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Banach's Fixed Point Theorem for Partial Metric Spaces

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Summary. - *In 1994, S.G. Matthews introduced the notion of a partial metric space and obtained, among other results, a Banach contraction mapping for these spaces. Later on, S.J. O'Neill generalized Matthews' notion of partial metric, in order to establish connections between these structures and the topological aspects of domain theory. Here, we obtain a Banach fixed point theorem for complete partial metric spaces in the sense of O'Neill. Thus, Matthews' fixed point theorem follows as special case of our result.*

1. Introduction and preliminaries

Throughout this paper the letters $\mathbb{R}, \mathbb{R}^+$ and $\mathbb{N}$ will denote the set of real numbers, the set of nonnegative real numbers and the set of natural numbers, respectively.

The notion of a partial metric space was introduced by S.G. Matthews in [4] as a part of the study of denotational semantics

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of dataflow networks. In particular, he established the precise relationship between partial metric spaces and the so-called weightable quasi-metric spaces, and proved a partial metric generalization of Banach's contraction mapping theorem.

Le us recall that a *partial metric* on a (nonempty) set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

i. $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;

ii. $p(x, x) \leq p(x, y)$;

iii. $p(x, y) = p(y, x)$;

iv. $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A *partial metric space* is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

In [5], S.J. O'Neill proposed one significant change to Matthews' definition of the partial metrics, and that was to extend their range from $\mathbb{R}^+$ to $\mathbb{R}$.

In the following, partial metrics in the O'Neill sense will be called *dualistic partial metrics* and a pair (X, p) such that X is a nonempty set and p is a dualistic partial metric on X will be called a *dualistic partial metric space.*

In this way, O'Neill developed several connections between partial metrics and the topological aspects of domain theory. Moreover, the pair $(\mathbb{R}, p)$, where $p(x, y) = x \vee y$ for all $x, y \in \mathbb{R}$, provides a paradigmatic example of a dualistic partial metric space that is not a partial metric space. Other examples of dualistic partial metric (or partial) metric spaces which are interesting from a computational point of view may be found in [1], [4], [6], [8], etc.

Each dualistic partial metric p on X generates a T_0 topology $\mathcal{T}(p)$ on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

From this fact it immediately follows that a sequence $(x_n)_n$ in a dualistic partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = lim_{n\to\infty} p(x, x_n)$.

According to [5] (compare [4]), a sequence $(x_n)_{n\in\mathbb{N}}$ in a dualistic partial metric space (X, p) is called a *Cauchy sequence* if there exists $\lim_{n,m\to\infty} p(x_n, x_m)$.

A dualistic partial metric space (X, p) is said to be *complete* if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X converges, with respect to $\mathcal{T}(p)$, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

As we indicated above, and motivated by applications in program verification, Matthews obtained in [4] a Banach fixed point theorem for complete partial metric spaces. Since (complete) dualistic partial metrics provide a new approach to generalizing both the domain theoretic and the metric approach to semantics (see [5], p. 314), it seems interesting to obtain a Banach fixed point theorem in the realm of dualistic partial metric spaces. In this paper we present a theorem of this type. In particular, Matthews' contraction mapping theorem will be deduced as a special case of our result.

2. Banach's fixed point theorem for complete dualistic partial metric spaces

Before stating our main result we establish some (essentially known) correspondences between dualistic partial metrics and quasi-metric spaces.

Our basic references for quasi-metric spaces are [2] and [3].

In our context by a *quasi-metric* on a set X we mean a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$:

i. $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$,

ii. $d(x, y) \leq d(x, z) + d(z, y)$.

A *quasi-metric space* is a pair (X, d) such that X is a (nonempty) set and d is a quasi-metric on X.

Each quasi-metric d on X generates a T_0-topology $\mathcal{T}(d)$ on X which has as a base the family of open d-balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If d is a quasi-metric on X, then the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$, is a metric on X.

The proof of the following auxiliary results are analogous to the proofs of [4], Theorems 4.1 and 4.2 and [5], Definition 2.6 and Lemma 2.7. However, we include such proofs in order to help to the reader.

LEMMA 2.1. *If (X,p) is a dualistic partial metric space, then the function $d_p : X \times X \to \mathbb{R}^+$ defined by*

$$d_p(x,y) = p(x,y) - p(x,x),$$

is a quasi-metric on X such that $\mathcal{T}(p) = \mathcal{T}(d_p)$.

Proof. Consider $x, y \in X$. Then $d_p(x,y) = p(x,y) - p(x,x)$ is always nonnegative because of $p(x,x) \leq p(x,y)$.

Now, we have to check that d_p is actually a quasi-metric on X. Let $x, y, z \in X$. It is obvious that $x = y$ provides that $d_p(x,y) = d_p(y,x) = 0$. Moreover, if $d_p(x,y) = d_p(y,x) = 0$ then $p(x,y) - p(x,x) = p(y,x) - p(y,y) = 0$. Hence we obtain that $x = y$, since $p(x,y) = p(x,x) = p(y,y)$. Furthermore

$$\begin{aligned} d_p(x,y) &= p(x,y) - p(x,x) \\ &\leq p(x,z) + p(z,y) - p(z,z) - p(x,x) \\ &= d_p(x,z) + d_p(z,y). \end{aligned}$$

Finally we show that $\mathcal{T}(d) = \mathcal{T}(d_p)$. Indeed, let $x \in X$ and $\varepsilon > 0$ and consider $y \in B_{d_p}(x,\varepsilon)$. Then $d_p(x,y) = p(x,y) - p(x,x) < \varepsilon$ and, hence, $p(x,y) < \varepsilon + p(x,x)$. Consequently $y \in B_p(x,\varepsilon)$ and $\mathcal{T}(d_p) \subseteq \mathcal{T}(d)$.

Conversely if $y \in B_p(x,\varepsilon)$ we have that $p(x,y) < \varepsilon + p(x,x)$. Thus $d_p(x,y) = p(x,y) - p(x,x) < \varepsilon$, $y \in B_{d_p}(x,y)$ and

$$\mathcal{T}(d) \subseteq \mathcal{T}(d_p).$$

□

LEMMA 2.2. *(compare [4], [5], [7]). A dualistic partial metric space (X,p) is complete if and only if the metric space $(X,(d_p)^s)$ is complete. Furthermore* $\lim_{n\to\infty}(d_p)^s(a,x_n) = 0$ *if and only if* $p(a,a) = \lim_{n\to\infty} p(a,x_n) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Proof. First we show that every Cauchy sequence in (X,p) is a Cauchy sequence in $(X,(d_p)^s)$. To this end let $(x_n)_n$ be a Cauchy sequence in (X,p). Then there exists $\alpha \in \mathbb{R}$ such that, given $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ with $|p(x_n,x_m) - \alpha| < \frac{\varepsilon}{2}$ for all $n,m \geq n_\varepsilon$. Hence

$$\begin{aligned} d_p(x_n,x_m) &= p(x_n,x_m) - p(x_n,x_n) \\ &= |p(x_n,x_m) - \alpha + \alpha - p(x_n,x_n)| \\ &\leq |p(x_n,x_m) - \alpha| + |\alpha - p(x_n,x_n)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $n,m \geq n_\varepsilon$. Similarly we show $d_p(x_m,x_n) < \varepsilon$ for all $n,m \geq n_\varepsilon$. We conclude that $(x_n)_n$ is a Cauchy sequence in $(X,(d_p)^s)$.

Next we prove that completeness of $(X,(d_p)^s)$ implies completeness of (X,p). Indeed, if $(x_n)_n$ is a Cauchy sequence in (X,p) then it is also a Cauchy sequence in $(X,(d_p)^s)$. Since the metric space $(X,(d_p)^s)$ is complete we deduce that there exists $y \in X$ such that $\lim_{n\to\infty}(d_p)^s(y,x_n) = 0$. By (2.1) we follow that $(x_n)_n$ is a convergent sequence in (X,p). Next we prove that $\lim_{n,m\to\infty} p(x_n,x_m) = p(y,y)$.

Since $(x_n)_n$ is a Cauchy sequence in (X,p) it is sufficient to see that $\lim_{n\to\infty} p(x_n,x_n) = p(y,y)$. Let $\varepsilon > 0$ then there exists $n_0 \in \mathbb{N}$ such that $(d_p)^s(y,x_n) < \frac{\varepsilon}{2}$ whenever $n \geq n_0$. Thus

$$\begin{aligned} |p(y,y) - p(x_n,x_n)| &\leq |p(y,y) - p(y,x_n)| + |p(y,x_n) - p(x_n,x_n)| \\ &= d_p(y,x_n) + d_p(x_n,y) \\ &< 2(d_p)^s(y,x_n) < \varepsilon, \end{aligned}$$

whenever $n \geq n_0$. This shows that (X,p) is complete.

Now we prove that every Cauchy sequence $(x_n)_n$ in $(X,(d_p)^s)$ is a Cauchy sequence in (X,p). Let $\varepsilon = \frac{1}{2}$. Then there exists $n_0 \in \mathbb{N}$ such that $d_p(x_n,x_m) < \frac{1}{2}$ for all $n,m \geq n_0$. Since

$$d_p(x_n,x_{n_0}) + p(x_n,x_n) = d_p(x_{n_0},x_n) + p(x_{n_0},x_{n_0}),$$

then

$$\begin{aligned} |p(x_n,x_n)| &= |d_p(x_{n_0},x_n) + p(x_{n_0},x_{n_0}) - d_p(x_n,x_{n_0})| \\ &\leq d_p(x_{n_0},x_n) + |p(x_{n_0},x_{n_0})| + d_p(x_n,x_{n_0}) \\ &\leq 2(d_p)^s(x_n,x_{n_0}) + |p(x_{n_0},x_{n_0})| \\ &< 1 + |p(x_{n_0},x_{n_0})|. \end{aligned}$$

Consequently the sequence $(p(x_n, x_n))_n$ is bounded in $\mathbb{R}$, and so there exists $a \in \mathbb{R}$ such that a subsequence $(p(x_{n_k}, x_{n_k}))_k$ is convergent to a, i.e. $\lim_{k\to\infty} p(x_{n_k}, x_{n_k}) = a$.

It remains to prove that $(p(x_n, x_n))_n$ is a Cauchy sequence in $\mathbb{R}$. Since $(x_n)_n$ is a Cauchy sequence in $(X, (d_p)^s)$, given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $(d_p)^s(x_n, x_m) < \frac{\varepsilon}{2}$ for all $n, m \geq n_\varepsilon$. Thus, for all $n, m \geq n_\varepsilon$,

$$\begin{aligned} |p(x_n, x_n) - p(x_m, x_m)| &= |d_p(x_m, x_n) - d_p(x_n, x_m)| \\ &\leq 2(d_p)^s(x_m, x_n) < \varepsilon \end{aligned}$$

because of

$$p(x_n, x_n) = d_p(x_m, x_n) + p(x_m, x_m) - d_p(x_n, x_m).$$

Therefore $\lim_{n\to\infty} p(x_n, x_n) = a$.

On the other hand,

$$\begin{aligned} |p(x_n, x_m) - a| &= |p(x_n, x_m) - p(x_n, x_n) + p(x_n, x_n) - a| \\ &\leq d_p(x_n, x_m) + |p(x_n, x_n) - a| < \varepsilon \end{aligned}$$

for all $n, m \geq n_\varepsilon$. Hence $\lim_{n,m\to\infty} p(x_n, x_m) = a$ and $(x_n)_n$ is a Cauchy sequence in (X, p).

We shall have established the lemma if we prove that $(X, (d_p)^s)$ is complete if so is (X, p). Let $(x_n)_n$ be a Cauchy sequence in $(X, (d_p)^s)$. Then $(x_n)_n$ is a Cauchy seuqence in (X, p), and so it is convergent to a point $y \in X$ with

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(y, x_n) = p(y, y).$$

Then, given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$p(y, x_n) - p(y, y) < \varepsilon \qquad \text{and} \qquad p(y, y) - p(x_n, x_n) < \varepsilon$$

whenever $n \geq n_\varepsilon$. As a consequence we have

$$d_p(y, x_n) = p(y, x_n) - p(y, y) < \varepsilon,$$

and

$$\begin{aligned} d_p(x_n, y) &= p(y, x_n) - p(x_n, x_n) \\ &\leq |p(y, x_n) - p(y, y)| + |p(y, y) - p(x_n, x_n)| < 2\varepsilon \end{aligned}$$

whenever $n \geq n_\varepsilon$. Therefore $(X, (d_p)^s)$ is complete.

Finally, it is a simple matter to check that $lim_{n\to\infty}(d_p)^s(a, x_n) = 0$ if and only if $p(a, a) = \lim_{n\to\infty} p(a, x_n) = \lim_{n,m\to\infty} p(x_n, x_m)$. □

THEOREM 2.3. *Let f be a mapping of a complete dualistic partial metric space (X, p) into itself such that there is a real number c with $0 \leq c < 1$, satisfying*

$$|p(f(x), f(y))| \leq c\,|p(x, y)|\,, \tag{1}$$

for all $x, y \in X$. Then f has a unique fixed point.

Proof. Fix $x \in X$. Then it is clear that for each $n \in \mathbb{N}$ we have

$$|p(f^n(x), f^n(x))| \leq c^n\,|p(x, x)|$$

and

$$\left|p(f^n(x), f^{n+1}(x))\right| \leq c^n\,|p(x, f(x))|\,.$$

Since, by (2.1),

$$d_p(f^n(x), f^{n+1}(x)) + p(f^n(x), f^n(x)) = p(f^n(x), f^{n+1}(x)),$$

we deduce that

$$d_p(f^n(x), f^{n+1}(x)) + p(f^n(x), f^n(x)) \leq c^n\,|p(x, f(x))|\,.$$

Hence

$$\begin{aligned} d_p(f^n(x), f^{n+1}(x)) &\leq c^n\,|p(x, f(x))| - p(f^n(x), f^n(x)) \\ &\leq c^n\,|p(x, f(x))| + |p(f^n(x), f^n(x))| \\ &\leq c^n(|p(x, f(x))| + |p(x, x)|). \end{aligned}$$

Now let $n, k \in \mathbb{N}$. Then

$$\begin{aligned} d_p(f^n(x), f^{n+k}(x)) &\leq d_p(f^n(x), f^{n+1}(x)) + \ldots + \\ &\qquad +d_p(f^{n+k-1}(x), f^{n+k}(x)) \\ &\leq (c^n + \ldots + c^{n+k-1})(|p(x, f(x))| + |p(x, x)|) \\ &\leq \frac{c^n}{1-c}(|p(x, f(x))| + |p(x, x)|). \end{aligned}$$

Similarly, we obtain that

$$d_p(f^{n+k}(x), f^n(x)) \leq \frac{c^n}{1-c}(|p(x, f(x))| + |p(x, x)|).$$

Consequently $(f^n(x))_n$ is a Cauchy sequence in the metric space $(X, (d_p)^s)$, which is complete by (2.2). So there is $a \in X$ such that $\lim_{n\to\infty}(d_p)^s(a, x_n) = 0$. We want to show that a is the unique fixed point of f. First note that, by (2.2), we obtain

$$p(a, a) = \lim_{n\to\infty} p(a, f^n(x)) = \lim_{n,m\to\infty} p(f^n(x), f^m(x)).$$

Moreover, since

$$\lim_{n,m\to\infty} d_p(f^n(x), f^m(x)) = \lim_{n\to\infty} p(f^n(x), f^n(x)) = 0,$$

we deduce, from (2.1), that

$$\lim_{n,m\to\infty} p(f^n(x), f^m(x)) = 0.$$

Therefore $p(a, a) = \lim_{n\to\infty} p(a, f^n(x)) = 0$. Now since

$$|p(f(a), f(a)| \leq c\,|p(a, a)| = 0,$$

it follows that $p(f(a), f(a)) = 0$. On the other hand, since

$$\left|p(f(a), f^{n+1}(a)\right| \leq c\,|p(a, f^n(x)|\,,$$

it follows that

$$\lim_{n\to\infty} p(f(a), f^n(x)) = 0.$$

Then (2.2) shows that $f(a)$ is a limit point of $(f^n(x))_n$ in $(X, (d_p)^s)$. Consequently $a = f(a)$. Finally let $b \in X$ such that $b = f(b)$. Then

$$|p(a, b)| = |p(f(a), f(b)| \leq c\,|p(a, b)|\,,$$

which implies that $a = b$. This concludes the proof. □

Corollary 2.4. *(Matthews) Let f be a mapping of a complete partial metric space (X, p) into itself such that there is a real number c with $0 \leq c < 1$, satisfying*

$$p(f(x), f(y)) \leq cp(x, y), \tag{2}$$

for all $x, y \in X$. Then f has a unique fixed point.

In the light of the preceding corollary one can ask if the contractive condition (1) in the statement of our theorem can be replaced by the corresponding contraction condition (2) above. The following easy example shows that it is not the case.

3. Example

Let $X = (-\infty, 2]$, and let p be the dualistic partial metric on X given by

$$p(x, y) = x \vee y.$$

for all $x, y \in X$. Since $(X, (d_p)^s)$ is a complete metric space, (X, p) is a complete dualistic partial metric space.

Let f be the mapping from X into itself defined by $f(x) = x - 1$, for all $x \in (-\infty, 2]$. It is immediate to see that $p(f(x), f(y)) \leq \frac{1}{2}p(x, y)$, for all $x, y \in X$. However f has no any fixed point, of course.

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