsilon$, so assume that $\|r\|_\infty \geq \epsilon$. Then we have

$$\|r\|_p^p = \int_\alpha^\beta (r(\varphi))^p d\varphi \geq (\beta - \alpha)(\min r(\varphi))^p \geq (\beta - \alpha)(C_\epsilon\|r\|_\infty)^p.$$

$\square$

THEOREM 3.9. $\mathcal{X}(K)$ *is complete in the metrics* d_p, $1 \leq p \leq \infty$ *and the compact sets in any of these metric spaces are precisely the closed bounded sets.*

Proof. We need only establish the result when $1 \leq p < \infty$. Both the completeness and characterization of compact sets follow easily from the lemma and the fact that closed bounded sets are compact in the metric d_∞. Indeed from the lemma any sequence that is bounded in d_p is also bounded in d_∞, and has a uniformly convergent sequence (by compactness in d_∞) which must also converge in d_p. $\square$

The following is immediate.

COROLLARY 3.10. *For* $1 \leq p \leq \infty$, *the metric spaces* $\{\mathcal{X}(K),\ d_p\}$ *are equivalent, that is they are homeomorphic as topological spaces.*

4. Perturbation and Dimension

In this section we study one way to perturb a body in $\mathcal{X}(K)$ and still remain in $\mathcal{X}(K)$. One of the consequences is that for almost all convex bodies $K \subset \mathcal{H}$, $\mathcal{X}(K)$ is infinite dimensional. In fact it is locally infinite dimensional; that is if $L \in \mathcal{X}(K)$, U is a neighborhood of L in $\mathcal{X}(K)$ and n is a natural number, U contains a subset homeomorphic to an open subset of $\mathbb{R}^n$. The only known exception is when the convex body is a parallel wedge W (a quadrilateral whose near and far sides are parallel). In this case $\mathcal{X}(W)$ is homeomorphic to a half-line. There are functions R which are concave toward the origin on

$[\alpha, \beta]$ which are not directed X-rays of convex bodies in $\mathcal{H}$. If L were a body with near side $r = 0$ and far side R, then $\mathcal{X}(L) = \{L\}$. (See [6] Examples 6.6 - 6.8.) It appears that $\mathcal{X}(K)$ may have dimension $0, 1$ or ∞, with the former two exceptional.

The main tool in the next two sections will be the operators $\mathcal{K}$, $\underline{\mathcal{K}}$ and $\overline{\mathcal{K}}$ defined in (4), (5) and (6). We will repeatedly use inequalities that these operators satisfy. The following were derived in [6] Lemma 3.8.

LEMMA 4.1.   *1. $\mathcal{K}(tf) = t^2 \mathcal{K}$, $\underline{\mathcal{K}}(tf) = t^2 \underline{\mathcal{K}} f$, $\overline{\mathcal{K}}(tf) = t^2 \overline{\mathcal{K}} f$.*

2. $\mathcal{K}(f + g) = \frac{f+g}{f} \mathcal{K} f + \frac{f+g}{g} \mathcal{K} g - 2fg \left[\frac{f'}{f} - \frac{g'}{g} \right]^2$.

3. $\frac{f+g}{f} \underline{\mathcal{K}} f + \frac{f+g}{g} \underline{\mathcal{K}} g - 2fg \left[\frac{D_1 f}{f} - \frac{D_1 g}{g} \right]^2 \le \underline{\mathcal{K}}(f+g) \le$

$\le \frac{f+g}{f} \underline{\mathcal{K}} f + \frac{f+g}{g} \overline{\mathcal{K}} g - 2fg \left[\frac{D_1 f}{f} - \frac{D_1 g}{g} \right]^2 \le \overline{\mathcal{K}}(f+g) \le$

$\frac{f+g}{f} \overline{\mathcal{K}} f + \frac{f+g}{g} \overline{\mathcal{K}} g - 2fg \left[\frac{D_1 f}{f} - \frac{D_1 g}{g} \right]^2$.

4. $\frac{f-g}{f} \underline{\mathcal{K}} f - \frac{f-g}{g} \overline{\mathcal{K}} g + 2fg \left[\frac{D_1 f}{f} - \frac{D_1 g}{g} \right]^2 \le \underline{\mathcal{K}}(f-g) \le$

$\le \frac{f-g}{f} \underline{\mathcal{K}} f - \frac{f-g}{g} \underline{\mathcal{K}} g + 2fg \left[\frac{D_1 f}{f} - \frac{D_1 g}{g} \right]^2$.

5. $\frac{f-g}{f} \overline{\mathcal{K}} f - \frac{f-g}{g} \overline{\mathcal{K}} g + 2fg \left[\frac{D_1 f}{f} - \frac{D_1 g}{g} \right]^2 \le \overline{\mathcal{K}}(f-g) \le$

$\le \frac{f-g}{f} \overline{\mathcal{K}} f - \frac{f-g}{g} \underline{\mathcal{K}} g + 2fg \left[\frac{D_1 f}{f} - \frac{D_1 g}{g} \right]^2$.

It is assumed that no expression is of the form $\infty - \infty$ and that all of the functions f, g, $f + g$, $f - g$ are positive where they occur.

In order to avoid repeatedly stating the same hypotheses (unless otherwise stated) r, R are the near and far side functions of a convex body $K \subset \mathcal{H}$ and $X = R - r$.

The following situation will be exceptional in our construction.

DEFINITION 4.2. *∂K contains parallel points of zero curvature (with respect to the source) at φ_0, $\alpha < \varphi_0 < \beta$ if $\overline{\mathcal{K}} r(\varphi_0) = \underline{\mathcal{K}} R(\varphi_0) = 0$ and the tangent lines to ∂K at φ_0 are parallel. More generally, we include the case where there exists a sequence $\{\varphi_j\}$, $\varphi_j \to \varphi_0$ such that*

$$\lim \overline{\mathcal{K}} r(\varphi_j) = \lim \underline{\mathcal{K}} R(\varphi_j) = 0,$$

and the tangent lines to ∂K at φ_0 are parallel. If the above holds with $\varphi_j \to \varphi_0$ replaced by $\varphi_j \downarrow \varphi_0$ ($\varphi_j \uparrow \varphi_0$) and tangent line replaced by right (respectively left) tangent line, we say that K has right (respectively left) parallel points of zero curvature at φ_0.

LEMMA 4.3. *If $\inf_{\varphi \in (\alpha,\beta)} \underline{\mathcal{K}}X = 0$, then ∂K has parallel (possibly left or right) points of zero curvature at some point φ_0. Conversely, if ∂K has parallel points of zero curvature at φ_0, $\lim_{\varphi \to \varphi_0} \underline{\mathcal{K}}R(\varphi) = 0$ and $\lim_{\varphi \to \varphi_0} \overline{\mathcal{K}}r(\varphi) = 0$, then $\inf_{\varphi \in (\alpha,\beta)} \underline{\mathcal{K}}X = 0$. (The limits involving r, R may be replaced by left or right hand limits and the conclusion remains valid with obvious alterations.)*

Proof. First suppose that $\underline{\mathcal{K}}X(\varphi_0) = 0$. Then Lemma 4.1.4 with $f - g = R - r$ gives

$$0 = \underline{\mathcal{K}}X(\varphi_0) \geq \frac{X}{R}\underline{\mathcal{K}}R(\varphi_0) - \frac{X}{r}\overline{\mathcal{K}}r(\varphi_0) + 2rR\left[\frac{D_1R(\varphi_0)}{R(\varphi_0)} - \frac{D_1r(\varphi_0)}{r(\varphi_0)}\right]^2.$$

All terms on the right are nonnegative. Hence all must vanish. Since $\frac{R'(\varphi)}{R(\varphi)} - \frac{r'(\varphi)}{r(\varphi)} = \cot(\omega - \varphi) - \cot(\psi - \varphi)$ where ω, ψ are the angles of inclination of the tangent line to the graphs of R, r at φ ([6] formula (1)), the vanishing of the last term implies that the tangent lines to ∂K at φ_0 are parallel. However if $\varphi_0 = \alpha$ or β and the tangent lines are parallel, then the expressions $\frac{D_1R(\varphi_0)}{R(\varphi_0)}$ and $\frac{D_1r(\varphi_0)}{r(\varphi_0)}$ would be infinite and opposite in sign so their difference could not vanish. Suppose that $\inf \underline{\mathcal{K}}X = 0$, but $\underline{\mathcal{K}}X(\varphi) > 0$ for all φ. Then choose a convergent sequence, say $\varphi_j \to \varphi_0$, such that $\underline{\mathcal{K}}X(\varphi_j) \to 0$. Applying Lemma 4.1.4 to $\underline{\mathcal{K}}X(\varphi_j)$ and taking the limit again gives the desired result. (If the sequence φ_j approached φ_0 from the left or right, then we would have left or right parallel points of zero curvature.) For the converse, suppose $\lim_{\varphi \to \varphi_0} \underline{\mathcal{K}}R(\varphi) = 0$ and $\lim_{\varphi \to \varphi_0} \overline{\mathcal{K}}r(\varphi) = 0$. Since R'', r'' exists almost everywhere, we may choose a sequence of points $\varphi_j \to \varphi_0$ such that $\mathcal{K}R(\varphi_j) = \underline{\mathcal{K}}R(\varphi_j) = \overline{\mathcal{K}}R(\varphi_j) \to 0$ and $\mathcal{K}r(\varphi_j) = \underline{\mathcal{K}}r(\varphi_j) = \overline{\mathcal{K}}r(\varphi_j) \to 0$. Then, Lemma 4.1.2 gives

$$\mathcal{K}X(\varphi_j) = \frac{X}{R}\mathcal{K}R(\varphi_j) - \frac{X}{r}\mathcal{K}r(\varphi_j) + 2rR\left[\frac{R'(\varphi_j)}{R(\varphi_j)} - \frac{r'(\varphi_j)}{r(\varphi_j)}\right]^2 \to 0$$

as $j \to \infty$. $\qquad\square$

LEMMA 4.4. *Suppose that* $\inf_{\varphi \in (\alpha, \beta)} \underline{\mathcal{K}} R(\varphi) = 0$. *Then either* $\inf_{\varphi \in (\alpha, \beta)} \underline{\mathcal{K}}(tr + X) > 0$ *for all* t, $0 \le t < 1$, *or* ∂K *contains parallel points (possibly left or right) of zero curvature.*

Proof. Since $X = R - r$, $tr + X = R - (1 - t)r$ and from Lemma 4.1.1 and 4.1.4

$$\underline{\mathcal{K}}(tr + X) = \underline{\mathcal{K}}(R - (1 - t)r) \ge \frac{R - (1 - t)r}{R} \underline{\mathcal{K}} R$$

$$-\frac{(1 - t)(R - (1 - t)r)}{r} \overline{\mathcal{K}} r + 2(1 - t)rR \left[\frac{D_1 R}{R} - \frac{D_1 r}{r} \right]^2,$$

where all summands on the right are nonnegative. If $\inf \underline{\mathcal{K}}(tr + X)$ vanishes for some t, $0 \le t < 1$, then the inf of each of the summands must vanish simultaneously and there exists a sequence $\varphi_j \to \varphi_0$, $\alpha \le \varphi_0 \le \beta$ such that

$$\lim_{\varphi_j \to \varphi_0} \underline{\mathcal{K}} R(\varphi_j) = \lim_{\varphi_j \to \varphi_0} \overline{\mathcal{K}} r(\varphi_j) = \lim_{\varphi_j \to \varphi_0} \left[\frac{D_1 R}{R} - \frac{D_1 r}{r} \right]^2 = 0.$$

The last identity cannot hold at α or β. Thus $\alpha < \varphi_0 < \beta$. Hence ∂K has parallel points of zero curvature at φ_0. The proof remains the same if the limit is replaced by a left or right hand limit. $\square$

LEMMA 4.5. *For each* t, $0 \le t < 1$, *there is a convex body* L_t *in* $\mathcal{H}$ *with directed X-ray* $tr + X$. *Consequently, if* F *is concave toward the origin and* > 0 *on* $[\alpha, \beta]$, F *can be uniformly approximated by functions that are directed X-rays of convex bodies in* $\mathcal{H}$. *If in addition* $F'_+ \to -\infty$ *at* α *(or* $F'_- \to +\infty$ *at* β*) and* $r \in C^2[\alpha, \beta]$*, then there exists a constant* $C > 0$*, independent of* t, $0 \le t \le 1$*, such that*

$$\overline{D}_2(F + tr) \le C[D_1(F + tr)]^2 \text{ for } \varphi \text{ sufficiently close to } \alpha \text{ (or } \beta).$$

Proof. By Theorem 2.4 the body K_t with near side tr and far side $tr + X$ is convex for $0 \le t \le 1$. Take $L_0 = K_X$, the X-ray body of K. Fix t, $0 < t < 1$ and choose t_1 so that $0 < t_1 < 1 - t$. Let L_t be the body with near side $t_1 r$ and farside $(t + t_1)r + X$. Since $0 < t + t_1 < 1$, the function $(t + t_1)r + X$ is concave toward the origin and L_t is a convex body in $\mathcal{H}$ with directed X-ray $(t + t_1)r + X - t_1 r = tr + X$.

If a function F is concave toward the origin and positive on $[\alpha, \beta]$, we can construct a convex body $L \subset \mathcal{H}$ whose far side is F. If f is the near side of L, then the bodies L_t with near side tf and far side $tf + (F - f)$, $0 \leq t < 1$ provide the desired uniform approximation.

Lemma 6.3 [6] establishes the desired inequality when $t = 0$. When $r \in C^2([\alpha, \beta])$, r, r', r'' are all bounded and the inequality easily extends to $0 < t \leq 1$ uniformly in t, for a possibly bigger constant C. $\qquad\square$

THEOREM 4.6. *Suppose ∂K contains no parallel points of zero curvature and n is a positive integer. Then every neighborhood of K in $\mathcal{X}(K)$ contains a subset homeomorphic to an open set in R^n. In particular since n is arbitrary, every neighborhood of K is infinite dimensional.*

Proof. By the previous lemma we may assume without loss of generality that R is the directed X-ray of a convex body in $\mathcal{H}$. Choose constants $a_1 > \cdots > a_n > 1$ and sufficiently close to 1 that the functions $f_j(\varphi) = \csc(a_j\varphi)$ are defined and positive on $[\alpha, \beta]$. It is not hard to see that the functions $f_1, \cdots, f_n$ are concave away from the origin and linearly independent. Indeed, one readily computes $\mathcal{K}f_j = (1 - a_j^2)f_j^2 < 0$. The family of convex bodies that we reconstruct will have near sides $r + \sum_{j=1}^n t_j f_j$ and far sides $R + \sum_{j=1}^n t_j f_j$ with the $t_j > 0$ and sufficiently small. It is clear from Lemma 4.1.1 and 4.1.3 that $\overline{\mathcal{K}}(r + \sum_{j=1}^n t_j f_j) \leq 0$, so there is no difficulty with the near side. We need to show that we can choose the t_j so that the far sides are all concave toward the origin. This is done by induction on n. Since K does not contain parallel points of zero curvature and is a directed X-ray, Lemma 4.4 gives that $\underline{\mathcal{K}}R > \delta > 0$ for some positive number δ. We show that for $t_1 > 0$ and sufficiently small, $\underline{\mathcal{K}}(R + tf_1) \geq \frac{\delta}{2} > 0$ for $0 \leq t \leq t_1$. From Lemma 4.1.3 we have

$$\underline{\mathcal{K}}(R + tf_1) = \frac{R + tf_1}{R}\underline{\mathcal{K}}(R) + t\frac{R + tf_1}{f_1}\mathcal{K}f_1 - 2tf_1 R\left[\frac{D_1 R}{R} - \frac{f_1'}{f_1}\right]^2$$

$$= \underline{\mathcal{K}}R + t\frac{R + tf_1}{f_1}\mathcal{K}f_1 + 2tf_1 R\left\{\frac{\underline{\mathcal{K}}R}{2R^2} - \left[\frac{D_1 R}{R} - \frac{f_1'}{f_1}\right]^2\right\}.$$

(The proof of the first step proceeds exactly like that of Theorem 6.5 [6], but we include it here for completeness.) Since f_1 is C^2 the

second summand, though negative, goes to zero uniformly as $t \to 0$. Since R is bounded below by a positive number, the last term can only cause difficulty if $D_1 R$ is unbounded. This can only occur at α or β. Thus we only need consider when $|D_1 R| \to \infty$ at these points. Expanding $\underline{\mathcal{K}}R$, we have

$$\underline{\mathcal{K}}R + 2tf_1 R \left\{ \frac{\underline{\mathcal{K}}R}{2R^2} - \left[\frac{D_1 R}{R} - \frac{f_1'}{f_1} \right]^2 \right\}$$

$$= \underline{\mathcal{K}}R + tf_1 \left[R - \overline{D}_2 R \right] + 4tf_1' D_1 R - 2tf_1 R \left(\frac{f_1'}{f_1} \right)^2.$$

Since $\underline{\mathcal{K}}R > 0$, $\overline{D}_2 R < R + 2\frac{(D_1 R)^2}{R}$. Using this and the estimate in Lemma 4.5 we have the above is

$$\geq (C - \frac{2tf_1}{R})(D_1 R)^2 + 4tf_1' R D_1 R - 2tf_1 R \left(\frac{f_1'}{f_1} \right)^2.$$

Since R is bounded below by a positive number, f_1 is bounded and $C > 0$, we have $C - \frac{2tf_1}{R} > 0$ for t sufficiently small. Thus there exists t_1 such that $\underline{\mathcal{K}}(R + tf_1) > \frac{\delta}{2}$ whenever $0 \leq t \leq t_1$. Moreover, from Lemma 4.5 the functions $R + tf_1$, $0 \leq t \leq 1$ satisfy the estimate of the previous lemma. The rest follows easily by induction. Assume that $t_1^*, \cdots t_{n-1}^*$ have been chosen such that

1. $\underline{\mathcal{K}}(R + \sum_{j=1}^{n-1} t_j f_j) > \frac{\delta}{2^{n-1}}$ whenever $0 \leq t_j \leq t_j^*$, and

2. the functions $R + \sum_{j=1}^{n-1} t_t f_j$ satisfy the uniform estimate of Lemma 4.5.

By induction there exists $t_n^* > 0$ such that $\underline{\mathcal{K}}(R + \sum_{j=1}^{n} t_t f_j) > \frac{\delta}{2^n} > 0$ whenever $0 \leq t_n \leq t_n^*$. Finally, if we let $\epsilon > 0$ be given and choose the positive numbers $t_1^*, \cdots, t_n^*$ such that $\max \left[\sum_{j=1}^{n} t_j^* f_j(\alpha), \sum_{j=1}^{n} t_j^* f_j(\beta) \right] < \epsilon$, we have a set of dimension n in $\mathcal{X}(K)$ contained in an ϵ neighborhood of K. This completes the proof. $\square$

We have left open the question of dimension when ∂K contains parallel points of zero curvature, but believe that unless K is a parallel wedge, $\mathcal{X}(K)$ is locally infinite dimensional. A way to prove this

would be to construct an infinite dimensional family of functions that are concave away from the origin which have zero curvature on the appropriate set, which when added to the far side of K preserves concavity toward the origin. We haven't worked out the details.

5. Boundedness

The set of convex bodies with a common parallel X-ray from a single direction is clearly unbounded since translation of the convex body parallel to the direction of the X-ray does not change the X-ray. In this section we show that there are nontrivial situations where $\mathcal{X}(K)$ is bounded, and hence compact by Theorem 3.3. We give both geometric and analytic characterizations of when $\mathcal{X}(K)$ is bounded and specific examples.

LEMMA 5.1.

$$\frac{\underline{\mathcal{K}}X}{X^2} \geq \frac{2rR}{X^2}\left[\frac{D_1R}{R} - \frac{D_1r}{r}\right]^2 = \frac{2r}{R}\left[\frac{D_1X}{X} - \frac{D_1r}{r}\right]^2$$

$$= \frac{2R}{r}\left[\frac{D_1X}{X} - \frac{D_1R}{R}\right]^2.$$

Proof. From Lemma 4.1.4 with $f - g = X = R - r$ we have

$$\frac{\underline{\mathcal{K}}X}{X^2} \geq \frac{\underline{\mathcal{K}}R}{RX} - \frac{\overline{\mathcal{K}}r}{rX} + \frac{2rR}{X^2}\left[\frac{D_1R}{R} - \frac{D_1r}{r}\right]^2.$$

Since $X = R - r$ and $X' = R' - r'$, one computes

$$\left[\frac{D_1R}{R} - \frac{D_1r}{r}\right]^2 = \frac{X^2}{r^2}\left[\frac{D_1X}{X} - \frac{D_1R}{R}\right]^2 = \frac{X^2}{R^2}\left[\frac{D_1X}{X} - \frac{D_1r}{r}\right]^2.$$

The result follows from this and the fact that $\underline{\mathcal{K}}R \geq 0$ and $\overline{\mathcal{K}}r \leq 0$. $\qquad\square$

LEMMA 5.2. *Suppose ℓ is a line (segment). If the body with far side ℓ and near side $\ell - X$ is convex, then the body with far side $t\ell$ and near side $t\ell - X$ is convex for all $t \geq 1$. Consequently, if $\mathcal{X}(K)$ contains a body with a flat far side, then $\mathcal{X}(K)$ is unbounded.*

Proof. We need to show that $\overline{\mathcal{K}}(t\ell - X) \leq 0$ for all $t \geq 1$. Since ℓ is a line, $\mathcal{K}\ell = 0$ and Lemma 4.1.5 gives

$$\overline{\mathcal{K}}(t\ell - X) \leq -\frac{t\ell - X}{X}\underline{\mathcal{K}}X + 2t\ell X\left[\frac{D_1 X}{X} - \frac{\ell'}{\ell}\right]^2 = \qquad (7)$$

$$= \underline{\mathcal{K}}X - t\ell X\left\{\frac{\underline{\mathcal{K}}X}{X^2} - 2\left[\frac{D_1 X}{X} - \frac{\ell'}{\ell}\right]^2\right\},$$

the last equality holding where $\underline{\mathcal{K}}X$ is finite.

On the other hand from Lemma 4.1.3 and Lemma 5.1 (with R replaced by ℓ)

$$0 = \underline{\mathcal{K}}(\ell) = \underline{\mathcal{K}}(X + (\ell - X)) \leq \frac{\ell}{X}\underline{\mathcal{K}}X + \frac{\ell}{\ell - X}\overline{\mathcal{K}}(\ell - X)$$

$$-2\frac{X\ell^2}{(\ell - X)}\left[\frac{D_1 X}{X} - \frac{\ell'}{\ell}\right]^2.$$

Consequently

$$2\ell X\left[\frac{D_1 X}{X} - \frac{\ell'}{\ell}\right]^2 \leq \frac{\ell - X}{X}\underline{\mathcal{K}}X$$

since $\overline{\mathcal{K}}(\ell - X) \leq 0$. The term on the left of the equal sign in (7) equals $-\infty$ whenever $\underline{\mathcal{K}}X = +\infty$, so we only need consider points where $\underline{\mathcal{K}}X$ is finite. Assuming this and substituting the last inequality into (7) gives

$$\overline{\mathcal{K}}(t\ell - X) \leq (1 - t)\underline{\mathcal{K}}X \leq 0, \text{ whenever } t \geq 1.$$

$\square$

LEMMA 5.3. *A necessary condition that there exists a body in $\mathcal{X}(K)$ whose far side is a line (segment) ℓ is that*

$$\frac{\overline{\mathcal{K}}X}{X^2} \geq 2\left[\frac{D_1 X}{X} - \frac{\ell'}{\ell}\right]^2,$$

with strict inequality at points where $\overline{\mathcal{K}}X > 0$. A sufficient condition for this to occur is that for some δ, $0 < \delta < \|X\|_\infty$,

$$\frac{\underline{\mathcal{K}}X}{X^2} \geq \frac{2}{1 - \delta X}\left[\frac{D_1 X}{X} - \frac{\ell'}{\ell}\right]^2.$$

Proof. From Lemma 4.1.5 at points where $\overline{\mathcal{K}}X$ is finite

$$\overline{\mathcal{K}}(\ell - X) \geq \overline{\mathcal{K}}X - \ell X \left\{ \frac{\overline{\mathcal{K}}X}{X^2} - 2\left[\frac{D_1 X}{X} - \frac{\ell'}{\ell}\right]^2 \right\}.$$

This expression is positive where the necessary condition fails.

For the sufficiency note that from the expression on the right hand side of the equal sign in (7) $\overline{\mathcal{K}}(t\ell - X) \leq 0$ for t sufficiently large provided that for some $\delta > 0$

$$\frac{\mathcal{K}X}{X^2} - 2\left[\frac{D_1 X}{X} - \frac{\ell'}{\ell}\right]^2 \geq \delta \frac{\mathcal{K}X}{X},$$

which is easily rewritten as the sufficient condition.   □

We turn now to the case of a flat near side. Suppose that a line segment ℓ is the near side of a convex body with directed X-ray X. Then Lemma 4.1.3 and 4.1.1 give

$$\underline{\mathcal{K}}(X + t\ell) = \underline{\mathcal{K}}X + t\ell X \left\{ \frac{\mathcal{K}X}{X^2} - 2\left[\frac{D_1 X}{X} - \frac{\ell'}{\ell}\right]^2 \right\}.$$

The last expression is ≥ 0 for all $t > 0$ if and only if

$$\frac{\mathcal{K}X}{X^2} \geq 2\left[\frac{D_1 X}{X} - \frac{\ell'}{\ell}\right]^2, \quad \text{when } \alpha < \varphi < \beta.$$

This gives a sufficient condition for unboundedness. In fact, the condition is also necessary.

THEOREM 5.4. *A necessary and sufficient condition that $\mathcal{X}(K)$ be unbounded is that there exists a line (segment) ℓ such that*

$$\frac{\mathcal{K}X}{X^2} \geq 2\left[\frac{D_1 X}{X} - \frac{\ell'}{\ell}\right]^2, \quad \text{when } \alpha < \varphi < \beta. \tag{8}$$

Moreover the inequality above is satisfied for a line (segment) ℓ if and only if for all $t \geq 0$, the body with near side $t\ell$ and far side $t\ell + X$ is convex.

Proof. It remains to prove the necessity. Suppose that $\mathcal{X}(K)$ is unbounded. Then for each integer $j \geq 1$ there exists a body $K_j \in \mathcal{X}(K)$ with near side function r_j with $\|r_j\|_\infty \to \infty$, and in light of Lemma 3.6 $\min r_j \geq j$. By Theorem 2.4 the functions $s_j = r_j/\|r_j\|_\infty$ are near side functions of convex bodies in $\mathcal{X}(K)$, and by compactness (Theorem 3.3), $\{s_j\}$ has a subsequence that converges uniformly to a function s which by completeness is a nearside function for a body in $\mathcal{X}(K)$. (Lemma 3.6 assures that $s > 0$ on $[\alpha, \beta]$.) We show that $s = \ell$ is a line segment. Indeed, the functions $S_j = (r_j + X)/\|r_j\|_\infty$ are concave toward the origin and converge uniformly to s also. Thus, s is both concave away from and toward the origin. Hence $s = \ell$ is a line segment. For simplicity of notation, we assume that the sequence $\{s_j\}$ itself converges uniformly to ℓ. From (2) we have for any $\alpha < \varphi_1 < \varphi_3 < \beta$ that:

$$s_j^2(\varphi_1) - s_j(\varphi_1)s_j(\varphi_3)\cos(\varphi_3 - \varphi_1) + s_j(\varphi_3)(s_j)'_+(\varphi_1)\sin(\varphi_3 - \varphi_1) \leq 0.$$

Since $s_j \to \ell$, taking the limit superior throughout the above inequality gives

$$\ell^2(\varphi_1) - \ell(\varphi_1)\ell(\varphi_3)\sin(\varphi_3 - \varphi_1) + \ell(\varphi_3)\sin(\varphi_3 - \varphi_1)\overline{\lim}(s_j)'_+(\varphi_1) \leq 0.$$

Since ℓ is a line

$$\ell^2(\varphi_1) - \ell(\varphi_1)\ell(\varphi_3)\sin(\varphi_3 - \varphi_1) + \ell(\varphi_3)\ell'(\varphi_1)\sin(\varphi_3 - \varphi_1) = 0,$$

and since φ_1 is arbitrary

$$\overline{\lim}(s_j)'_+(\varphi) \leq \ell'(\varphi), \ \varphi \in (\alpha, \beta).$$

Repeating this argument but using (3) we obtain the same inequality with $(s_j)'_+$ replaced by $(s_j)'_-$. Thus

$$\overline{\lim} \, D_1 s_j(\varphi) \leq \ell'(\varphi).$$

In a similar way for $S_j = s_j + \frac{X}{\|r_j\|_\infty}$ we have

$$S_j(\varphi_1)^2 - S_j(\varphi_1)S_j(\varphi_3)\cos(\varphi_3 - \varphi_1) + S_j(\varphi_3)(S_j)'_+(\varphi_1)\sin(\varphi_3 - \varphi_1) \geq 0.$$

Noting that $S_j \to \ell$ also, upon taking the limit inferior and using that ℓ is a line we obtain $\underline{\lim}\left[(s_j)'_+(\varphi_1) + \frac{X'}{\|r_j\|_\infty}\right] \geq \ell'(\varphi.)$ Since X'

is bounded on compact subsets of (α, β), proceeding as above we obtain

$$\ell'(\varphi) = \lim D_1 s_j(\varphi), \text{ when } \alpha < \varphi < \beta,$$

where the convergence is pointwise and uniform on compact subsets of (α, β). From Lemma 5.1 and the fact that $\min r_j \geq j$

$$\frac{\underline{K}X}{X^2} \geq \frac{2r_j}{R_j}\left[\frac{D_1 X}{X} - \frac{D_1 r_j}{r_j}\right]^2$$

$$= \frac{2r_j}{r_j + X}\left[\frac{D_1 X}{X} - \frac{D_1 s_j}{s_j}\right]^2 \geq \frac{2j}{j + X}\left[\frac{D_1 X}{X} - \frac{D_1 s_j}{s_j}\right]^2.$$

Letting $j \to \infty$ establishes (8) completing the proof.

$\square$

We can rephrase the characterization so that one can determine whether $\mathcal{X}(K)$ is bounded from the directed X-ray. To this end first note that if ψ is the angle of inclination of a line ℓ, then $\frac{\ell'(\varphi)}{\ell(\varphi)} = \cot(\psi - \varphi)$.

THEOREM 5.5. $\mathcal{X}(K)$ *is unbounded if and only if*

$$\sup_{\varphi \in (\alpha, \beta)}\left[\varphi - \cot^{-1}\left(-\frac{D_1 X}{X} - \sqrt{\frac{\underline{K}X}{2X^2}}\right)\right] \leq \qquad (9)$$

$$\inf_{\varphi \in (\alpha, \beta)}\left[\varphi - \cot^{-1}\left(-\frac{D_1 X}{X} + \sqrt{\frac{\underline{K}X}{2X^2}}\right)\right],$$

If ℓ is a line (segment) with angle of inclination ψ with ψ between the sup *and* inf *above, then the body with near side $t\ell$ and far side $t\ell + X$ is convex for all $t \geq 0$.*

Proof. Suppose that ℓ is a line with angle of inclination ψ satisfying (8). We take $\alpha > \psi > \beta - \pi$ so that $0 < \varphi - \psi < \pi$. Since $\frac{\ell'}{\ell} = \cot(\psi - \varphi) = -\cot(\varphi - \psi)$ we may rewrite (8) as

$$-\frac{D_1 X}{X} - \sqrt{\frac{\underline{K}X}{2X^2}} \leq \cot(\varphi - \psi) \leq -\frac{D_1 X}{X} + \sqrt{\frac{\underline{K}X}{2X^2}}.$$
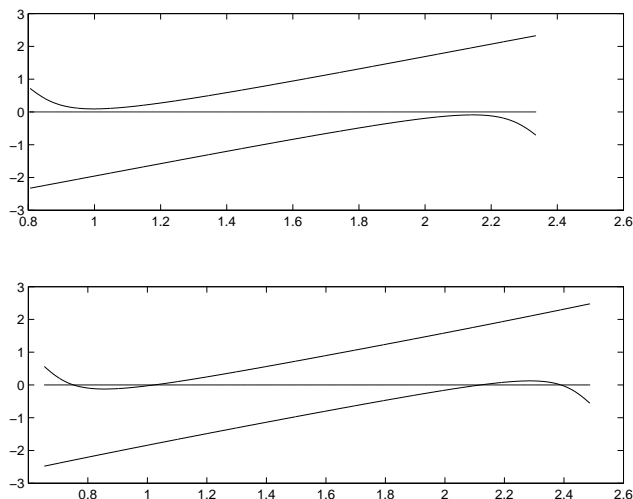
Figure 1: Top: $a = \frac{\rho}{c} = 0.7$, $\mathcal{X}(K_{\rho,c})$ is unbounded. Bottom: $a = 0.8$, $\mathcal{X}(K_{\rho,c})$ is bounded.

Since $0 < \varphi - \psi < \pi$, we may apply $\cot^{-1}$ throughout reversing all inequalities. Then solve for ψ. Since ψ is constant, we obtain (9). The last statement of the theorem follows from Theorem 5.4.

$\square$

In order that the theorem on boundedness be of interest there must exist nontrivial examples where $\mathcal{X}(K)$ is bounded. We conclude with one. Consider convex bodies $K_{c,\rho}$, $c > \rho > 0$ which are circles of radius $\rho > 0$ with center on the vertical axis at a distance c from the origin. One readily computes that the near side function is $r = c\sin(\varphi) - \sqrt{\rho^2 - c^2\cos^2\varphi}$, the far side function is $R = c\sin(\varphi) + \sqrt{\rho^2 - c^2\cos^2\varphi}$ and the directed X-ray

$$X = 2\sqrt{\rho^2 - c^2\cos^2\varphi} = 2c\sqrt{a^2 - \cos^2\varphi}, \ a = \frac{\rho}{c}.$$

Using the alternate formula $\mathcal{K}f = f^3\left(\frac{1}{f} + (\frac{1}{f})''\right)$, (4), one computes

$$\frac{\mathcal{K}X}{X^2} = 1 + \frac{(1-a)^2\cos^2\varphi + \sin^2\varphi(a^2 + \cos^2\varphi)}{(a^2 - \cos^2\varphi)^2}.$$

So the quantitities $\frac{X'}{X}$ and $\frac{\mathcal{K}X}{X^2}$ depend only on a. Graphs of $(\varphi, f(\varphi))$ and $(\varphi, g(\varphi))$ (in rectangular coordinates) where f, g are the functions on the left and right hand side of (9) are shown in Figure 1. From the graph and Theorem 5.5, we see that $\mathcal{X}(K_{c,\rho})$ is unbounded when $a = \frac{\rho}{c} = .7$ and bounded when $a = .8$. Further computations shows that there exists a number μ, $0.74 < \mu < 0.75$ such that $\mathcal{X}(K_{c,\rho})$ is unbounded when $a \le \mu$ and bounded when $a > \mu$.

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