Rend. Istit. Mat. Univ. Trieste
Vol. XXXV, 195–235 (2003)

Concentration of Local Energy for Two-dimensional Wave Maps

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Summary. - *We construct some particular kind of solution to the two - dimensional equivariant wave map problem with inhomogeneous source term in space-time domain of type*

$$\Omega_\alpha(t) = \{x \in \mathbb{R}^2 : |x|^\alpha < t\},$$

where $\alpha \in (0, 1]$. *More precisely,we take the initial data* (u_0, u_1) *at time* T *in the space* $H^{1+\varepsilon} \times H^\varepsilon$ *with some* $\varepsilon > 0$. *The source term is in* $L^1((0,T); H^\varepsilon(\Omega_\alpha(t)))$ *and we show that the* $H^{1+\varepsilon}$ *- norm of the solution blows-up, when* $t \to 0_+$ *and* $\alpha \in (0, 1-\varepsilon)$.

1. Introduction

The wave maps arise in various problems of mathematical physics (see Higgs field model in [4], relativity models in [1]). To be more precise, let (N, g) be n - dimensional manifold endowed with Riemannian metric structure, i.e. positive definite bilinear form g in every point of N. We call N target manifold. Let $M = \mathbb{R}^{m+1}$ be the Minkowski space-time equipped with the metric $h = (-1, 1, ..., 1)$. The wave map is a map that satisfies the equation

$$D^\alpha \partial_\alpha u = 0, \tag{1}$$

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Keywords: equivariant wave maps, H^s-spaces, blow-up of solution
AMS Subject Classification: 35L05, 35J10, 35P25, 35B40.
The authors are partially supported by Research Training Network (RTN) HYKE, financed by the European Union, contract number : HPRN-CT-2002-00282.

where

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha}, (x^\alpha) = (t,x) \in \mathbb{R}^{1+m}, \ \alpha = 0, 1, \cdots, m.$$

Moreover, D^α is the covariant pull - back derivative in the bundle u^*TN. As usual, the Greek indices α, β run from 0 to m. We use summation convention over repeated indices.

By the Nash embedding theorem, we may assume that the target N is embedded in some $\mathbb{R}^d$ for d large enough. So, our u is given by d - dimensional vector $u = (u_1, ..., u_d)$. Then the intrinsic equation (1) can be rewritten in extrinsic form

$$u_{tt} - \Delta u - B(u)(\partial_\alpha u, \partial^\alpha u) = 0, \tag{2}$$

where

$$B(p) : T_pN \times T_pN \to T_pN^\perp$$

is the second fundamental form on $N \subset \mathbb{R}^d$.

Given any function

$$F : (t,x) \in \mathbb{R}^{m+1} \to F(t,x) \in T_{u(t,x)}N$$

we can consider the following inhomogeneous version of (2)

$$u_{tt} - \Delta u - B(u)(\partial_\alpha u, \partial^\alpha u) = F. \tag{3}$$

In this work we study the Cauchy problem for (3) subject to the initial conditions

$$\begin{aligned} u(0,x) &= u_0(x) \in H^s(\mathbb{R}^m; N), \\ \partial_t u(0,x) &= u_1(x) \in H^{s-1}(\mathbb{R}^m; TN) \end{aligned} \tag{4}$$

More precisely, we identify $H^s = H^s(\mathbb{R}^m; N), s \geq 0$ with the space of functions

$$u(x) \in H^s(\mathbb{R}^m; \mathbb{R}^d),$$

satisfying $u(x) \in N$ for almost every $x \in \mathbb{R}^m$ (here we use the embedding $N \hookrightarrow \mathbb{R}^d$).

If $U \subset \mathbb{R}^m$ is an open set, then $H^s(U) = H^s(U; N), s \geq 0$ is the space of functions

$$u(x) \in H^s(U; \mathbb{R}^d),$$

satisfying $u(x) \in N$ for almost every $x \in U$.

From the results in [7], [8], [15] it follows that given any $s > m/2$, any data

$$(u_0, u_1) \in H^s \times H^{s-1}$$

and any source term

$$F \in L^1((0, T_0); H^{s-1}),$$

one can find a finite time interval $[0, T], 0 < T < T_0$ so that there exists a unique solution

$$u \in C([0, T]; H^s)$$

to the Cauchy problem (2), (4).

It is well - known from the result of Shatah [12] that the classical C^∞ solutions to the homogeneous wave map problem (2) might blow - up if $m = n \geq 3$ and $N = \mathbb{S}^n$.

On the other hand, Tao [14] have shown that the Cauchy problem for (2) is ill - posed, when $n = 1, m \geq 2$ and $N = \mathbb{S}^1$. The corresponding inhomogeneous problem (3) is treated in [2], where the blow-up result (in H^1 norm) is established, when

$$F \in L^p L^q, \ \frac{1}{p} + \frac{2}{q} > 2.$$

In [3] the case $n = m = 2$ is considered and it is shown that the solution map for the wave map problem is not uniformly continuous.

If the target is hyperboloid (or manifold with negative curvature) results due to Grillakis, Struwe (see [5] and [10]), show that C^∞-large initial data admit the existence of a global C^∞-solution. The key point in this approach is the following property, called the *non-concentration of energy*:

$$\int_{\Omega(t)} |\nabla_{t,x} u(t, x)|^2 \, \mathrm{d}x \to 0, \tag{5}$$

as $t \to 0$ and $\Omega(t) = \{x \in \mathbb{R}^2 : |x| < t\}$. The property (5) plays crucial role in the work [10].

In this work we study the property (5) for more general domains: $\Omega_\alpha(t) = \{x \in \mathbb{R}^2 : |x|^\alpha < t\}$, where $\alpha \in (0, 1]$ and the target is S^2.

We shall consider the inhomogeneous problem (3), assuming u is an equivariant wave map. We shall compare the following local norms: $\|u(t,.)\|_{H^{1+\varepsilon}(\Omega_\alpha(t))}$ and $\int_0^t \|F(s,.)\|_{H^\varepsilon(\Omega_\alpha(s))}\,\mathrm{d}s$, where $\varepsilon \geq 0$. These norms, when $\varepsilon = 0, \alpha = 1$, are closely connected with the energy estimate:

$$\|u(t,.)\|_{H^1(\Omega(t))} \leq C \sum_{k=0}^{1} \left\| \partial_t^k u(T,.) \right\|_{H^{1-k}(\Omega(T))} + \\ + C \int_t^T \|F(s,.)\|_{H^0(\Omega(s))}\,\mathrm{d}s, \quad (6)$$

where $0 < t < T$. This energy estimate combined with the conformal energy estimate (see [10]) lead to (5).

If one tries to improve the regularity in (6) by $\varepsilon > 0$, then a natural question arises: can we establish an estimate of type

$$\|u(t,.)\|_{H^{1+\varepsilon}(\Omega_\alpha(t))} \leq C \sum_{k=0}^{1} \left\| \partial_t^k u(T,.) \right\|_{H^{1-k+\varepsilon}(\Omega_\alpha(T))} + \\ + C \int_t^T \|F(s,.)\|_{H^\varepsilon(\Omega_\alpha(s))}\,\mathrm{d}s \quad ? \quad (7)$$

This question of course is particularly interesting in the case of dimension $n = 2$, when the energy space H^1 coincides with the critical space with respect to scaling and local existence; the behavior of (7) as $\varepsilon \to 0$ can be regarded as a measure of the instability of H^1 from the point of view of local existence. We aim at showing that (7) is not true and even more, we shall see that

$$\lim_{t \to 0_+} \|u(t,.)\|_{H^{1+\varepsilon}(\Omega_\alpha(t))} = \infty,$$

while the expression

$$\sum_{k=0}^{1} \left\| \partial_t^k u(T,.) \right\|_{H^{1+\varepsilon}(\Omega_\alpha(T))} + \int_0^T \|F(s,.)\|_{H^\varepsilon(\Omega_\alpha(s))}\,\mathrm{d}s$$

is bounded for $0 < \alpha < 1$ and $\varepsilon < \min\left(\frac{1}{2}, 1-\alpha\right)$.

More precisely, we have the following.

THEOREM 1.1. *Let* $n = m = 2$ *and* $N = \mathbb{S}^2$. *One can find positive numbers* $\varepsilon > 0$, $T > 0$ *initial data*

$$(u_0, u_1) \in H^{1+\varepsilon} \times H^{\varepsilon}$$

and a source term

$$F \in L^1((0,T); H^{\varepsilon}(\Omega_\alpha(t)))$$

so that the Cauchy problem for (3) *with initial data*

$$u(T,x) = u_0(x), \quad \partial_t u(T,x) = u_1(x) \tag{8}$$

has a solution in the domain $K_\alpha(T) = \{(t,x); t \in (0,T], x \in \Omega_\alpha(t)\}$, *where* $\alpha \in (0, 1-\varepsilon)$, *such that*

$$u \in C((0,T]; H^{1+\varepsilon}(\Omega_\alpha(t)))$$

and

$$\lim_{t \to 0_+} \|u(t)\|_{H^{1+\varepsilon}(\Omega_\alpha(t))} = \infty.$$

REMARK 1.1. *The condition* $u \in C((0,T]; H^{1+\varepsilon}(\Omega_\alpha(t)))$ *means that the function defined as* $v(t) = \|u(t,.)\|_{H^{1+\varepsilon}(\Omega_\alpha(t))}$, *belongs to* $C((0,T])$.

The above Theorem shows that the naive intuitive argument based on approximation of (weak) H^1 solutions in the light cone $K(T) = \{(t,x); t \in (0,T], x \in \Omega(t)\}$ by means of more regular sequences of $H^{1+\varepsilon}$ solutions in the slightly distorted "cones"

$$K_\alpha(T) = \{(t,x); t \in (0,T], x \in \Omega_\alpha(t)\},$$

might have a concentration of local energy effect manifested by the relation

$$\lim_{t \to 0_+} \|u(t)\|_{H^{1+\varepsilon}(\Omega_\alpha(t))} = \infty.$$

The plan of the work is the following. In section 2 we consider a special type of wave maps: so called equivariant wave maps. In section 3 we construct special equivariant solutions to the inhomogeneous problem and estimate the H^1 norm of the solution and $L^1((0,T); L^2(\Omega_\alpha(t)))$ of the source terms. Higher regularity estimates for the concrete solution are discussed in section 4, where the complete proof of the main Theorem is presented. In the Appendix some technical lemmas are established.

The authors are grateful to Piero D'Ancona for discussions during the preparation of the work.

2. Harmonic maps and special harmonic maps on the sphere

The harmonic maps correspond to the case, when the manifold M is $m-$ dimensional Riemannian manifold equipped with metric σ. The target is (N, g) and as before it is n - dimensional manifold endowed with Riemannian metric g. The harmonic map is a map that satisfies the equation

$$\sum_{j=1}^{m} D^j \partial_j u = 0, \tag{9}$$

where

$$\partial_j = \frac{\partial}{\partial x^j}$$

and $x^j, j = 1, \cdots, m$ are the local coordinates on M. Moreover, D^j is the covariant pull - back derivative in the bundle u^*TN.

To introduce the energy functional we suppose that $y^1, \cdots, y^n$ are local coordinates on N provided

$$y^1, \cdots, y^n \in Y$$

with Y being a small neighborhood of $0 \in \mathbb{R}^n$. Given any small neighborhood X of $0 \in \mathbb{R}^m$ and any map

$$U : x = (x^1, \cdots, x^m) \in X \to y = (y^1, \cdots, y^n) \in Y$$

we can define locally the energy functional

$$E(U) = \int_X \sum_{a,b=1}^{n} \sum_{j,k=1}^{m} \sigma^{jk}(x) g_{ab}(y) \partial_j y^a(x) \partial_k y^b(x) \sqrt{\sigma}\, \mathrm{d}x.$$

To simplify further the calculations we use the summation convention for repeated indices so

$$E(U) = \int_X \sigma^{jk}(x) g_{ab}(y) \partial_j y^a(x) \partial_k y^b(x) \sqrt{\sigma}\, \mathrm{d}x. \tag{10}$$

The Euler-Lagrange equation associated with this functional has the form

$$\begin{aligned} &- 2g_{ab}\partial_k(\sigma^{jk}\sqrt{\sigma}\partial_j y^b) - 2\partial_c g_{ab}\partial_k y^c \sigma^{jk}\sqrt{\sigma}\partial_j y^b + \\ &\qquad + (\partial_a g_{bc}\partial_j y^c \sigma^{jk}\sqrt{\sigma}\partial_k y^b) = 0. \end{aligned} \tag{11}$$

Since the Laplace-Beltrami operator Δ_M has local representation

$$\Delta_M = \sum_{j,k=1}^{m} \frac{1}{\sqrt{\sigma}} \partial_j \sqrt{\sigma} \sigma^{jk} \partial_k, \tag{12}$$

where

$$\sigma = \det\left(\sigma_{jk}\right),$$

we may write :

$$g_{ab}\Delta_M \ y^b + \partial_c g_{ab} \partial_k y^c \sigma^{jk} \partial_j y^b - \frac{1}{2}(\partial_a g_{bc} \partial_j y^c \sigma^{jk} \partial_k y^b) = 0, \tag{13}$$

The Christoffel symbols are given by the following expression:

$$\gamma_{c;ab} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}). \tag{14}$$

If we write

$$\partial_c g_{ab} \partial_k y^c \sigma^{kj} \partial_j y^b = \frac{1}{2}(\partial_c g_{ba} \partial_k y^c \sigma^{kj} \partial_j y^b + \partial_b g_{ac} \partial_k y^c \sigma^{kj} \partial_j y^b)$$

and use the expression of Christoffel symbols, then we arrive at the following equation

$$g_{ab}\Delta_M \ y^b + \gamma_{a;bc} \partial_k u^b \sigma^{jk}(x) \partial_j u^c = 0.$$

Raising the index a, we obtain

$$\Delta_M \ y^a + \gamma^a_{bc} \partial_k y^b \sigma^{jk}(x) \partial_j y^c = 0. \tag{15}$$

By the Nash embedding theorem, we may assume that the target N is embedded in some $\mathbb{R}^d$ for d large enough. So, our u is given by d - dimensional vector $u = (u_1, ..., u_d)$ and the local coordinates $y^1, \cdots, y^n$ on N enables one to parameterize locally the manifold N as follows

$$u = u(y), \quad y \in Y.$$

To simplify the further calculations we shall assume that $d = n + 1$, i.e. N is a surface in $\mathbb{R}^d$. The Riemannian metric g on N is induced by the Euclidean metric on $\mathbb{R}^d$ i.e.

$$g_{ab} = \langle \partial_{y^a} u, \partial_{y^b} u \rangle_{\mathbb{R}^d}, \tag{16}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ is the scalar product in $\mathbb{R}^d$.

Then the intrinsic equation (15) can be rewritten in extrinsic form

$$\Delta_M \; u + \sum_{j,k=1}^{m} \sigma^{jk} B(u)(\partial_j u, \partial_k u) = 0, \tag{17}$$

where

$$B(p) : T_p N \times T_p N \to T_p N^{\perp}$$

is the second fundamental form on $N \subset \mathbb{R}^d$ and Δ_M is the Laplace-Beltrami operator on the manifold M. Recall that the second fundamental form is defined by

$$B(u)(v, w) = \sum_{a,c=1}^{n} b_{ac}(u) v^a w^c \nu(u) \tag{18}$$

for any two vectors $v, w \in T_p N$ with coefficients b_{ac} defined as follows (see [9], Chapter 7, section 3, example 3.3)

$$b_{ac} = -\langle \partial_{y^a} \partial_{y^c} u(y), \nu(u(y)) \rangle_{\mathbb{R}^d}, \; a, c = 1, \cdots, n, \tag{19}$$

where $\nu(u)$ is the unit normal at $u \in N$. In the above local representation we have used a local basis $dy^1, \cdots, dy^n$ in $T_p N$, which is dual to the basis of vector fields $\partial_{y^1}, \cdots, \partial_{y^n}$ so that

$$v = \sum_{j=1}^{n} v_j dy^j, \quad w = \sum_{j=1}^{n} w_j dy^j.$$

To verify the above assertion it is sufficient to rewrite the energy functional in (10) as follows

$$E(U) = \int_X \sigma^{jk}(x) \langle \partial_j u(y(x)) \partial_k u(y(x)) \rangle_{\mathbb{R}^d} \sqrt{\sigma} \, \mathrm{d}x. \tag{20}$$

Taking the extremum of this integral over $u \in H^1$, such that $u(y) \in N$ we obtain the Euler - Lagrange equation with Lagrange multiplier

$$\Delta_M \; u = -\mu \nu(u), \tag{21}$$

where the Lagrange multiplier μ can be obtained by scalar multiplication with u, i.e.

$$\mu = -\frac{1}{\sqrt{\sigma}} \langle \partial_j \sqrt{\sigma}\ \sigma^{jk} \partial_k u, \nu \rangle_{\mathbb{R}^d}.$$

Using the property

$$\partial_k u \in T_u(N), \quad \nu(u) \in T_u(N)^\perp,$$

we get

$$\mu = -\langle \sigma^{jk} \partial_{y^a} \partial_{y^c} u, \nu \rangle_{\mathbb{R}^d} \partial_j y^a \partial_k y^c.$$

Now the definition (19) of the second fundamental form leads to (17).

From now on we restrict our attention to the simplest case when $M = \mathbb{S}^{m-1} = N$. Then we can assume that the standard metric on $M = N$ are induced by the embedding

$$\mathbb{S}^{m-1} \subset \mathbb{R}^m.$$

If $\omega \in \mathbb{S}^{m-1}$, then $\omega = (\omega_1, \cdots, \omega_m) \in \mathbb{R}^m$ and $|\omega| = 1$. We shall denote by $\kappa = (\kappa_1, \cdots, \kappa_{m-1})$ any local coordinates on $\mathbb{S}^{m-1}$ and by $x = (x_1, \cdots, x_m)$ the coordinates on $\mathbb{R}^m$. The standard metric on $\mathbb{S}^{m-1}$ is induced by the embedding and has the form

$$\sum_{j,k=1}^{m-1} \sigma_{jk}(\kappa) d\kappa_j d\kappa_k$$

with respect to local coordinates $\kappa_1, \cdots, \kappa_{m-1}$. Introducing spherical coordinates

$$r = |x|, \ \omega = \frac{x}{|x|} \in \mathbb{S}^{m-1},$$

we have the following decomposition of the Laplace operator in $\mathbb{R}^m$

$$\Delta_x = \partial_r^2 + \frac{m-1}{r} \partial_r + \frac{1}{r^2} \Delta_{\mathbb{S}^{m-1}}. \tag{22}$$

Then the intrinsic form of harmonic map equation in (15) implies that a local map

$$\kappa = (\kappa_1, \cdots, \kappa_{m-1}) \longrightarrow \lambda = \lambda(\kappa) = (\lambda^1, \cdots, \lambda^{m-1})$$

is (locally) a harmonic map if

$$\Delta_M \ \lambda^a + \sum_{j,k=1}^{m-1} \sum_{b,c=1}^{m-1} \gamma^a_{bc}(\lambda)\sigma^{jk}(\kappa)\partial_{\kappa_k}\lambda^b\partial_{\kappa_j}\lambda^c = 0, \tag{23}$$

where $a = 1, \cdots, m-1$.

The embedding

$$\mathbb{S}^{m-1} \subset \mathbb{R}^m$$

enables us to consider the corresponding diffeomorphism

$$\lambda \in \mathbb{R}^{m-1} \longrightarrow \theta = \theta(\lambda) \in \mathbb{S}^{m-1} \tag{24}$$

that maps a small neighborhood of the origin in $\mathbb{R}^{m-1}$ onto small neighborhood on the sphere $\mathbb{S}^{m-1}$. Then the equation (21) shows that a map

$$\kappa = (\kappa_1, \cdots, \kappa_{m-1}) \longrightarrow \theta = \theta(\kappa) = (\theta_1, \cdots, \theta_m) \in \mathbb{S}^{m-1}$$

is (locally) a harmonic map if

$$\Delta_{\mathbb{S}^{m-1}} \ \theta = -K\theta, \tag{25}$$

where $K > 0$ is a constant.

Lemma 2.1. *Let*

$$\kappa = (\kappa_1, \cdots, \kappa_{m-1}) \longrightarrow \theta = \theta(\kappa) = (\theta_1, \cdots, \theta_m) \in \mathbb{S}^{m-1}$$

be a local C^2 solution to

$$\Delta_{\mathbb{S}^{m-1}} \ \theta = -K\theta, \tag{26}$$

where $K > 0$ is a constant. Then

$$K = \sum_{j,k=1}^{m-1} \sigma^{jk}(\kappa)\langle \partial_{\kappa_j}\theta, \partial_{\kappa_k}\theta\rangle_{\mathbb{R}^m} \tag{27}$$

and

$$K = \sum_{b,c=1}^{m-1} \sum_{j,k=1}^{m-1} \sigma_{bc}(\lambda)\sigma^{jk}(\kappa)\partial_{\kappa_j}\lambda^b\partial_{\kappa_k}\lambda^c, \tag{28}$$

where

$$\lambda(\kappa) = \theta^{-1}\theta(\kappa) \tag{29}$$

and θ^{-1} is the diffeomorphism inverse to (24).

Remark 2.1. *The map* (29) *in the above Lemma can be extended as a map*

$$\omega \in \mathbb{S}^{m-1} \longrightarrow \lambda(\omega) = (\lambda_1(\omega), \cdots \lambda_{m-1}(\omega)), \tag{30}$$

since $\kappa = (\kappa_1, \cdots, \kappa_{m-1})$ *are local coordinates on* $\mathbb{S}^{m-1}$.

Proof. Multiplying the equation (26) by θ, we get

$$K = -\langle \Delta_{\mathbb{S}^m}\ \theta, \theta \rangle_{\mathbb{R}^m},$$

where (see (12))

$$\Delta_M = \sum_{j,k=1}^{m-1} \frac{1}{\sqrt{\sigma}}\ \partial_{\kappa_j}\ \sqrt{\sigma}\ \sigma^{jk}\ \partial_{\kappa_k}. \tag{31}$$

From the relation

$$\langle \theta(\kappa), \theta(\kappa) \rangle_{\mathbb{R}^m} = 1$$

we obtain

$$\langle \partial_{\kappa_k}\theta(\kappa), \theta(\kappa) \rangle_{\mathbb{R}^m} = 0$$

so

$$K = -\sum_{j,k=1}^{m-1}\ \sigma^{jk}\ \langle \partial_{\kappa_j}\partial_{\kappa_k}\ \theta, \theta \rangle_{\mathbb{R}^m} = \sum_{j,k=1}^{m-1}\ \sigma^{jk}\ \langle \partial_{\kappa_j}\ \theta, \partial_{\kappa_k}\theta \rangle_{\mathbb{R}^m}$$

and this proves the first relation (27). The second relation (28) follows from

$$\sigma_{bc}(\lambda) = \langle \partial_{\lambda^b}\theta(\lambda), \partial_{\lambda^c}\theta(\lambda) \rangle_{\mathbb{R}^m}$$

and the chain rule

$$\partial_{\kappa_j}\theta = \sum_{b=1}^{m-1} \partial_{\lambda^b}\theta \partial_{\kappa_j}\lambda^b.$$

This completes the proof of the Lemma. $\square$

To find solution to the equation (25) we follow the idea from [6] and shall look for polynomial functions

$$x = (x_1, \cdots, x_m) \in \mathbb{R}^m \longrightarrow P(x) = y = (y_1, \cdots, y_n) \in \mathbb{R}^n, \tag{32}$$

such that $P(x) = (P_1(x), \cdots, P_n(x))$ and $P_j(x)$ are harmonic polynomial in x homogeneous of order $L \geq 1$, i.e.

$$\Delta_{\mathbb{R}^m} P_j(x) = 0 \tag{33}$$

and such that

$$(x_1)^2 + \cdots (x_m)^2 = 1 \ \Rightarrow (P_1(x))^2 + \cdots (P_n(x))^2 = 1. \tag{34}$$

The homogeneity argument shows that (34) is consequence of

$$(P_1(x))^2 + \cdots (P_n(x))^2 = \left((x_1)^2 + \cdots (x_m)^2\right)^L. \tag{35}$$

Once the above problem (33) and (35) is solved, we can introduce polar coordinates

$$r = |x|, \ \omega = \frac{x}{|x|}$$

and set

$$u(\omega) = P(\omega).$$

Using the decomposition of the Laplace operator together with (33) and the relation

$$P_j(x) = r^L P_j(\omega)$$

and rewrite (33) as

$$r^{L-2} \left(L(L-1) + (m-1)L + \Delta_{\mathbb{S}^{m-1}}\right) P_j(\omega) = 0$$

so $u(\omega) = P(\omega)$ satisfies

$$\Delta_{\mathbb{S}^{m-1}} u = -L(L+m-2)u \tag{36}$$

so the equation (25) is valid with $\mu = L(L+m-2)$.

First, we consider the case $n = m = 2$. Then $\kappa \in [0, 2\pi)$ can be considered as a local coordinate on $M = \mathbb{S}^1$ while $\lambda \in [0, 2\pi)$ is the local coordinate on $N = \mathbb{S}^1$. Then $\Delta_{\mathbb{S}^1} = \partial_{\kappa\kappa}$ and setting

$$u(\kappa) = (\cos\lambda, \sin\lambda), \quad \lambda = \lambda(\kappa),$$

the equation (36) becomes

$$\begin{aligned} -\sin\lambda \partial_{\kappa\kappa}\lambda - \cos\lambda \, (\partial_\kappa \lambda)^2 &= -L^2 \cos\lambda, \\ \cos\lambda \partial_{\kappa\kappa}\lambda - \sin\lambda \, (\partial_\kappa \lambda)^2 &= -L^2 \sin\lambda. \end{aligned}$$

An obvious solution is

$$\lambda = L\kappa.$$

An alternative approach based on solution of the system (33) and (35) can be found using the embeddings

$$\mathbb{S}^1 \subset \mathbb{R}^{1+1} = \mathbb{C}.$$

If x_1, x_2 are the coordinates on $\mathbb{R}^2$ and we can define the polynomial vector valued function

$$z = x_1 + ix_2 \longrightarrow P(z) = z^L.$$

Since $\Delta_{\mathbb{C}} = 4\partial_z \partial_{\bar{z}}$, we see that $P(z)$ are harmonic polynomials of order L so (33) is satisfied. The property (35) follows from the obvious relation

$$|z^L|^2 = |z|^{2L}.$$

For $L = 2$ we obtain in particular

$$\begin{aligned} P_1(x) &= (x_1)^2 - (x_2)^2, \\ P_2(x) &= 2x_1x_2. \end{aligned} \tag{37}$$

Next, we consider the case $m = n = 3$. For $L = 1$ we can take $P_j(x) = x_j$ and see that (33) and (35) are satisfied. For $L = 2$ we use the argument of the previous case $m = n = 2$ and see that all polynomials (see (37))

$$(x_2)^2 - (x_3)^2, \quad (x_3)^2 - (x_1)^2, \quad (x_1)^2 - (x_2)^2$$

as well

$$x_1x_2, \; x_2x_3, \; x_3x_1$$

are harmonic ones. For this we choose

$$P_1(x) = a\left((x_2)^2 - (x_3)^2\right) + b\left(x_1x_2 + x_2x_3 + x_3x_1\right),$$

$$P_2(x) = a\left((x_3)^2 - (x_1)^2\right) + b\left(x_1x_2 + x_2x_3 + x_3x_1\right),$$

$$P_3(x) = a\left((x_1)^2 - (x_2)^2\right) + b\left(x_1x_2 + x_2x_3 + x_3x_1\right),$$

where a, b are suitable constants chosen so that (35) is fulfilled. Note that

$$
\begin{aligned}
(P_1(x))^2 + (P_2(x))^2 + (P_3(x))^2 = \\
&2a^2((x_1)^4 + (x_2)^4 + (x_3)^4) + \\
&+ (3b^2 - 2a^2)\left((x_1x_2)^2 + (x_2x_3)^2 + (x_3x_1)^2\right).
\end{aligned}
$$

Comparing this relation with

$$
\begin{aligned}
\left((x_1)^2 + (x_2)^2 + (x_3)^2\right)^2 = \\
&((x_1)^4 + (x_2)^4 + (x_3)^4) + 2\left((x_1x_2)^2 + (x_2x_3)^2 + (x_3x_1)^2\right)
\end{aligned}
$$

we see that it is sufficient to take

$$2a^2 = 1, \quad 3b^2 - 2a^2 = 2,$$

i.e.

$$a = \frac{1}{\sqrt{2}}, \quad b = 1.$$

With this choice we have

$$(P_1(x))^2 + (P_2(x))^2 + (P_3(x))^2 = \left((x_1)^2 + (x_2)^2 + (x_3)^2\right)^2$$

so (35) is satisfied with $L = 2$. For higher dimensional case $L \geq 3$ or for $n \geq m \geq 3$ the existence of harmonic polynomial maps satisfying (33) and (35) is discussed in [6]. For our considerations concerning the concentration of local energy for two dimensional wave maps only the case $n = m = 2$ is sufficient.

3. Equivariant wave maps and construction of special solutions

In this section we shall derive briefly the wave map equation and shall construct a special class of equivariant wave maps that solve the inhomogeneous problem (3).

The equation (1) is the Euler-Lagrange equation related to the density

$$\langle \partial_\alpha u, \partial^\alpha u \rangle_{g(u)} \tag{38}$$

that in small neighborhood of a fixed $u_0 \in N$ has the form

$$\langle \partial_\alpha u, \partial^\alpha u \rangle_{g(u)} = h^{\alpha\beta} g_{ab} \partial_\alpha u^a \partial_\beta u^b.$$

Here and below the Greek indices α, β vary from 0 to m, while the Latin indices a, b, c, d vary from 1 to n. A summation convention for repeated indices is also assumed.

The corresponding Lagrangian is given by:

$$L[u] = \int_{\mathbb{M}} h^{\alpha\beta} g_{ab} \partial_\alpha u^a \partial_\beta u^b. \tag{39}$$

Since we assumed M to be the Minkowski space R^{1+m} with the standard metric

$$h = diag(-1, 1, ..., 1),$$

we can simplify the Lagrangian :

$$L[u] = \int_{\mathbb{R}^{1+m}} g_{ab} \partial^\alpha u^a \partial_\alpha u^b. \tag{40}$$

Then the Euler - Lagrange equations become:

$$-2\partial_\alpha (g_{ab} \partial^\alpha u^b) + \partial_\alpha u^c \partial^\alpha u^b \partial_a g_{bc} = 0, \tag{41}$$

or equivalently:

$$-g_{ab} \partial_\alpha \partial^\alpha u^b - \partial_c g_{ab} \partial_\alpha u^c \partial^\alpha u^b + \frac{1}{2} (\partial_a g_{bc} \partial_\alpha u^c \partial^\alpha u^b) = 0.$$

In terms of D'Alembertian we may write :

$$g_{ab} \Box u^b + \partial_c g_{ab} \partial_\alpha u^c \partial^\alpha u^b - \frac{1}{2} (\partial_a g_{bc} \partial_\alpha u^c \partial^\alpha u^b) = 0, \tag{42}$$

where $\Box = -\partial_\alpha \partial^\alpha = \partial_0^2 - \partial_1^2 - ... - \partial_n^2$. The Christoffel symbols are given by the following expression:

$$\Gamma_{c;ab} = \frac{1}{2} (\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}). \tag{43}$$

If we write

$$\partial_c g_{ab} \partial_\alpha u^c \partial^\alpha u^b = \frac{1}{2} (\partial_c g_{ba} \partial_\alpha u^c \partial^\alpha u^b + \partial_b g_{ac} \partial_\alpha u^c \partial^\alpha u^b)$$

and use the expression of Christoffel symbols, then we arrive at the following equation

$$g_{ab}\Box u^b + \Gamma_{a;bc}\partial_\alpha u^b \partial^\alpha u^c = 0.$$

Raising the index a, we obtain

$$\Box u^a + \Gamma^a_{bc}\partial_\alpha u^b \partial^\alpha u^c = 0. \tag{44}$$

In order to handle the inhomogeneous case, a minor modification of the density (38) is sufficient:

$$\langle \partial_\alpha u, \partial^\alpha u\rangle_{g(u)} + \langle F, u\rangle_{g(u)}, \tag{45}$$

where

$$F : x = (x^0, x^1, \cdots, x^m) \in \mathbb{R}^{1+m} \to F(x) \in T_{u(x)}N$$

is the given source term. The corresponding inhomogeneous problem has the form

$$\Box u^a + \Gamma^a_{bc}\partial_\alpha u^b \partial^\alpha u^c = F^a. \tag{46}$$

As in the previous section we can rewrite these equations in extrinsic form. To this purpose assume that N is a $n-$ dimensional surface in $\mathbb{R}^{n+1}$ with metric induced by the Euclidean metric on $\mathbb{R}^{n+1}$. Thus u is a $d = n+1$-dimensional vector $u = (u_1, ..., u_d)$; on the other hand, on N we can take local coordinates $y^1, \cdots, y^n$ so that N is described locally by a chart

$$u = u(y), \qquad y \in Y \subset \mathbb{R}^n.$$

The Riemannian metric g on N is induced by the Euclidean metric on $\mathbb{R}^d$ (see (16) of the previous section)

$$g_{ab} = \langle \partial_{y^a} u, \partial_{y^b} u\rangle_{\mathbb{R}^d}, \tag{47}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ is the scalar product in $\mathbb{R}^d$.

Then the wave map (locally) is a function

$$x = (x^0, x^1, \cdots, x^m) \in X \subset \mathbb{R}^{m+1} \longrightarrow y = y(x) \in \mathbb{R}^n$$

defined in a small neighborhood X of the origin in $\mathbb{R}^{n+1}$, satisfying the intrinsic equation (46), i.e.

$$\Box y^a + \Gamma^a_{bc}\partial_\alpha y^b \partial^\alpha y^c = F^a. \tag{48}$$

It is easy to verify that the wave map

$$v(x) := u(y(x)), \quad x \in X$$

satisfies the extrinsic equation

$$\Box\ v + \sum_{\alpha,\beta=0}^{m} h^{\alpha\beta} B(v)(\partial_\alpha v, \partial_\beta v) = 0, \tag{49}$$

where

$$B(p) : T_pN \times T_pN \to T_pN^\perp$$

is the second fundamental form of $N \subset \mathbb{R}^d$. We recall the explicit form (18) of the second fundamental form from the previous section:

$$B(u)(v,w) = \sum_{a,c=1}^{n} b_{ac}(u) v^a w^c \nu(u) \tag{50}$$

for any two vectors $v, w \in T_pN$, and with coefficients b_{ac} defined as follows (see (19))

$$b_{ac} = -\langle \partial_{y^a}\partial_{y^c} u(y), \nu(u(y))\rangle_{\mathbb{R}^d}, \quad a, c = 1, \cdots, n; \tag{51}$$

$\nu(u)$ denotes as usual the unit normal at $u \in N$.

To verify the above claim it is sufficient to rewrite the energy functional in (39) as follows

$$L[u] = \int_X h^{\alpha\beta} \langle \partial_\alpha u(y(x)) \partial_\beta u(y(x))\rangle_{\mathbb{R}^d}\, \mathrm{d}x. \tag{52}$$

Taking the extremum of this integral over $u \in H^1$, under the constraint $u(y) \in N$ we obtain the Euler-Lagrange equation with Lagrange multiplier

$$\Box u = -\mu\nu(u), \tag{53}$$

where the Lagrange multiplier μ can be obtained by scalar multiplication with u:

$$\mu = h^{\alpha\beta} \langle \partial_\alpha \partial_\beta u, \nu \rangle_{\mathbb{R}^d}.$$

Using the property

$$\partial_{y^k} u \in T_u(N), \quad \nu(u) \in T_u(N)^\perp,$$

we get

$$\mu = h^{\alpha\beta} \langle \ \partial_{y^a} \partial_{y^c} u, \nu \rangle_{\mathbb{R}^d} \partial_\alpha y^a \partial_\beta y^c.$$

Now the definition (19) of the second fundamental form leads to (17).

In the case when $N = \mathbb{S}^n$, we have $\nu(u) = u$, and equation (49) simplifies to

$$\Box \ u - \sum_{\alpha,\beta=0}^{m} h^{\alpha\beta} \langle \partial_\alpha u, \partial_\beta \rangle_{\mathbb{R}^{n+1}} \ u = 0. \tag{54}$$

To recall the equivariant wave map ansatz, we shall assume that N is a smooth $n-$dimensional rotationally symmetric, wrapped product manifold defined as $N = \{(\phi, \lambda); \phi \in (0, \phi^*), \lambda \in \mathbf{S}^{n-1}\}$ with metric

$$d\phi^2 + g(\phi)^2 d\sigma^2, \tag{55}$$

where $d\sigma^2$ is the standard metric on $\mathbf{S}^{n-1}$, i.e.

$$d\sigma^2 = \sigma_{jk}(\lambda) d\lambda_j d\lambda_k,$$

while $(\lambda_1, \dots, \lambda_{n-1})$ are the local coordinates on $\mathbf{S}^{n-1}$. In these local coordinates we have

$$\begin{aligned} g_{\phi\lambda_j} &= 0, \\ g_{\phi\phi} &= 1, \\ g_{\lambda_i\lambda_j} &= g^2(\phi)\sigma_{ij}(\lambda). \end{aligned} \tag{56}$$

If at least two of indices a,b,c are ϕ, then (56) implies that $\Gamma_{a;bc} = 0$. If only one of indices a, b, c is ϕ, then $\Gamma_{\phi,\lambda_i\lambda_j} = -g'(\phi)g(\phi)\sigma_{ij}$ and $\Gamma_{\lambda_i,\lambda_j\phi} = g'(\phi)g(\phi)\sigma_{ij}$. Finally, $\Gamma_{\lambda_i,\lambda_j\lambda_k} = g^2(\phi)\gamma_{i,jk}$ where

$$\gamma_{i,jk} = \frac{1}{2}\left(\partial_{\lambda_j}\sigma_{ik} + \partial_{\lambda_k}\sigma_{ij} - \partial_{\lambda_i}\sigma_j k\right),$$

are the Christoffel symbols for the metric σ. The equivariant wave map satisfy the following ansatz

$$u_\phi(t,x) = \phi(t,r),\ u_{\lambda_j} = \lambda_j(\omega), \tag{57}$$

where

$$\omega \in \mathbf{S}^{m-1} \mapsto \lambda_j(\omega) \in \mathbb{R}, j = 1, \cdots, n-1$$

is the map of (29).

Recall that this map in the local coordinates

$$\kappa = (\kappa_1, \cdots, \kappa_{m-1})$$

on $\mathbb{S}^{m-1}$ defines a solution to the equation

$$\Delta_{\mathbb{S}^{m-1}} \lambda^j + K\lambda^j = 0,$$

where $K = L(L+m-2)$, $L \geq 1$ is an integer and

$$K = \sigma_{bc}(\lambda)\sigma^{jk}(\kappa)\partial_{\kappa_j}\lambda^b \partial_{\kappa_k}\lambda^c, \tag{58}$$

due to (28) of Lemma 2.1

Choosing $a = \phi$ in (44) we obtain

$$\Box\phi + \Gamma^{\phi}_{\lambda^b\lambda^c}(u)\partial_\alpha\lambda^b\partial^\alpha\lambda^c = 0,$$

where

$$\Gamma^{\phi}_{\lambda^b\lambda^c}(u) = -g'(\phi)g(\phi)\sigma_{bc}(\lambda).$$

Note that

$$\partial_\alpha\lambda^b\partial^\alpha\lambda^c = \sigma^{jk}(\kappa)\frac{\partial_{\kappa_j}\lambda^b\partial_{\kappa_k}\lambda^c}{r^2},$$

so from (58) we find

$$\Box\phi + \frac{Kg'(\phi)g(\phi)}{r^2} = 0. \tag{59}$$

The corresponding inhomogeneous problem is

$$\Box\phi + \frac{Kg'(\phi)g(\phi)}{r^2} = f. \tag{60}$$

In the special case, when the target is the two - dimensional sphere $\mathbf{S}^2$, the metric on $\mathbf{S}^2$ has the form $d\phi^2 + \sin^2\phi d\lambda^2$. Let u :

$\mathbb{R}\times\mathbb{R}^2 \to \mathbf{S}^2$ be an equivariant wave map. Then $u = (u_1, u_2, u_3)$ with

$$u_1 = \cos(\phi) cos(\lambda), \quad u_2 = \cos(\phi)\sin(\lambda), \quad u_3 = \sin(\phi). \tag{61}$$

Introducing polar coordinates (r,κ) in $\mathbb{R}^2$, we have $x_1 = r\cos\kappa$, $x_2 = r\sin\kappa$; so the equivariant ansatz (57) shows that $\phi = \phi(t,r)$ satisfies (59) and $\lambda = \lambda(\kappa)$ is a harmonic map between $\mathbf{S}^1$ and $\mathbf{S}^1$. For the simplest case of identity map, i.e. $\lambda(\kappa) = \kappa$ the equation (60) becomes

$$\Box\phi + \frac{\sin(2\phi)}{2r^2} = f, \tag{62}$$

where $\Box\phi = (\partial_t^2 - \partial_r^2 - \frac{1}{r}\partial_r)\phi$. The vector - valued function u in (61) solves the equation

$$u_{tt} - \Delta u + \left(|u_t|^2 - |\nabla_x u|^2\right) u = F, \tag{63}$$

provided ϕ solves the inhomogeneous equation (62). Indeed, we have the relations

$$\begin{aligned} u_t &= \partial_t\phi\partial_\phi u, \\ u_r &= \partial_r\phi\partial_\phi u, \\ u_{tt} &= -\phi_t^2 u + \phi_{tt}\partial_\phi u, \\ \partial_r^2 u &= -\left(\partial_r\phi\right)^2 u + \partial_{rr}\phi\partial_\phi u \end{aligned}$$

and the representation formula

$$\Box = \partial_t^2 - \partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\kappa^2. \tag{64}$$

From

$$|\nabla_x u|^2 = |\partial_r u|^2 + r^{-2}|\partial_\kappa u|^2$$

and (61) we get

$$|\nabla_x u|^2 = |\partial_r\phi|^2 + \frac{\cos^2\phi}{r^2}|\partial_t u|^2 = |\partial_t\phi|^2,$$

so

$$|\partial_t u|^2 - |\nabla_x u|^2 = \phi_t^2 - \phi_r^2 - \frac{\cos^2\phi}{r^2}. \tag{65}$$

Further, from (64) and (64) we find

$$\Box u = -\phi_t^2\, u + \phi_{tt}\, \partial_\phi u + \phi_r^2\, u - \phi_{rr}\, \partial_\phi u - \frac{\phi_r}{r}\, \partial_\phi u - \frac{1}{r^2}\partial_{\kappa\kappa} u. \quad (66)$$

Next, we need the following.

LEMMA 3.1. *We have the relation*

$$\partial_{\kappa\kappa} u = -\left(\cos\phi\right)^2 \; u + \frac{\sin(2\phi)}{2}\, \partial_\phi u. \quad (67)$$

Proof. Consider the vectors $e = (\cos\lambda, \sin\lambda, 0)$ and $e_3 = (0,0,1)$. Then

$$u = e\cos\phi + e_3 \sin\phi, \quad \partial_\phi u = -e\sin\phi + e_3\cos\phi$$

and from these relations we get immediately

$$e = u\cos\phi - \partial_\phi u\, \sin\phi.$$

This relation and the identity $\partial_{\kappa\kappa} u = -e\cos\phi$ imply (67). This completes the proof. □

Combining the above Lemma and (65), we obtain

COROLLARY 3.1. *If u is defined by* (61)*, then the following relation*

$$\Box u + \left(|u_t|^2 - |\nabla_x u|^2\right) u = \left(\Box\phi + \frac{\sin(2\phi)}{2r^2}\right) \partial_\phi u \quad (68)$$

holds.

We conclude this section by a final remark. If ϕ solves the inhomogeneous Cauchy problem for (60), i.e.

$$\begin{aligned} \Box\phi + \frac{\sin(2\phi)}{2r^2} &= f, \quad (69)\\ \phi(0,x) = \phi_0(|x|),\ \partial_t\phi_0(0,x) &= \phi_1(|x|) \end{aligned}$$

then we get immediately a solution of the corresponding extrinsic problem

$$\begin{aligned} u_{tt} - \Delta u + \left(|u_t|^2 - |\nabla_x u|^2\right) u &= F, \quad (70)\\ u(0,x) &= u_0(x)\\ \partial_t u_0(0,x) &= u_1(x), \end{aligned}$$

where

$$u_0 = u(\phi_0),\ u_1 = u(\phi_1),\ F = f\partial_\phi u \tag{71}$$

with $u = u(\phi)$, defined according to (61). An analogous connection exists between the Sobolev spaces associated with these two problems (see Lemmas 5.1 and 5.2 in the Appendix).

4. Solution to the Cauchy problem

From now on we will use the following notation: if f and g are two function we write $f \lesssim g$, if there exists a constant C, such that $f \leq Cg$ and $f \sim g$, if there exist constants A and B, such that $Ag \leq f \leq Bg$. Consider the equation (69) for equivariant wave maps with initial data

$$\phi(1,r) = \phi_0(r), \quad \phi_t(1,r) = \phi_1(r), \tag{72}$$

where $\phi = \phi(t,r)$ depends only on t and r. We shall construct a solution of the following special form:

$$\phi(t,r) = Q\left(\frac{v(r)}{t}\right); \tag{73}$$

the function Q must satisfy a suitable ordinary differential equation, which we derive now. The definition of ϕ implies:

$$\begin{aligned}
\partial_t\phi &= -\frac{v(r)}{t^2}Q'\left(\frac{v(r)}{t}\right),\\
\partial_t^2\phi &= \frac{(v(r))^2}{t^4}Q''\left(\frac{v(r)}{t}\right) + \frac{2v(r)}{t^3}Q'\left(\frac{v(r)}{t}\right),\\
\partial_r\phi &= \frac{v'(r)}{t}Q'\left(\frac{v(r)}{t}\right),\\
\partial_r^2\phi &= \frac{(v'(r))^2}{t^2}Q''\left(\frac{v(r)}{t}\right) + \frac{v''(r)}{t}Q'\left(\frac{v(r)}{t}\right).
\end{aligned}$$

Plugging these quantities into (69) we see that ϕ satisfies the identity:

$$\begin{aligned}\left(\partial_t^2-\Delta\right)\phi(t,x)+\frac{\sin 2\phi}{2r^2}=\frac{(v(r))^2}{t^4}Q''\left(\frac{v(r)}{t}\right)+\\+\frac{2v(r)}{t^3}Q'\left(\frac{v(r)}{t}\right)-\frac{(v'(r))^2}{t^2}Q''\left(\frac{v(r)}{t}\right)+\\-\frac{v''(r)}{t}Q'\left(\frac{v(r)}{t}\right)-\frac{v'(r)}{rt}Q'\left(\frac{v(r)}{t}\right)+\frac{\sin 2Q}{2r^2}.\end{aligned}\tag{74}$$

Our main idea is to regard all the terms involving the time derivatives as a *source term*, i.e., to choose

$$f=\partial_t^2u=\frac{(v(r))^2}{t^4}Q''\left(\frac{v(r)}{t}\right)+\frac{2v(r)}{t^3}Q'\left(\frac{v(r)}{t}\right);\tag{75}$$

then the equation (74)

$$(\partial_t^2-\Delta)\phi(t,x)+\frac{\sin 2\phi}{2r^2}=f$$

simplifies to

$$-\Delta\phi(t,x)+\frac{\sin 2\phi}{2r^2}=0,$$

and, recalling our choice of ϕ, this leads to the following equation for Q:

$$\frac{(v'(r))^2}{t^2}Q''+\frac{v(r)}{t}\frac{v'(r)+rv''(r)}{rv(r)}Q'-\frac{\sin 2Q}{2r^2}\ =0.\tag{76}$$

As $v=v(r)$ we may choose a solution of the following ordinary differential equation:

$$\frac{v'(r)+rv''(r)}{rv(r)}=\frac{c}{r^2},\tag{77}$$

where the positive constant c is a parameter to be chosen. This is an equation of Euler-type:

$$r^2v''(r)+rv'(r)-cv(r)=0.$$

A special solution to (77) is $v=r^\alpha$, provided we take $c=\alpha^2$. With these choices, the equation for Q becomes

$$\alpha^2\frac{1}{r^2}\frac{r^{2\alpha}}{t^2}Q''+\alpha^2\frac{1}{r^2}\frac{r^\alpha}{t}Q'-\frac{\sin 2Q}{2r^2}\ =0.\tag{78}$$

Setting for brevity

$$z = \frac{r^\alpha}{t},$$

we can rewrite (78) as follows:

$$\alpha^2 z^2 Q'' + \alpha^2 z Q' - \frac{\sin 2Q}{2} = 0. \tag{79}$$

Now, making change of the variable

$$s = \frac{1}{\alpha} \ln z,$$

we get

$$Q''(s) - \frac{\sin 2Q(s)}{2} = 0. \tag{80}$$

We have not yet chosen the initial data for Q. Multiplying the equation by Q' we obtain

$$(Q')^2 + \frac{1+\cos 2Q}{2} = const \tag{81}$$

and this means that the quantity

$$I(s) = (Q')^2 + \frac{1+\cos 2Q}{2} \equiv (Q')^2 + (\cos Q)^2$$

is constant on the integral curves of (80) or, in other words, is a first integral of the equation. Now we may choose the initial data for Q such that $I(s)$ is equal to 1: indeed, it is sufficient to take

$$Q(0) = \frac{\pi}{2}, \quad Q'(0) = 1 \qquad \Longrightarrow \qquad I(s) = (Q')^2 + (\cos Q)^2 = 1.$$

The last equation has the two solutions

$$\tan\left(\frac{Q}{2}\right) = c_0 e^{\pm s} \ ,$$

and our choice of the initial data for Q implies $c_0 = 1$. Discarding the solution with sign $-$ we finally obtain

$$Q(z) = 2 \ \arctan\left(z^{\frac{1}{\alpha}}\right). \tag{82}$$

Our next step is to study the regularity properties of the remainder $f(t,r)$ defined by

$$f(t,r) = \frac{r^{2\alpha}}{t^4} Q''\left(\frac{r^\alpha}{t}\right) + 2\frac{r^\alpha}{t^3} Q'\left(\frac{r^\alpha}{t}\right) \tag{83}$$

for $r^\alpha < t$. In fact, we shall prove the following.

LEMMA 4.1. *For any* $\alpha \in [\alpha_0, 1)$ *with some fixed* $\alpha_0, 0 < \alpha_0 < 1$, *and for any* $T > 0$ *we have*

$$f \in L^1((0,T); {H^0}_{rad}(\Omega_\alpha(t))).$$

Proof. As before set

$$z = \frac{r^a}{t}.$$

From the equations (83) and (79) for Q we have:

$$f = \frac{1}{t^2}\left(zQ' + \frac{\sin 2Q}{2\alpha^2}\right). \tag{84}$$

So, the norm of f in ${H^0}_{rad}(\Omega_\alpha(t)) = {L^2}_{rad}(\Omega_\alpha(t))$ for fixed $t \in (0,T)$ is:

$$\|f(t,.)\|^2_{{L^2}_{rad}(\Omega_\alpha(t))} = \frac{1}{t^4}\int_0^{t^{\frac{1}{\alpha}}} \left(zQ'(z) + \frac{\sin 2Q}{2\alpha^2}\right)^2 r\,\mathrm{d}r, \tag{85}$$

where in terms of z we have

$$Q'(z) = \frac{2}{\alpha}\,\frac{z^{\frac{1}{\alpha}-1}}{1+z^{\frac{2}{\alpha}}}$$

so we get

$$\begin{aligned}
\|f(t,.)\|^2_{{L^2}_{rad}(\Omega_\alpha(t))} &\leq \frac{C}{t^4}\int_0^{t^{\frac{1}{\alpha}}} \left(\frac{z^{\frac{2}{\alpha}}}{(1+z^{\frac{2}{\alpha}})^2} + (\sin 2Q)^2\right) r\,\mathrm{d}r \\
&\leq \frac{C}{t^4}\int_0^{t^{\frac{1}{\alpha}}} \frac{z^{\frac{2}{\alpha}}}{(1+z^{\frac{2}{\alpha}})^2}\, r\,\mathrm{d}r + \\
&\qquad + \frac{C}{t^4}\int_0^{t^{\frac{1}{\alpha}}} (\sin 2Q)^2\, r\,\mathrm{d}r \\
&\leq \frac{C}{t^4}\left(\int_0^{t^{\frac{1}{\alpha}}} \frac{r^3}{t^{\frac{2}{\alpha}}}\,\mathrm{d}r + t^{\frac{2}{\alpha}}\right) = \frac{C}{t^{4-\frac{2}{\alpha}}}
\end{aligned}$$

where we have used the inequality $\frac{1}{1+z^{\frac{2}{\alpha}}} \leq 1$ and $C = C(\alpha_0, T) > 0$ is a constant, independent of α, t. Finally, we have the following inequality:

$$\|f(t,\cdot)\|_{L^2{}_{rad}(\Omega_\alpha(t))} \quad \leq \quad \frac{C}{t^{2-\frac{1}{\alpha}}}. \tag{86}$$

Now it is obvious that the function $t^{-2+1/\alpha}$ is in $L^1(0,T)$, if $\alpha < 1$. This completes the proof. □

The next lemma shows that $\phi(t,.)$ belongs to the energy space H^1.

Lemma 4.2. *The solution $\phi(t,r)$ of* (69)*, defined according to* (73) *belongs to $H^1{}_{rad}(\Omega_\alpha(t))$ for every fixed $t > 0$.*

Proof. For any fixed $t > 0$ we have

$$\begin{aligned}
\|\phi(t,.)\|^2_{\dot{H}^1_{rad}} &= \int_0^{t^{\frac{1}{\alpha}}} |\partial_r\phi|^2 r \,\mathrm{d}r \\
&\leq C\int_0^{t^{\frac{1}{\alpha}}} \left(\frac{z^{\frac{2}{\alpha}}}{(1+z^{\frac{2}{\alpha}})^2}\right)\frac{1}{r}\,\mathrm{d}r \\
&\leq C\int_0^{t^{\frac{1}{\alpha}}} \frac{r}{t^{\frac{2}{\alpha}}}\,\mathrm{d}r = Const. \qquad (87)
\end{aligned}$$

Note that the reverse inequality also holds. Indeed, we have

$$\begin{aligned}
\|\phi(t,.)\|^2_{\dot{H}^1_{rad(loc)}(\Omega_\alpha(t))} &= \int_{\Omega_\alpha(t)} |\partial_r\phi|^2 r \,\mathrm{d}r \\
&= C\int_0^{t^{\frac{1}{\alpha}}} \left(\frac{z^{\frac{2}{\alpha}}}{(1+z^{\frac{2}{\alpha}})^2}\right)\frac{1}{r}\,\mathrm{d}r \\
&\geq C\int_0^{t^{\frac{1}{\alpha}}} \frac{r}{t^{\frac{2}{\alpha}}}\,\mathrm{d}r = Const \qquad (88)
\end{aligned}$$

where we used inequalities

$$\frac{1}{2} \leq \frac{1}{1+z^{\frac{2}{\alpha}}} \leq 1, \quad \forall z \in [0,1].$$

The solution of the equation has the form

$$\phi = 2 \ \arctan\left(z^{\frac{1}{\alpha}}\right) \lesssim z^{\frac{1}{\alpha}} \equiv w$$

so we have the bound

$$\|u(t,.)\|^2_{L^2{}_{rad}(\Omega_\alpha(t))} \lesssim \|w(t,.)\|^2_{L^2{}_{rad}(\Omega_\alpha(t))} .$$

Then we get

$$\|\phi(t,.)\|_{L^2{}_{rad}(\Omega_\alpha(t))} \leq C \, \|w(t,.)\|^2_{L^2{}_{rad}(\Omega_\alpha(t))} = \int_0^{t^{\frac{1}{\alpha}}} \frac{r^2}{t^{\frac{2}{\alpha}}} r \, \mathrm{d}\, r = Ct^{\frac{1}{\alpha}}.$$

So our lemma is proved. □

In the next section we will improve the regularity estimates of the solution and the source term.

5. Higher regularity of the solution

To study higher regularity properties of the solution, constructed in the previous section, we will need the definition of the Sobolev spaces H^s for fractional $s > 0$.

Definition 5.1. *(see for instance[Triebel 2.5.1]) We say that the function $f \in H^s(\mathbb{R}^2), s > 0$, if the following condition holds:*

$$\|f\|_{H^s(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R}^2)} + \\ + \left(\int_{\mathbb{R}^2} |h|^{-(1+2s)} \left\| \Delta^{[s]+1}{}_h f(x) \right\|^2_{L^2(\mathbb{R}^2)} \mathrm{d}h \right)^{1/2} < C. \quad (89)$$

Here $\Delta^k{}_h f(x)$ is the difference of order k defined inductively as follows: the difference of order 0 and 1 of the function f are given by $\Delta^0{}_h f(x) = f(x)$, $\Delta^1{}_h f(x) = f(x+h) - f(x)$. Then the difference of order k is defined inductively: $\Delta^k{}_h f(x) = \Delta^1{}_h(\Delta^{k-1}{}_h f(x))$.

To define Sobolev spaces in domain $\Omega \subset \mathbb{R}^2$ we recall that the function $\tilde{f}$ (defined in $\mathbb{R}^2$) is an extension of the function f (defined in Ω), if $\tilde{f}_{\restriction \Omega} = f$. Then the definition of the Sobolev space $H^s(\Omega)$ is the following:

Definition 5.2. *We say that the function* $f \in H^s(\Omega), s > 0,$ *if the following condition holds:*

$$\|f\|_{H^s(\Omega)} = \inf_{\tilde{f}} \|\tilde{f}\|_{H^s(\mathbb{R}^2)} < \infty, \tag{90}$$

where the inf *is taken over all extensions* $\tilde{f} \in H^s(\mathbb{R}^2)$.

In our case we will need some refined version of this definition, i.e. we want to understand what happens when the function is radially symmetric and $s \in (0,2)$. First we start with the case $s \in (0,1)$.

Definition 5.3. *Let* $0 < s < 1$. *We say that* $f = f(r)$ *belongs to the space* $H^s_{rad}(\mathbb{R}^2)$, *if the following condition holds:*

$$\|f\|_{H^s{}_{rad}(\mathbb{R}^2)} = \|f\|_{L^2{}_{rad}(\mathbb{R}^2)} + \\ + \left(\int_0^1 |h|^{-(1+2s)} \int_0^\infty \left|\Delta^1{}_h f(r)\right|^2 r \,\mathrm{d}r \,\mathrm{d}h\right)^{1/2} < C. \tag{91}$$

The definition of the Sobolev spaces in a bounded domain in this case is analogous to the previous one. Next we have to study the case $s \in (1,2)$.

Definition 5.4. *Let* $s = 1+\varepsilon \in (1,2)$. *We say that* $f = f(r)$ *belongs to the space* $H^{1+\varepsilon}_{rad}(\mathbb{R}^2)$, *if* $f \in H^1_{rad}(\mathbb{R}^2)$ *and the following condition holds:*

$$\|f\|_{H^{1+\varepsilon}{}_{rad}(\mathbb{R}^2)} = \|f\|_{H^1{}_{rad}(\mathbb{R}^2)} + \\ + \left(\int_0^1 |h|^{-(1+2\varepsilon)} \int_0^\infty \left|\Delta^1{}_h \partial_r f(r)\right|^2 r \,\mathrm{d}r \,\mathrm{d}h\right)^{1/2} < C. \tag{92}$$

In the special case, when the domain Ω is a sphere with center at the origin and of radius $R > 0$ and the function $f(x) = f(|x|)$ is radially symmetric, we have the following obvious inequality

$$\|f\|_{H^s(|x|<R)} \geq \|f\|_{L^2(|x|<R)} + \\ + \left(\int_{|h|<R/2} |h|^{-(1+2s)} \int_0^{R/2} \left|\Delta^{[s]+1}{}_h f(r)\right|^2 r \,\mathrm{d}r \,\mathrm{d}h\right)^{1/2}. \tag{93}$$

On the other hand, for $R = 1$ and for any integer k the norm $\|f\|_{H^k(|x|<1)}$ is equivalent to the norm

$$\sum_{|\alpha|\leq k} \|\partial_x^\alpha \tilde{f}\|_{L^2(|x|<2)}.$$

Here

$$\tilde{f}(r) = \begin{cases} f(r), & \text{if } 0 \leq r < 1; \\ f(2-r)\varphi(r-1), & \text{if } 1 < r < 2. \end{cases} \tag{94}$$

Here $\varphi(s)$ is the standard cut – off function, such that $\varphi(s) = 1$ for $|s| \leq 1/2$ and $\varphi(s) = 0$ for $|s| \geq 1$.

Using this fact and an interpolation argument, we see that for any real $s > 0$

$$\|f\|_{H^s(|x|<1)} \sim \|\tilde{f}\|_{H^s(\mathbb{R}^2)}. \tag{95}$$

To prove Theorem 1.1 we can use the equivalence of the H^s norms of the solutions u and ϕ obtained in Lemma 5.1 (the corresponding relation between the source terms are given in Lemma 5.2) and reduce the proof to the analysis of the solution $\phi = \phi(t,r)$ to the Cauchy problem (69). Making a shift in time we can impose initial data conditions of type (72).

Then the key point to prove our main result is to improve the regularity result of the previous section. In fact, we shall show that the source term $f = f(t,r)$, defined in (83), is in the class $L^1((0,T); H^\varepsilon_{rad}(\Omega_\alpha(t)))$, while the solution $\phi = \phi(t,r)$ defined in (73) belongs to

$$C((0,T]; H^{1+\varepsilon}_{rad}(\Omega_\alpha(t)))$$

for $0 < \varepsilon < 1 - \alpha$. In this way the proof of Theorem 1.1 can be reduced to the proof of the following.

Theorem 5.1. *Let $\alpha \in (0,1)$ and $T > 0$. For any $\varepsilon > 0$ such that*

$$\varepsilon < \min\left(\frac{1}{2}, 1-\alpha\right)$$

we have the properties:

$$f \in L^1((0,T); H^\varepsilon_{rad}(\Omega_\alpha(t))); \tag{96}$$

$$\phi \in C((0,T]; H^{1+\varepsilon}_{rad}(\Omega_\alpha(t))); \tag{97}$$

$$\lim_{t\to 0_+} \|\phi(t,\cdot)\|_{H^{1+\varepsilon}_{rad}(\Omega_\alpha(t))} = \infty. \tag{98}$$

Proof. There is no loss of generality if we assume $T = 1$. First of all, we will estimate the $H^{\varepsilon}_{rad}(\Omega_\alpha(t))$-norm of $f = f(t,r)$ for fixed $t \in (0,1)$; recall that in the previous section we have estimated the $L^2_{rad}(\Omega_\alpha(t))$-norm of $f = f(t,r)$. Applying Definition (90), we see that it is sufficient to estimate the quantity

$$\int_0^1 \frac{1}{h^{1+2\varepsilon}} \int_0^{2\theta} |\Delta_h^1 \tilde{f}(t,r)|^2 r \; \mathrm{d}r \; \mathrm{d}h, \tag{99}$$

where $\theta = t^{1/\alpha}$, the extension $\tilde{f}$ is constructed as follows:

$$\tilde{f}(t,r) = \begin{cases} f(t,r), & \text{if } 0 < r < \theta; \\ f(t,r)\varphi((r-\theta)/\theta)), & \text{if } r > \theta, \end{cases} \tag{100}$$

and $\varphi(s)$ is a standard cut-off function, such that $\varphi(s) = 1$ for $|s| \le 1/2$ and $\varphi(s) = 0$ for $|s| \ge 1$. Note that

$$|\Delta_h^1 \varphi_\theta(r)| \le C \frac{h}{\theta} \tag{101}$$

with $\varphi_\theta(r) = \varphi((r-\theta)/\theta))$ and C is a constant, independent of $\theta \in (0,1)$.

In the quantity (99) we can split the integral in r in the two pieces $0 < r < \theta$ and $\theta < r < 2\theta$; we shall estimate only the first piece, i.e.,

$$I(t) = \int_0^1 \frac{1}{h^{1+2\varepsilon}} \int_0^{\theta} |\Delta_h^1 \tilde{f}(t,r)|^2 r \, \mathrm{d}r \, \mathrm{d}h, \tag{102}$$

since the estimate of the second piece

$$\int_0^1 \frac{1}{h^{1+2\varepsilon}} \int_\theta^{2\theta} |\Delta_h^1 \tilde{f}(t,r)|^2 r \; \mathrm{d}r \; \mathrm{d}h,$$

is completely similar and uses only (101) in addition to the argument presented below.

Recalling (84), we know that the function f is given by the following expression:

$$f(t,r) = \frac{1}{t^2} \left(\frac{2r\theta}{\alpha(r^2+\theta^2)} + \frac{1}{2\alpha^2} \sin 2Q \right)$$

where $\theta = t^{1/\alpha}$. Now change the order of integration in $I(t)$, first with respect to h, then with respect to r, and divide the integral in two parts:

$$I_1 = \int_0^\theta \int_0^{\theta - r} \frac{1}{h^{1+2\varepsilon}} \left|\Delta_h^1 f(t,r)\right|^2 r \; \mathrm{d}h \; \mathrm{d}r, \tag{103}$$

$$I_2 = \int_0^\theta \int_{\theta - r}^1 \frac{1}{h^{1+2\varepsilon}} \left|\Delta_h^1 \tilde{f}(t,r)\right|^2 r \; \mathrm{d}h \; \mathrm{d}r. \tag{104}$$

This allows to simplify the estimate for the function $f = f(t,r)$. Indeed, writing explicitly the integral I_1 and using the trivial estimate $(a+b)^2 \leq 2(a^2+b^2)$, we arrive at the following expression:

$$I_1 \leq C \int_0^\theta \frac{r}{t^4} \int_0^{\theta - r} \frac{1}{h^{(1+2\varepsilon)}} \left| \frac{(r+h)\theta}{(r+h)^2+\theta^2} - \frac{r\theta}{r^2+\theta^2} \right|^2$$

$$+ \frac{1}{h^{(1+2\varepsilon)}} \left|(\sin 2Q(r+h) - \sin 2Q(r))\right|^2 \; \mathrm{d}h \; \mathrm{d}r \lesssim$$

$$\lesssim \int_0^\theta \frac{r}{t^4} \int_0^{\theta - r} \frac{\theta^2}{h^{(1+2\varepsilon)}} \left| \frac{h(\theta^2 - r^2 - rh)}{((r+h)^2+\theta^2)(r^2+\theta^2)} \right|^2$$

$$+ \frac{1}{h^{(1+2\varepsilon)}} \frac{1}{2\alpha^2} \left|\sin 2Q(r+h) - \sin 2Q(r)\right|^2 \; \mathrm{d}h \; \mathrm{d}r. \tag{105}$$

The function $\theta^2 - (r^2 + rh) \geq 0$ for $h \in [0, \theta - r]$ has a maximum at $h = 0$, so we have

$$\left|\theta^2 - (r^2 + rh)\right| \leq (\theta^2 - r^2).$$

On the other hand, we know that

$$\frac{1}{((r+h)^2+\theta^2)(r^2+\theta^2)} \leq \frac{1}{(r^2+\theta^2)^2}$$

and

$$\left|\sin 2Q(r+h) - \sin 2Q(r)\right| \leq \left|2Q(r+h) - 2Q(r)\right| \leq 2\left|Q'(r+\omega h)h\right|$$

for some $0 < \omega < 1$. Note that

$$Q'(r+\omega h) = \frac{2\theta}{(r+\omega h)^2+\theta^2},$$

so we see that the following inequality is true:

$$\frac{2\theta}{(r+\omega h)^2+\theta^2} \leq \frac{2\theta}{(r^2+\theta^2)}.$$

Hence, the integral I_1 can be estimated by the following chain of inequalities:

$$\begin{aligned}
I_1 &\lesssim \int_0^\theta \frac{r}{t^4}\left(\int_0^{\theta-r} h^{(1-2\varepsilon)}\frac{(\theta^2-r^2)^2}{(r^2+\theta^2)^4}\theta^2 + \frac{h^{(1-2\varepsilon)}}{2\alpha^2}\frac{4\theta^2}{(r^2+\theta^2)^2}\,\mathrm{d}h\right)\mathrm{d}r \\
&\lesssim \frac{\theta^2}{t^4}\int_0^\theta r(\theta-r)^{(2-2\varepsilon)}\frac{\theta^4}{(r^2+\theta^2)^4}\,\mathrm{d}r + \\
&\qquad + \frac{1}{t^4}\int_0^{2\theta} r(\theta-r)^{(2-2\varepsilon)}\frac{4\theta^2}{(r^2+\theta^2)^2}\,\mathrm{d}r \\
&\lesssim \frac{1}{\theta^2 t^4}\int_0^\theta r(\theta-r)^{(2-2\varepsilon)}\,\mathrm{d}r + \\
&\qquad + \frac{1}{\theta^2 t^4}\int_0^\theta r(\theta-r)^{(2-2\varepsilon)}\,\mathrm{d}r.
\end{aligned} \tag{106}$$

The function $(\theta-r)^{2-2\varepsilon}$ is decreasing in the interval $[0,\theta]$, so we have that $(\theta-r)^{2-2\varepsilon} \leq \theta^{2-2\varepsilon}$. This remark gives the estimate

$$I_1 \lesssim \frac{\theta^{2-2\varepsilon}}{t^4} \tag{107}$$

and recalling that $\theta = t^{1/\alpha}$ we obtain

$$I_1 \lesssim \frac{1}{t^{4-\frac{2(1-\varepsilon)}{\alpha}}} . \tag{108}$$

Consider now the second integral:

$$I_2 = \int_0^\theta \int_{\theta-r}^1 \frac{1}{h^{1+2\varepsilon}}\left|\Delta_h^1 \tilde{f}(t,r)\right|^2 r\ \mathrm{d}h\ \mathrm{d}r. \tag{109}$$

The obvious estimate

$$|a-b|^2 \le 2(a^2+b^2)$$

implies that

$$\left|\Delta_h^1 \tilde{f}(t,r)\right|^2 \le 2\left|\tilde{f}(t,r)\right|^2 + 2\left|\tilde{f}(t,r+h)\right|^2.$$

Further, we have the estimates

$$|\tilde{f}(t,r)| \le \frac{C}{t^2}\left(\frac{r\theta}{r^2+\theta^2}+1\right),$$

$$|\tilde{f}(t,r+h)| \le \frac{C}{t^2}\left(\frac{r\theta}{(r+h)^2+\theta^2}+1\right) + \frac{C}{t^2}\frac{h\theta}{(r+h)^2+\theta^2},$$

and using the inequalities

$$\frac{r\theta}{(r+h)^2+\theta^2} \le \frac{r\theta}{r^2+\theta^2} \le \frac{1}{2},$$

$$\frac{rh}{(r+h)^2+\theta^2} \le \frac{rh}{h^2+\theta^2} \le \frac{1}{2}$$

we obtain

$$\left|\Delta_h^1 \widetilde{f}(t,r)\right|^2 \le \frac{C}{t^2}.$$

Then I_2 can be estimated as follows:

$$\begin{aligned}
I_2 &\le \frac{C}{t^4}\int_0^\theta r \int_{\theta-r}^1 \frac{1}{h^{1+2\varepsilon}}\ \mathrm{d}h\ \mathrm{d}r \\
&\le C\int_0^\theta \frac{r}{t^4}\left(\frac{1}{(\theta-r)^{2\varepsilon}}-1\right)\mathrm{d}r \\
&\lesssim \frac{1}{t^4}\left(\int_0^\theta \frac{r}{(\theta-r)^{2\varepsilon}}\,\mathrm{d}r\right) \\
&\lesssim \frac{1}{t^4}\theta^{2-2\varepsilon} \\
&\lesssim \frac{1}{t^{4-\frac{2-2\varepsilon}{\alpha}}}. \qquad (110)
\end{aligned}$$

Now, Definition 5.3 shows that

$$\|f\|^2_{H^{\varepsilon}{}_{rad}(\Omega_\alpha(t))} \leq \left\|\tilde{f}\right\|^2_{L^2{}_{rad}(|x|<2\theta)} + \\ + \int_0^1 h^{-(1+2\varepsilon)} \int_0^{2\theta} \left|\Delta^1{}_h \tilde{f}(r)\right|^2 r \ \mathrm{d}r \ \mathrm{d}h. \quad (111)$$

The estimate for the first term on the right-hand side (see(86)) is

$$\|f(t,\cdot)\|_{L^2{}_{rad}(|x|<2\theta)} \quad \leq \quad \frac{C}{t^{2-\frac{1}{\alpha}}}. \quad (112)$$

On the other hand, we have just established the following estimate of the second term:

$$\int_0^1 \frac{1}{h^{1+2\varepsilon}} \int_0^{2\theta} |\Delta_h^1 \widetilde{f}(t,r)|^2 r \ \mathrm{d}r \ \mathrm{d}h \lesssim \frac{1}{t^{4-\frac{2-2\varepsilon}{\alpha}}}. \quad (113)$$

Summing up we obtain

$$\|f\|_{H^{\varepsilon}{}_{rad}(\Omega_\alpha(t))} \lesssim \frac{1}{t^{2-\frac{1-\varepsilon}{\alpha}}}. \quad (114)$$

With this estimate we have that if $\varepsilon < 1-\alpha$, then the function $f = f(t,r)$ lies in the desired space, i.e. $f \in L^1((0,1); H^{\varepsilon}_{rad}(\Omega_\alpha(t)))$.

Our next step is to show that the solution $\phi = \phi(t,r)$, defined in (73), belongs to $H^{1+\varepsilon}_{rad}(\Omega_\alpha(t))$. To this purpose, we need an estimate for the function $\phi = \phi(t,r)$ similar to the one just proved for f. We proceed in a similar way: first of all, we extend ϕ as it was done in (100) and consider the corresponding extension $\widetilde{\phi}$. The estimate of the $H^1_{rad}(\Omega_\alpha(t))$-norm of $\phi = \phi(t,r)$ at a fixed time $t > 0$ was obtained in Lemma 4.2. Now we will estimate from above and from below the integral

$$J = \int_0^1 h^{-(1+2\varepsilon)} \int_0^{2\theta} \left|\Delta^1{}_h \partial_r \widetilde{\phi}(t,r)\right|^2 r \ \mathrm{d}r \ \mathrm{d}h.$$

Thus, in particular, we shall prove that the solution is in the desired space for strictly positive t.

As above, it is sufficient to consider the integral

$$J_0 = \int_0^1 h^{-(1+2\varepsilon)} \int_0^{\theta} \left|\Delta^1{}_h \partial_r \widetilde{\phi}(t,r)\right|^2 r \ \mathrm{d}r \ \mathrm{d}h,$$

and again, we can split the integral J_0 as follows:

$$J_0 = J_1 + J_2,$$

where

$$J_1 = \int_0^\theta h^{-(1+2\varepsilon)} \int_0^{\theta - r} \left| \Delta^1{}_h \partial_r \widetilde{\phi}(t,r) \right|^2 r \ \mathrm{d}r \ \mathrm{d}h,$$

while

$$J_2 = \int_0^\theta h^{-(1+2\varepsilon)} \int_{\theta - r}^1 \left| \Delta^1{}_h \partial_r \widetilde{\phi}(t,r) \right|^2 r \ \mathrm{d}r \ \mathrm{d}h.$$

We have the following upper bound for J_1:

$$\begin{aligned}
J_1 &= \int_0^\theta 2\theta^2 r \int_0^{\theta - r} \frac{1}{h^{(1+2\varepsilon)}} \left| \frac{1}{(r+h)^2 + \theta^2} - \frac{1}{r^2 + \theta^2} \right|^2 \ \mathrm{d}h \ \mathrm{d}r \\
&\lesssim \int_0^\theta 2\theta^2 r \int_0^{\theta - r} \frac{(2r+h)^2 h^2}{h^{(1+2\varepsilon)} (r^2+\theta^2)^4} \ \mathrm{d}h \ \mathrm{d}r \\
&\lesssim \int_0^\theta 2\theta^2 r \int_0^{\theta - r} \frac{(r+\theta)^2 h^{1-2\varepsilon}}{\theta^8} \ \mathrm{d}h \ \mathrm{d}r \\
&\lesssim \frac{1}{\theta^6} \int_0^\theta r(\theta - r)^{2-2\varepsilon} (\theta + r)^2 \ \mathrm{d}r \\
&\lesssim \frac{1}{\theta^{2\varepsilon}}. \qquad (115)
\end{aligned}$$

In a similar way we obtain the lower bound for J_1

$$\begin{aligned}
J_1 &\gtrsim \int_0^\theta \theta^2 r \int_0^{\theta - r} \frac{(2r+h)^2 h^2}{h^{(1+2\varepsilon)} (r^2+\theta^2)^4} \ \mathrm{d}r \ \mathrm{d}h \\
&\gtrsim \int_0^\theta \theta^2 r \int_0^{\theta - r} \frac{h^4}{h^{(1+2\varepsilon)} \theta^8} \mathrm{d}h \ \mathrm{d}r.
\end{aligned}$$

Taking the smaller domain of integration

$$\{r \in [0, \theta/2], h \in [\theta/4, \theta/2]\}$$

we get

$$J_1 \gtrsim \theta^{-3-2\varepsilon} \int_0^{\theta/2} r \int_{\theta/4}^{\theta/2} \ \mathrm{d}h \ \mathrm{d}r \sim \frac{C}{\theta^{2\varepsilon}}. \qquad (116)$$

Concerning the second integral J_2, we shall establish only an upper bound, since 0 is a sufficient lower bound. We proceed in a way similar to the study of I_2 and find

$$
\begin{aligned}
J_2 &\lesssim \int_0^\theta \theta^2 r \int_{\theta-r}^1 \frac{1}{h^{(1+2\varepsilon)}} \left| \frac{1}{(r+h)^2+\theta^2} + \frac{1}{r^2+\theta^2} \right|^2 \mathrm{d}h \, \mathrm{d}r \\
&\lesssim \int_0^\theta \theta^2 r \int_{\theta-r}^1 \frac{1}{h^{(1+2\varepsilon)}} \frac{1}{(r^2+\theta^2)^2} \, \mathrm{d}h \ \mathrm{d}r \\
&\lesssim \frac{1}{\theta^2} \int_0^\theta r \left(\frac{1}{(\theta-r)^{2\varepsilon}} - 1 \right) \, \mathrm{d}r \\
&\lesssim \frac{1}{\theta^{2\varepsilon}}
\end{aligned}
\tag{117}
$$

provided $2\varepsilon < 1$.

In conclusion, the estimates from above proved for J_1, J_2 yield (97), while property (98) follows from the lower bound for J_1 obtained in (116). This completes the proof of the theorem. □

5.1. Appendix

In this section we will prove the two technical lemmas needed in the proof of Theorem 1.1. We begin with a definition:

DEFINITION 5.5. *The space* $\mathbb{H}(\varepsilon) := \dot{H}^{1+\varepsilon}_{rad} \cap \dot{H}^1_{rad}$ *is the Hilbert space obtained by completing* $C_0^\infty(\mathbb{R}^2)$ *with respect to the norm* $\|v\|_{\mathbb{H}} := \|v\|_{\dot{H}^1} + \|v\|_{\dot{H}^{1+\varepsilon}}$.

The definition of the space $H^{1+\varepsilon}(\Omega)$ for $\Omega \subset \mathbb{R}^2$ is the following (see e.g. [16], Definition 4.2.1.1):

DEFINITION 5.6. *We say that the distribution* u *belongs to the space* $H^{1+\varepsilon}(\Omega)$ *if there exists an extension* $\tilde{u} \in H^{1+\varepsilon}(\mathbb{R}^2)$ *and in this case the* $H^{1+\varepsilon}(\Omega)$*-norm of* u *is given by:*

$$
\|u\|_{H^{1+\varepsilon}(\Omega)} = \inf_{\tilde{u}} \|\tilde{u}\|_{H^{1+\varepsilon}(\mathbb{R}^2)},
$$

where inf *is taken over all extensions* $\tilde{u}$ *of* u.

Our first result applies to the function $u(\phi)$ defined as in (61) as follows

$$u_1 = \cos(\phi)cos(\lambda), \quad u_2 = \cos(\phi)\sin(\lambda), \quad u_3 = \sin(\phi) \tag{118}$$

with

$$\phi = \phi(t, r) = 2 \ \arctan\left(\frac{r}{t^{1/\alpha}}\right).$$

Then we have:

LEMMA 5.1. *For every function $\phi = \phi(r) \in \mathbb{H}(\varepsilon) = \dot{H}^{1+\varepsilon}_{rad} \cap \dot{H}^1_{rad}$ such that $\phi'(r) \geq 0$ (almost everywhere in $r > 0$) we have: the estimates*

$$\|u(\phi)\|_{\mathbb{H}(\varepsilon)} \leq C\|\phi\|_{\mathbb{H}(\varepsilon)}, \quad \forall \ \ \varepsilon \in [0, 1], \tag{119}$$

$$\|u(\phi)\|_{\mathbb{H}(\varepsilon)} \geq C_1\|\phi\|_{\mathbb{H}(\varepsilon)}, \quad \forall \ \ \varepsilon \in [0, 1]. \tag{120}$$

Proof. The proof relies on the fact that the function u takes its values on the unit sphere, i.e., the vector-valued function u has norm 1 as a vector in $\mathbb{R}^3$. The first inequality then is just a special case of Theorem 1 in Section 5.3.6 of the book of Runst and Sickel ([11]), and we shall not reproduce it here.

Consider now the second estimate from below We will study separately the cases $\varepsilon = 0$ and $\varepsilon = 1$ first. In the case $\varepsilon = 0$ we have

$$\begin{aligned} \|u(\phi)\|^2_{\dot{H}^1_{rad}} &= \int_0^\infty |\partial_r(u(\phi(r)))|^2 r \ \mathrm{d}r \\ &= \int_0^\infty |u'(\phi(r))|^2 |\phi'(r)|^2 r \ \mathrm{d}r \\ &= \int_0^\infty |\phi'(r)|^2 r \ \mathrm{d}r \\ &= \|\phi\|^2_{\dot{H}^1_{rad}}. \end{aligned}$$

For the case $\varepsilon = 1$ we have analogously:

$$\begin{aligned}
\|u(\phi)\|^2_{\dot{H}^2_{rad}} &= \int_0^\infty |\partial_r^2(u(\phi(r)))|^2 r\,\mathrm{d}r \\
&= \int_0^\infty |u''(\phi(r))(\phi'(r))^2 + \\
&\qquad\qquad +u'(\phi)\phi''(r)|^2 r\,\mathrm{d}r \\
&= \int_0^\infty (|u''(\phi(r))|^2(\phi'(r))^2 + \\
&\quad 2\left(u'(\phi(r)), u''(\phi(r))\right)\phi'(r)\phi''(r) + \\
&\quad\quad +|u'(\phi(r))|^2(\phi''(r))^2)r\ \mathrm{d}r \\
&= \int_0^\infty \left((\phi'(r))^2 + (\phi''(r))^2\right) r\ \mathrm{d}r \\
&\geq \|\phi\|^2_{\dot{H}^2_{rad}},
\end{aligned}$$

since the vectors $u'(\phi(r))$ and $u''(\phi(r))$ are orthogonal.

We now consider the fractional case $0 < \varepsilon < 1$. We have

$$\begin{aligned}
\|u(\phi)\|^2_{\dot{H}^{1+\varepsilon}_{rad}} &= \int_0^1 \frac{1}{h^{1+2\varepsilon}} \int_0^\infty |\triangle_h \partial_r(u(\phi(r)))|^2 r\ \mathrm{d}r\,\mathrm{d}h \\
&= \int_0^1 \frac{1}{h^{1+2\varepsilon}} \int_0^\infty |u'(\phi(r+h))(\phi'(r+h)) + \\
&\quad -u'(\phi(r))\phi'(r)|^2 r\ \mathrm{d}r\,\mathrm{d}h \\
&= \int_0^1 \frac{1}{h^{1+2\varepsilon}} \int_0^\infty r|u'(\phi(r+h))|^2(\phi'(r+h))^2 + \\
&\quad -2r\left(u'(\phi(r+h)), u'(\phi(r))\right)\phi'(r+h)\phi'(r) \\
&\quad +|u'(\phi(r))|^2(\phi'(r))^2 r\ \mathrm{d}r\,\mathrm{d}h \\
&\geq \int_0^1 \frac{1}{h^{1+2\varepsilon}} \int_0^\infty ((\phi'(r+h))^2 - 2\phi'(r+h)\phi'(r) + \\
&\quad +(\phi''(r))^2)r\ \mathrm{d}r\,\mathrm{d}h \\
&= \|\phi\|^2_{\dot{H}^{1+\varepsilon}_{rad}},
\end{aligned}$$

where we used the Cauchy-Schwartz inequality and the positivity of the first derivative of the function ϕ. □

We now extend the above result to bounded domains $\Omega \subset \mathbb{R}^2$. To this end we need to prove the following proposition:

PROPOSITION 5.1. *For a bounded domain Ω and under, the conditions of the previous lemma, we have the following estimates:*

$$\|u(\phi)\|_{H^{1+\varepsilon}(\Omega)} \leq C\|\phi\|_{H^{1+\varepsilon}(\Omega)}, \quad \forall \quad \varepsilon \in [0,1], \tag{121}$$

$$\|u(\phi)\|_{H^{1+\varepsilon}(\Omega)} \geq C_1\|\phi\|_{H^{1+\varepsilon}(\Omega)}, \quad \forall \quad \varepsilon \in [0,1], \tag{122}$$

with C, C_1 independent of ε.

Proof. We have

$$\begin{aligned} \|u(\phi)\|_{H^{1+\varepsilon}(\Omega)} &= \inf_{\widetilde{u(\phi)}} \|\widetilde{u(\phi)}\|_{H^{1+\varepsilon}} \\ &\lesssim \|u(\tilde{\phi})\|_{\mathbb{H}(\varepsilon)} + \|\tilde{\phi}\|_{L^2(\Omega')} \\ &\lesssim \|\tilde{\phi}\|_{\mathbb{H}(\varepsilon)} + \|\tilde{\phi}\|_{L^2(\Omega')} \\ &= \|\tilde{\phi}\|_{H^{1+\varepsilon}} \\ &\lesssim \|\phi\|_{H^{1+\varepsilon}(\Omega)} + \delta, \end{aligned} \tag{123}$$

where we used the fact that $u(\tilde{\phi})$ is an extension of $u(\phi)$; notice that we are allowed to choose the extension of ϕ in a slightly larger domain Ω' such that $\tilde{\phi} = 0$ in $\mathbb{R}^2 \setminus \Omega'$ and $\|\tilde{\phi}\|_{H^{1+\varepsilon}} - \|\phi\|_{H^{1+\varepsilon}(\Omega)} < \delta$, for any fixed $\delta > 0$. Since δ is arbitrary, this concludes the proof of the first inequality.

To prove the opposite inequality, we use the fact that $|u| = 1$ as a vector in $\mathbb{R}^3$. Then, taking the same extension $\tilde{\phi}$ on a larger domain $\Omega \subseteq \Omega'$ as above, we have

$$\|\tilde{\phi}\|_{L^2(\Omega')} \lesssim \|u(\tilde{\phi})\|_{L^2(\Omega')}. \tag{124}$$

Moreover, recalling the proof of the previous Lemma, we have also

$$\|\tilde{\phi}\|_{\mathbb{H}(\varepsilon)} \lesssim \|u(\tilde{\phi})\|_{\mathbb{H}(\varepsilon)} \lesssim \|\widetilde{u(\phi)}\|_{\mathbb{H}(\varepsilon)} + \delta. \tag{125}$$

Taking the sum between these two inequalities we conclude the proof. □

LEMMA 5.2. *Let $\phi \in C([0,T]; H^s)$ and $f \in L^1((0,T); H^{s-1})$ for some $s > 1$. Then*

$$F = f\partial_\phi u(\phi) \in L^1((0,T); H^{s-1}).$$

Proof. We first estimate some norms with respect to space variables at a fixed time $t > 0$. The L^2-norm of the term $F = f\partial_\phi u(\phi)$ can be computed immediately as follows:

$$\|f\partial_\phi u(\phi)\|_{L^2} = \|f\|_{L^2} \,. \tag{126}$$

Next we use the fact that for $\mu > 1$ the space H^μ is an algebra, and this gives the estimate

$$\|f\partial_\phi u(\phi)\|_{H^\mu} \lesssim \|f\|_{H^\mu} \|\partial_\phi u(\phi)\|_{H^\mu} \,. \tag{127}$$

Thus, if we consider f as an operator from L^2 into L^2 and from H^μ into H^μ, an interpolation argument implies that f is bounded on H^s for each $0 < s < \mu$, with a norm bounded by $\|f\|_{H^s}$. Hence we have

$$\|f\partial_\phi u(\phi)\|_{H^s} \lesssim \|f\|_{H^s} \|\partial_\phi u(\phi)\|_{H^s} \,. \tag{128}$$

But from the previous lemma we may control the H^s norm of $u'(\phi)$ with the H^s norm of ϕ. In conclusion, the result follows by integrating the above estimate and applying Hölder inequality with respect to time. □

An analogous result holds on the restricted cones:

LEMMA 5.3. *Let $\phi \in C([0,T]; H^s(\Omega_\alpha(t)))$ and, for some $s > 1$, $f \in L^1((0,T); H^{s-1}(\Omega_\alpha(t)))$. Then*

$$F = f\partial_\phi u(\phi) \in L^1((0,T); H^{s-1}(\Omega_\alpha(t))).$$

Proof. The argument is identical to the above one (use Proposition 5.1). □

REFERENCES

[1] Y. CHOQUET-BRUHAT, *Global wave maps on curved space times*, Lecture Notes in Phys. **537** (2000), 1–29.

[2] P. D'ANCONA AND V. GEORGIEV, *Low regularity solutions for the wave map equation into the 2-D sphere*, will appear in Math. Zeit.

[3] P. D'ANCONA AND V. GEORGIEV, *On the continuity of the solution operator to the wave map system*, will appear in Comm. Pure Appl. Math.

[4] J. Ginibre and G. Velo, *The Cauchy problem for the* O(N), **C**P($N-1$), *and* $G_{\mathbf{C}}(N, p)$ *models*, Ann. Physics **142** (1982), no. 2, 393–415.
[5] M. G. Grillakis, *On the wave map problem*, Contemp. Math., vol. 263, Amer. Math. Soc., Providence, RI, 2000.
[6] H. Karcher and J. C. Wood, *Nonexistence results and growth properties for harmonic maps and forms*, J. Reine Angew. Math. **353** (1984), 165–180.
[7] S. Klainerman and M. Machedon, *On the regularity properties of a model problem related to wave maps*, Duke Math. J. **87** (1997), no. 3, 553–589.
[8] S. Klainerman and S. Selberg, *Bilinear estimates and applications to nonlinear wave equations*, Commun. Contemp. Math. **4** (2002), no. 2, 223–295.
[9] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. II*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1996, Reprint of the 1969 original, A Wiley-Interscience Publication.
[10] S. Müller and M. Struwe, *Global existence of wave maps in* $1+2$ *dimensions with finite energy data*, Topol. Methods Nonlinear Anal. **7** (1996), no. 2, 245–259.
[11] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 3, Walter de Gruyter & Co., Berlin, 1996.
[12] J. Shatah and A. S. Tahvildar-Zadeh, *On the Cauchy problem for equivariant wave maps*, Comm. Pure Appl. Math. **47** (1994), no. 5, 719–754.
[13] M. Struwe, *Equivariant wave maps in two space dimensions*, Comm. Pure Appl. Math. **56** (2003), no. 7, 815–823, Dedicated to the memory of Jürgen K. Moser.
[14] T. Tao, *Ill-posedness for one-dimensional wave maps at the critical regularity*, Amer. J. Math. **122** (2000), no. 3, 451–463.
[15] D. Tataru, *On global existence and scattering for the wave maps equation*, Amer. J. Math. **123** (2001), no. 1, 37–77.
[16] H. Triebel, *Interpolation theory, function spaces, differential operators*, North Holland Co., Amsterdam, 1978.

Received October 13, 2003.