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# When every Vector Bundle is a Direct Sum of Line Bundles?

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Summary. - *Here we prove the following result. Let $X$ be a reduced and connected projective variety. Every vector bundle on $X$ is isomorphic to a direct sum of line bundles if and only if every irreducible component of $X$ is isomorphic to $\mathbf{P}^1$, every singular point of $X$ is an ordinary node and every irreducible component of $X$ contains at most two singular points of $X$.*

## 1. Non-split vector bundles

It is very natural to ask on which connected algebraic varieties (or complex analytic spaces, or stacks, or ...) every vector bundle is isomorphic to a direct sum of line bundles. The aim of this short paper is the proof of the following result.

Theorem 1.1. *Let $X$ be a reduced and connected projective variety. Every vector bundle on $X$ is isomorphic to a direct sum of line bundles if and only if every irreducible component of $X$ is isomorphic to $\mathbf{P}^1$, every singular point of $X$ is an ordinary node and every irreducible component of $X$ contains at most two singular points of $X$, i.e. if and only if $X$ is a chain of $\mathbf{P}^1$'s.*

Theorem 1.1 is well-known (and easier to prove) when $X$ is irreducible.

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Remark 1.2. *Let $X$ be a reduced and connected projective curve. Fix any integer $n \geq 2$. The proof of Theorem 1.1 (plus the use of a general extension of $\mathcal{O}_C$ by $\mathcal{O}_C$ when $C$ is any reduced and connected projective curve such that $h^1(C, \mathcal{O}_C) > 0$) will show that every rank $n$ vector bundle on $X$ is isomorphic to a direct sum of $n$ line bundles if and only if $X$ is a chain of $\mathbf{P}^1$'s.*

We work over an algebraically closed field $\mathbb{K}$.

Let $Y$ be a reduced curve and $P \in Y$. We recall that the local ring $\mathcal{O}_{Y,P}$ is seminormal (i.e. $Y$ has a seminormal singularity) if and only if all formal branches of $Y$ at $P$ are smooth and the number of such branches is equal to the dimension of the Zariski tangent space of $Y$ at $P$.

Lemma 1.3. *Let $C$ be an integral projective curve such that $g := p_a(C) > 0$. Fix any integer $d \geq 2g+1$ and any $L \in Pic^d(C)$. Notice that $h^1(C, L) = 0$ and that $L$ is very ample. Let $h_L : C \to \mathbf{P}^n$, $n := d-g$, be the complete embedding associated to the linear system $|L|$. Then $h_L^*(T\mathbf{P}^n(-1))$ is not a direct sum of line bundles.*

*Proof.* Since $C$ is locally Cohen-Macaulay, we have

$$h^1(C, L) = h^0(C, Hom(L, \omega_C))$$

([1], p. 1 or Th. 1.15 at p. 167). Since $\deg(\omega_C) = 2g-2$ even when $C$ is not Gorenstein, we have $h^1(C, L) = 0$. Now we prove the last assertion. Assume that $h_L^*(T\mathbf{P}^n(-1))$ is a direct sum of $n$ line bundles. By the Euler's sequence $T\mathbf{P}^n(-1)$ is a rank $n$ spanned vector bundle such that $\det(T\mathbf{P}^n(-1)) \cong \mathcal{O}_{\mathbf{P}^n}(1)$. Hence $h_L^*(T\mathbf{P}^n(-1))$ is a rank $n$ vector bundle with determinant isomorphic to $L$ and hence with degree $d$. Since $h_L(X)$ spans $\mathbf{P}^n$, $h_L^*(T\mathbf{P}^n(-1))$ contains no trivial factor (hint: use the Euler's sequence of $T\mathbf{P}^n$ to show that a trivial factor of $h_L^*(T\mathbf{P}^n(-1))$ corresponds to a hyperplane of $\mathbf{P}^n$ containing $h_L(C)$). Since $g > 0$, every non-trivial spanned line bundle on $C$ has degree at least 2. Hence $h_L^*(T\mathbf{P}^n(-1))$ has degree at least $2n = 2d - 2g > d$, contradiction. □

Lemma 1.4. *Let $Y$ be a reduced projective curve, $D$ a union of some of the irreducible components of $Y$ and $F$ a rank $n$ vector bundle on $D$. Let $T$ be the closure of $Y \backslash D$ in $Y$. Fix any rank $n$ vector* *bundle $A$ on $T$. Then there exists a vector bundle $E$ on $Y$ such that $E|D \cong F$ and $E|T \cong A$.*

*Proof.* Since the set $D \cap T$ is finite, there is an open neighborhood $U$ of $D \cap T$ in $D$ such that $F \cap U \cong {\mathcal{O}_U}^{\oplus n}$ and an open neighborhood $V$ of $D \cap T$ in $T$ such that $A|V \cong {\mathcal{O}_V}^{\oplus n}$. Glue together $F$ and $A$ along $V \cup U$ to get the existence of at least one such bundle. □

LEMMA 1.5. *Fix an integer $n \geq 2$. Let $Y$ be a reduced and connected projective curve such that every rank $n$ vector bundle on $Y$ is a direct sum of line bundles. Then every rank $n$ vector bundle on every connected union $Z$ of irreducible components of $Y$ is isomorphic to a direct sum of line bundles of $Z$*

*Proof.* We may assume $Z$ connected. Fix any rank $n$ vector bundles $F$ on $Z$. By Lemma 1.4 there is a rank $n$ vector bundle $E$ on $Y$ such that $E|Z \cong F$. By assumption $E \cong L_1 \oplus \cdots \oplus L_n$ for some $L_i \in \mathrm{Pic}(Y)$. Hence $F \cong L_1|Z \oplus \cdots \oplus L_n|Z$. □

REMARK 1.6. *In the set-up of Lemma 1.4 we see that if $T, D$ are irreducible components of $Y$ such that $T \cap D \neq \emptyset$, then $p_a(T \cup D) = 0$ and hence $\sharp(T \cap D) = 1$ and $T \cup D$ has an ordinary node at the point $T \cap D$.*

LEMMA 1.7. *Let $Y$ be the connected projective curve with a unique singular point, $P$, a seminormal singularity at $P$ and 3 irreducible components $T_1, T_2, T_3$ such that $T_i \cong \mathbf{P}^1$ for every $i$, i.e. let $Y$ be isomorphic to a general union of 3 lines of $\mathbf{P}^3$ through a common point. Notice that $p_a(Y) = 0$ and that for all integers $d_1, d_2, d_3$ there is a unique $L \in Pic(Y)$ such that $deg(L|T_i) = d_i$ for every $i$. For every integer $n \geq 2$ there is a rank $n$ vector bundle $E_n$ on $Y$ such that $E_n$ is not isomorphic to a direct sum of $n$ line bundles.*

*Proof.* The first assertion is obvious: just use that $h^1(Y, \mathcal{O}_Y) = 0$. Let $\mathcal{O}_Y(d_1, d_2, d_3)$ denote the unique (up to isomorphisms) line bundle $L$ on $Y$ such that $\deg(L|T_i) = d_i$ for every $i$. For every $i \in \{1, 2, 3\}$ set $Z_i = T_j \cup T_k$, where $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$. Since $h^1(Z_i, \mathcal{O}_{Z_i}) = 0$, every $L \in \mathrm{Pic}(Z_i)$ is uniquely determined by the two integers $\deg(L|T_j)$ and $\deg(L|T_k)$. We use the obvious notation $\mathcal{O}_{Z_i}(d_j, d_k)$. Every vector bundle on $Z_i$ is isomorphic to a direct sum on line bundles ([3], Prop. 3.1). Take as $E_2$ a general line bundle obtained gluing together with general data at $P$ the vector bundles $\mathcal{O}_{T_i}(1) \oplus \mathcal{O}_{T_i}$. We have $E_2|Z_i \cong \mathcal{O}_{Z_i}(1,0) \oplus \mathcal{O}_{Z_i}(0,1)$ for all $i$. These 3 relations imply that $E_2$ is not isomorphic to a direct sum of 2 line bundles. Now take $n \geq 3$ and set $E_n = E_2 \oplus {\mathcal{O}_Y}^{\oplus(n-2)}$. Since $Y$ is projective, the Krull-Schmidt unique decomposition theorem for vector bundles on $X$ is true ([2], Th. 3). Hence $E_n$ is not isomorphic to a direct sum of $n$ line bundles. $\square$

*Proof of Theorem 1.1.* The “ if ” part is [3], Prop. 3.1. Now we prove the “ only if ” part. We assume that every vector bundle on $X$ is a direct sum of line bundles. Assume the existence of at least one component of $X$ which is either a curve with arithmetic genus $> 0$ or a variety of dimension $\geq 2$. Since $X$ is projective, this assumption implies the existence of an integral projective curve $C \subseteq X$ such that $g := p_a(C) > 0$. Since $X$ is projective, there is a very ample line bundle $R$ on $X$ such that $h^1(X, \mathcal{I}_C \otimes R) = 0$. Hence the complete embedding $h_R : X \to \mathbf{P}^m$ induced by the complete linear system $|R|$ maps $C$ isomorphically into a linear subspace $M$ of $\mathbf{P}^m$ and the associated map $C \to M$ is induced by the complete linear system $|R|C\,|$. Taking instead of $R$ a large power of it, we may also assume $d := \deg(R|C) \geq 2p_a(C)+1$. Under this assumption we have $n := \dim(M) = d - g$. Notice that $h_R^*(T\mathbf{P}^m(-1))|C \cong {\mathcal{O}_C}^{\oplus(m-n)} \oplus h_{R|C}^*(TM(-1))$. By Lemma 1.3 $h_{R|C}^*(TM(-1))$ is not isomorphic to a direct sum of line bundles. Hence $h_R^*(T\mathbf{P}^m(-1))$ is not a direct sum of line bundles (use Krull-Schmidt unique decomposition theorem ([2], Th. 3)), contradiction. Hence every irreducible component of $X$ is isomorphic to $\mathbf{P}^1$. By Lemma 1.5 and Remark 1.6 we also obtain that the graph of the irreducible components of $X$ has no closed path and that any two irreducible components of $X$ are transversal. Fix any such irreducible component $T$ and assume that $T$ contains at least 3 singular points of $X$. Hence there are 3 distinct points $P_i$, $i = 1,2,3$ of $T$ and 3 irreducible components $T_i$, $i = 1,2,3$, of $X$ such that $T \cap T_i = \{P_i\}$ for all $i$ and the curve $T \cup T_1 \cup T_2 \cup T_3$ is a nodal tree. By [3], Remark at the bottom of p. 390, there is a rank two vector bundle on $T \cup T_1 \cup T_2 \cup T_3$ which is not isomorphic to a direct sum of two line bundles. Hence the contradiction comes from Lemma 1.4. Now fix $P \in \mathrm{Sing}(X)$. To conclude the proof of the “ only if ” part it is sufficient to show that $X$ has a seminormal singularity at $P$ and that it has only two formal branches at $P$. Assume that $X$ has not a seminormal singularity at $P$. Since all formal branches of $X$ at $P$ are smooth, there is an integer $n \geq 2$ and $n+1$ irreducible components $T_i$, $1 \leq i \leq n+1$, of $X$ such that $T_i \neq T_j$ for all $i \neq j$, $P \in T_i$ for all $i$ and the curve $Z := \cup_{i=1}^{n+1} T_i$ has $n$-dimensional Zariski tangent space at $P$. Since $p_a(Z) = 1 > 0$, the contradiction comes from Lemma 1.5 and the hint given in Remark 1.2. By Lemma 1.7 $X$ has only nodal singularities, concluding the proof. □

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