Rend. Istit. Mat. Univ. Trieste
Vol. XXXV, 81–115 (2003)

Functions that are the Directed X-Ray of a Planar Convex Body

W. Black, J. Kimble, D. Koop, D. C. Solmon (*)

Summary. - *We characterize functions that are the directed X-ray of a planar convex body from a source that is a positive distance from the body. In addition to a concavity condition the necessary and sufficient conditions involve the structure of points of zero curvature and a priori estimates for derivatives of the directed X-ray near supporting rays and points of zero curvature. The techniques employed also lead to explicit methods for constructing families of planar convex bodies with a common directed X-ray.*

1. Introduction

Since P. C. Hammer [7] posed his question about determining a convex body from X-rays, a vast literature has sprung up around the subject. See Gardner [6] for an overview of work on this question and historical comments. The results obtained depend on the geometric arrangement of the X-ray data. Some researchers assume that the

---

(*) Authors' addresses: William Black, Department of Mathematics, University of California at Santa Barbara, Santa Barbara CA 93106, USA, e-mail: blackw@mail.math.ucsb.edu
Jennifer Kimble, Department of Mathematics, Mississippi University for Women, Columbus MS 39701, USA, email: jak@muw.edu
David Koop, Department of Mathematics, Calvin College, Grand Rapids MI 49546, USA, e-mail: dakoop@cs.wisc.edu
Donald C. Solmon, Department of Mathematics, Oregon State University, Corvallis OR 97331, USA, e-mail: solmon@math.orst.edu
Keywords: directed x-ray, convex body
AMS Subject Classification: 52A10, 52A38
Supported by NSF Grant DMS-9820499.

X-ray data are arranged in parallel rays, while others (as we do here) assume the X-rays emanate from a point source, so called directed X-rays. Much of the work on Hammer's problem, especially those dealing with directed X-rays, involves finding minimal numbers and locations of the sources for which uniqueness obtains. See Falconer [4], Gardner [5] and Volčič [12]. In this paper we address a situation where there is only one directed X-ray, so there are infinitely many convex bodies with the same directed X-ray. We address two related questions. Given a convex body in the plane, how can one constr! uct other convex bodies with the same directed X-ray? Given a function X, when is X the directed X-ray of a convex body in the plane? We always assume that the source is outside the convex body. The first question is fairly simple, while the second is more complicated. However, both can be addressed by a similar approach.

Since there will be only one source, it is convenient to choose polar coordinates with the origin at the source and the convex body K in the open upper half plane $\{(x,y) : y > 0\}$. A ray $\varphi = \varphi_0$ which meets the interior of K intersects the boundary of K at two points $(r(\varphi_0), \varphi_0)$ (a near side point) and $(R(\varphi_0), \varphi_0)$ (a far side point) with $0 < r(\varphi_0) < R(\varphi_0)$. The set of near side points of K determines a function $r(\varphi)$ and the set of far side points determines a function $R(\varphi)$. The directed X-ray of K is the function $X(\varphi) = R(\varphi) - r(\varphi)$. Clearly, K is uniquely determined by any pair of the three functions $r,\ R,\ X$. Both questions can be addressed by studying properties of near side and far side functions. Indeed, suppose that K is a convex body with near side function r and far side function R. Let s be a positive continuous function. Then the body with near side $r + s$ and far side $R+s$ has the same directed X-ray as K. The question is whether the new body is convex, that is whether $r+s$ is the near side function, $R + s$ is the far side function of a convex body. Similarly, given a function Y it is easy to construct a body with directed X-ray Y. Simply form the body with near side s and far side $s + Y$. The probelm is then to determine whether this new body is convex. In this case s must be chosen to be a proper near side function, and then we must determine whether $s + Y$ is a proper far side function.

Longinetti [10] proved that the graph of the directed X-ray of a convex body in the plane is concave toward the source. We will show that in addition to being concave toward the source, X and its derivatives must satisfy certain a priori estimates near the supporting rays and near points of zero curvature. In addition points of zero curvature on the graph of the directed X-ray must form a (possibly degenerate) line segment. We include several examples of functions satisfying Longinetti's condition that are not directed X-rays of a convex body in the upper half plane.

Given the simple nature of the approach outlined above, the technical problem is to determine properties of functions that are proper near side and far side functions of a convex body (functions that are respectively concave away from the origin, concave toward the origin), and also to study which properties are preserved under the algebraic operations of addition and scalar multiplication.

The structure of the paper is the following. First we make precise the notion of functions concave away from or toward the origin and summarize their properties. Next, we introduce and derive properties of quadratic forms and differential operators (curvature operators) that will be used to determine the direction of concavity of a function. In the fourth section we explore the special structure that arises when there are points of zero curvature on the graph of the directed X-ray. Next, we discuss families of convex bodies with a common directed X-ray. In the last two sections, we present the results characterizing functions that are a directed X-ray. It is convenient to deal first with the case where there are no points of zero curvature and then handle that case separately.

This work began during the NSF sponsored REU program in the summers of 2000, 2001 at Oregon State University. Preliminary results appeared in [1, 8]. The results in Section 5 on families of convex bodies with a common directed X-ray have been expanded upon in [2]. [3] will deal with geometric and topological properties of such families.

2. Functions concave toward or away from the origin

The purpose of this section is to present some results about functions concave away from or toward the origin that will be used in the sequel. First we introduce terminology that will be used throughout.

We take (r, φ) to be polar coordinates of a point in the plane with φ measured counterclockwise from the positive x-axis. Unless otherwise stated we always assume that $r \geq 0$. Polar functions $r = r(\varphi)$, $\varphi \in [\alpha, \beta]$ are assumed to be continuous and nonnegative on the closed interval $[\alpha, \beta]$ and positive on (α, β). Unless otherwise stated $0 < \alpha < \beta < \pi$.

Consider a polar function f. We will say that f is *concave toward the origin* on an interval I if for all choices of φ_1, $\varphi_2 \in I$, the line sement joining the points $(f(\varphi_1), \varphi_1)$ and $(f(\varphi_2), \varphi_2)$ separates the origin from the graph of f on the interval I. If the graph of f separates the origin from the line segment for all choices, then we say that f is *concave away from the origin.*

Throughout the paper we use the term *convex body* to refer to a compact, convex subset of the plane with nonempty interior. The symbol $\mathcal{H}$ is used for the open upper half plane $\{(x, y) : y > 0\}$.

Consider a convex body $K \subset \mathcal{H}$ with supporting rays $\varphi = \alpha$ and $\varphi = \beta$. As we discussed in the introduction, a ray $\varphi = \varphi_0$, $\alpha < \varphi_0 < \beta$, meets K in two points and the set of such points determines the near and far side functions of K (with respect to the source). At the end points $r(\alpha) = \min(\{r : (r, \alpha) \in K\}$, and $R(\alpha) = \max\{r : (r, \alpha) \in K\}$. Similarly, for $r(\beta)$ and $R(\beta)$. So, $0 < r(\varphi) \leq R(\varphi)$ and the inequality is strict when $\varphi \in (\alpha, \beta)$. It is geometrically obvious that near side functions of convex bodies are concave away from the origin, while far side functions are concave toward the origin.

Suppose $a \notin K$ is a point in the plane and $\theta = (\cos\varphi, \sin\varphi)$ is a direction. If $\ell(a, \theta)$ is the ray emanating from a with direction θ, then the *directed X-ray* of K with source a and direction θ is the length of the segment $\ell(a, \theta) \cap K$. Since we will only deal with a single source, we locate it at the origin. The directed X-rays of K are then given by the function $X(\varphi) = R(\varphi) - r(\varphi)$.

By an appropriate rotation of coordinates the graph of a near side function $r = r(\varphi)$ is also the graph of a convex function $y = f(x)$ in rectangular coordinates. Also, given any point $(R(\varphi_0), \varphi_0)$ on the far side we may rotate coordinates so that a neighborhood of the point is the graph of a concave function in rectangular coordinates.

Upon rotating r so that its graph is also the graph of a function

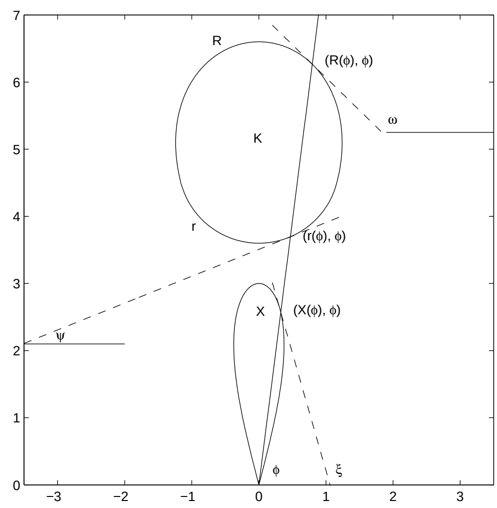

Figure 1: Convex body K, near side r, far side R and directed X-ray X. Directed X-ray angle ϕ and angles of inclination ξ, ψ, ω.

$y = f(x)$ in rectangular coordinates, one then computes

$$r'_-(\varphi) = r(\varphi)\frac{f'_+(x)\sin\varphi + \cos\varphi}{f'_+(x)\cos\varphi - \sin\varphi} = r(\varphi)\cot(\psi_\varphi - \varphi), \qquad (1)$$

where $f'_+(x) = \tan(\psi_\varphi)$ and ψ_φ is the angle of inclination of the right hand tangent line to the graph of f at $x = r(\varphi)\cos\varphi$. This simple observation allows derivation of many properties of functions concave toward or away from the origin from well known properties of convex and concave functions. (See Royden [11] p. 113-117 as a reference on properties of convex and concave functions.)

The angles of inclination of the tangent lines (or left or right tangent lines) to the graphs of r, R and X will play an important role. These will be denoted respectively by ψ, ω and ξ throughout. See Figure 1. The standard counterclockwise measurement of the polar angle introduces some confusion in the notion of left and right derivative. For any function of a real variable g we define $g'_+(t) = \lim_{h\to 0+} \frac{g(t+h)-g(t)}{h}$, and $g'_-(t) = \lim_{h\to 0+} \frac{g(t)-g(t-h)}{h}$. The results in the first lemma are derived via rotation from known properties of convex and concave functions. The proof is omitted.

LEMMA 2.1. *Suppose that r is concave away from the origin and R is concave toward the origin on $[\alpha, \beta]$. Then r and R are absolutely continuous on $[\gamma, \delta]$ for all $[\gamma, \delta] \subset (\alpha, \beta)$. In addition the following hold.*

1. *For all $\varphi \in (\alpha, \beta)$, the left and right hand derivatives of r and R exist,*
$$r'_-(\varphi) \leq r'_+(\varphi) \text{ and } R'_-(\varphi) \geq R'_+(\varphi).$$
Equality holds in both inequalities with the possible exception of a countable set.

2. *For all $\varphi \in (\alpha, \beta]$,*
$$r'_-(\varphi) = \lim_{\psi \to \varphi-} r'_-(\psi) = \lim_{\psi \to \varphi-} r'_+(\psi), \text{ and}$$
$$R'_-(\varphi) = \lim_{\psi \to \varphi-} R'_-(\psi) = \lim_{\psi \to \varphi-} R'_+(\psi).$$

3. *For all $\varphi \in [\alpha, \beta)$,*
$$r'_+(\varphi) = \lim_{\psi \to \varphi+} r'_+(\psi) = \lim_{\psi \to \varphi+} r'_-(\psi), \text{ and}$$
$$R'_+(\varphi) = \lim_{\psi \to \varphi+} R'_+(\psi) = \lim_{\psi \to \varphi+} R'_-(\psi).$$
r'_- and R'_+ are lower semicontinuous, while r'_+ and R'_- are upper semicontinuous on (α, β).

4. *The functions r'_- and r'_+ are stictly increasing. Each of the functions r'_-, r'_+, R'_-, R'_+ is differentiable almost everywhere on $[\alpha, \beta]$. Moreover $(r'_-)' = (r'_+)'$ and $(R'_-)' = (R'_+)'$ almost everywhere.*

5. *The functions r'_-, r'_+, R'_-, R'_+ are all bounded on compact subsets of (α, β). If $r(\alpha) > 0$, $R(\alpha) > 0$, $r(\beta) > 0$, $R(\beta) > 0$, then these derivatives are bounded on $[\alpha, \beta]$ unless one of the rays $\varphi = \alpha$, $\varphi = \beta$ is tangent to the graph of r or R. In this case, when $\varphi = \alpha$ is tangent to the graph of r (R), then $r'_+(\alpha) = -\infty$, ($R'_+(\alpha) = +\infty$). When $\varphi = \beta$ is tangent to the graph of r (R), then $r'_-(\beta) = +\infty$, ($R'_-(\beta) = -\infty$).*

We will refer to points φ at which the graph of f has a unique tangent line as *smooth* points of f, or smooth points of the graph of f. Points where $f'_- \neq f'_+$ will be called *nonsmooth* points.

3. A quadratic form and the curvature operators

Here we define and give some properties of quadratic forms Q and Q_h. These forms will be used to determine whether a polar function is concave toward or away from the origin. The quadratic form Q was introduced in [9]. Taking appropriate limits with the quadratic form Q_h leads to operators we call lower and upper curvature operators which also determine the direction of concavity. The latter have the advantage of depending on only one variable. Algebraic properties and inequalities involving the curvature operators are also developed.

DEFINITION 3.1. *Suppose that* $0 < \alpha \leq \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \beta < \pi$ *and* $r_j = r(\varphi_j), j = 1, 2, 3$. *We define the quadratic form* Q *by*

$$Q(r_1, r_2, r_3) = r_1 r_2 \sin(\varphi_2 - \varphi_1) + r_2 r_3 \sin(\varphi_3 - \varphi_2) - r_1 r_3 \sin(\varphi_3 - \varphi_1).$$

For simplicity we often write Qr instead of $Q(r_1, r_2, r_3)$. When this is done, it is always assumed that the three points φ_j are ordered as above. There are two interpretations of Q which make its geometric properties evident. Let $\theta_j = (\cos\varphi_j, \sin\varphi_j)$ $j = 1, 2, 3$, be unit vectors. The segment joining $r_1\theta_1$ and $r_2\theta_2$ lies above (below) the segment from $r_1\theta_1$ to $r_3\theta_3$ according as the cross product $(r_2\theta_2 - r_1\theta_1) \times (r_3\theta_3 - r_1\theta_1)$ points upward (or downward) from the xy plane. One readily computes that

$$Q(r_1, r_2, r_3) = (r_2\theta_2 - r_1\theta_1) \times (r_3\theta_3 - r_1\theta_1) \cdot k,$$

where k is the upward pointing unit normal vector to the xy plane. Another interpretation can be given in terms of area. The area of the triangle T_{jk} with vertices O, $r_j\theta_j$, $r_k\theta_k$ (O denoting the origin) is given by $A(T_{jk}) = |\frac{r_j\theta_j \times r_k\theta_k}{2}|$. One can check that $Q(r_1, r_2, r_3) \geq 0$, (≤ 0) according as $A(T_{12}) + A(T_{23}) - A(T_{13}) \geq 0$, (≤ 0). Thus a continuous polar curve r is concave away from (concave toward) the origin on an interval I if and only if $Qr \leq 0$ ($Qr \geq 0$) for all ordered choices of $\varphi_1, \varphi_2, \varphi_3 \in I$. [9]

For the most part we will only need a symmetric variant of Q. For $h > 0$ define

$$Q_h f(\varphi) = Q(f(\varphi - h), f(\varphi), f(\varphi + h))$$
$$= (f(\varphi - h) + f(\varphi + h))f(\varphi)\sin h - f(\varphi - h)f(\varphi + h)\sin 2h.$$

When dealing with this operator it is implicitly assumed that the interval $[\varphi - h, \varphi + h]$ is a subset of the domain of f. This will often be indicated by the phrase "for sufficiently small $h > 0$".

LEMMA 3.2. *The quadratic form Q_h satisfies the following.*

1. $Q_h(tr) = t^2 Q_h(r)$, *for all real numbers t.*

2. *If the curve $r(\varphi)$ is concave away from the origin on $[\alpha, \beta]$, then $Q_h r(\varphi) \leq 0$ for all $\varphi \in (\alpha, \beta)$ and sufficiently small $h > 0$. Conversely, if each $\varphi_0 \in (\alpha, \beta)$ has a neighborhood $N(\varphi_0)$ such that $Q_h r(\varphi) \leq 0$ for all $\varphi \in N(\varphi_0)$ and all $h > 0$ such that $(\varphi - h, \varphi + h) \subset N(\varphi)$, then r is concave away from the origin on $[\alpha, \beta]$.*

3. *If the curve $R(\varphi)$ is concave toward the origin on $[\alpha, \beta]$, then $Q_h R(\varphi) \geq 0$ for all $\varphi \in (\alpha, \beta)$ and sufficiently small $h > 0$. Conversely, if each $\varphi_0 \in (\alpha, \beta)$ has a neighborhood $N(\varphi_0)$such that $Q_h R(\varphi) \geq 0$ for all $\varphi \in N(\varphi_0)$ and all $h > 0$ such that $(\varphi - h, \varphi + h) \subset N(\varphi_0)$, then R is concave toward the origin on $[\alpha, \beta]$.*

4. *If $Q_h r(\varphi) = 0$ for fixed φ and $h > 0$, then the points $(r(\varphi - h), \varphi - h)$, $(r(\varphi), \varphi)$, $(r(\varphi + h), \varphi + h)$ are collinear. In particular if r is concave toward or concave away from the origin on $[\varphi - h, \varphi + h]$, then the graph of r on this interval is a line segment.*

5. $Q_h(r + s) = (r(\varphi - h) + s(\varphi - h) + r(\varphi + h) + s(\varphi + h))\times$

$$\left[\frac{Q_h(r)}{r(\varphi - h) + r(\varphi + h)} + \frac{Q_h(s)}{s(\varphi - h) + s(\varphi + h)}\right]$$
$$-\frac{\sin 2h\,[r(\varphi - h)s(\varphi + h) - r(\varphi + h)s(\varphi - h)]^2}{(r(\varphi - h) + r(\varphi + h))(s(\varphi - h) + s(\varphi + h))}.$$

Proof. The first conclusion is trivial while the next three follow easily from properties of Q established in [9]. The identity for $Q_h(r+s)$ is an elementary but tedious computation which we sketch. From the definition of Q_h we have

$$Q_h(r+s) = (r(\varphi-h)+s(\varphi-h))(r(\varphi)+s(\varphi))\sin h$$
$$+(r(\varphi)+s(\varphi))(r(\varphi+h)+s(\varphi+h))\sin h$$
$$-(r(\varphi-h)+s(\varphi-h))(r(\varphi+h)+s(\varphi+h))\sin 2h.$$

Solving for $r(\varphi)$ and $s(\varphi)$ in the definition of $Q_h r$ and $Q_h s$ gives

$$r(\varphi)+s(\varphi) = \frac{1}{\sin h}\times$$
$$\left[\frac{Q_h r + r(\varphi-h)r(\varphi+h)\sin 2h}{r(\varphi-h)+r(\varphi+h)} + \frac{Q_h s + s(\varphi-h)s(\varphi+h)\sin 2h}{s(\varphi-h)+s(\varphi+h)}\right].$$

Now replace $r(\varphi)+s(\varphi)$ by the above expression. One obtains the first two terms on the right hand side of Lemma 3.2.5 and the additional term

$$\sin 2h\,(r(\varphi-h)+s(\varphi-h)+r(\varphi+h)+s(\varphi+h))\times$$
$$\left[\frac{r(\varphi-h)r(\varphi+h)}{r(\varphi-h)+r(\varphi+h)} + \frac{s(\varphi-h)s(\varphi+h)}{s(\varphi-h)+s(\varphi+h)} - 1\right].$$

A straightforward algebraic computation shows that this expression is equal to the last term in Lemma 3.2.5. □

COROLLARY 3.3. *If r and s are both concave away from the origin on $[\alpha,\beta]$, then so is the function $r+s$. If R is concave toward the origin, r is concave away from the origin and $R \geq r > 0$, then $R-r$ is concave toward the origin.*

Proof. From Lemma 3.2.2, $Q_h(r) \leq 0$ and $Q_h(s) \leq 0$, while from Lemma 3.2.5 $Q_h(r+s) \leq 0$. Lemma 3.2.2 now shows that $r+s$ is concave away from the origin. To obtain the second statement apply Lemma 3.2.3 and 3.2.5 to $R-r$. □

Another immediate consequence of Lemma 3.2 is a result of Longinetti [10].

Corollary 3.4. *Let K be a convex body with directed X-ray X. Then X is concave toward the origin.*

We now use Q_h to define the curvature operators. The reason for this terminology will become clear later.

Definition 3.5. *The curvature operator, $\mathcal{K}$, lower curvature operator $\underline{\mathcal{K}}$, and upper curvature operator $\overline{\mathcal{K}}$ are defined by*

1. $\mathcal{K}f(\varphi) = \lim_{h\to 0+} \frac{Q_h f(\varphi)}{h^3}$;
2. $\underline{\mathcal{K}}f(\varphi) = \underline{\lim}_{h\to 0+} \frac{Q_h f(\varphi)}{h^3} = \lim_{\delta\to 0} \inf_{0<h<\delta} \frac{Q_h f(\varphi)}{h^3}$;
3. $\overline{\mathcal{K}}f(\varphi) = \overline{\lim}_{h\to 0+} \frac{Q_h f(\varphi)}{h^3} = \lim_{\delta\to 0} \sup_{0<h<\delta} \frac{Q_h f(\varphi)}{h^3}$.

Using l'Hopital's Rule one easily verifies that when f is C^2

$$\mathcal{K}f = f^2 + 2(f')^2 - ff'' = f^3 \left[\frac{1}{f} + \left(\frac{1}{f}\right)''\right]. \tag{2}$$

It is well known that a C^2 function f is concave toward (concave away from) the origin on an open interval I if and only if $\mathcal{K}f(\varphi) \geq 0$ ($\mathcal{K}f(\varphi) \leq 0$) for all $\varphi \in I$. From the second form of the operator it is clear that

$$\mathcal{K}(\csc(a\varphi + b)) = (1 - a^2)\csc^2(a\varphi + b), \tag{3}$$

and

$$\mathcal{K}(\operatorname{csch}(a\varphi + b)) = (1 + a^2)\operatorname{csch}^2(a\varphi + b).$$

Both forms of this operator and a variant of the formula in Lemma 3.8.2 below appeared in Longinetti [10]. We turn now to developing other formulae for, and inequalities involving, the curvature operators. First some notation.

Definition 3.6. *Let $f : (a, b) \to R$. The symmetric lower and upper first and second derivatives of f at a point t are given respectively by*

$$\underline{D}_1 f(t) = \underline{\lim}_{h\to 0+} \frac{f(t+h) - f(t-h)}{2h};$$

$$\overline{D}_1 f(t) = \overline{\lim}_{h\to 0+} \frac{f(t+h) - f(t-h)}{2h};$$

$$\underline{D}_2 f(t) = \underline{\lim}_{h\to 0+} \frac{f(t-h)+f(t+h)-2f(t)}{h^2};$$
$$\overline{D}_2 f(t) = \overline{\lim}_{h\to 0+} \frac{f(t-h)+f(t+h)-2f(t)}{h^2}.$$

The values taken on by these functions may be $+\infty$ or $-\infty$.

REMARK 3.7. *When f has a finite right and left derivative at t (as is the case with functions concave toward or away from the origin) it is easy to see that $\underline{D}_1 f(t) = \overline{D}_1 f(t) = \frac{f'_-(t)+f'_+(t)}{2}$. Since this situation will occur frequently, we define*

$$D_1 f(t) = \frac{f'_-(t)+f'_+(t)}{2}.$$

Of course $D_1 f(t) = f'(t)$ when f is differentiable at t. From Lemma 2.1 we see that when R is concave toward the origin $\underline{D}_2 R = -\infty$ at nonsmooth points, and when r is concave away from the origin $\overline{D}_2 r = +\infty$ at nonsmooth points. Also, for functions that are concave toward or away from the origin, $\overline{D}_2 f = \underline{D}_2 f = (f'_+)' = (f'_-)'$ at almost every point.

LEMMA 3.8. *Let f, g be positive continuous functions on (α, β). Assume that f, g have finite left and right derivatives at φ. The following identities and inequalities hold at φ provided none of the expressions are of the form $\infty - \infty$. It is assumed that $f - g > 0$ in the inequalities where that expression occurs.*

1. $\mathcal{K}(tf) = t^2\mathcal{K}f,\ \underline{\mathcal{K}}(tf) = t^2\underline{\mathcal{K}}f,\ \overline{\mathcal{K}}(tf) = t^2\overline{\mathcal{K}}f$ *for all* $t \geq 0$.

2. $\mathcal{K}(f+g) = \frac{f+g}{f}\mathcal{K}f + \frac{f+g}{g}\mathcal{K}g - 2fg\left[\frac{D_1 f}{f} - \frac{D_1 g}{g}\right]^2.$

3. $\frac{f+g}{f}\underline{\mathcal{K}}f + \frac{f+g}{g}\underline{\mathcal{K}}g - 2fg\left[\frac{D_1 f}{f} - \frac{D_1 g}{g}\right]^2 \leq \underline{\mathcal{K}}(f+g)$
$$\leq \frac{f+g}{f}\underline{\mathcal{K}}f + \frac{f+g}{g}\overline{\mathcal{K}}g - 2fg\left[\frac{D_1 f}{f} - \frac{D_1 g}{g}\right]^2.$$

4. $\frac{f-g}{f}\underline{\mathcal{K}}f - \frac{f-g}{g}\overline{\mathcal{K}}g + 2fg\left[\frac{D_1 f}{f} - \frac{D_1 g}{g}\right]^2 \leq \underline{\mathcal{K}}(f-g)$
$$\leq \frac{f-g}{f}\underline{\mathcal{K}}f - \frac{f-g}{g}\underline{\mathcal{K}}g + 2fg\left[\frac{D_1 f}{f} - \frac{D_1 g}{g}\right]^2.$$

5. $\frac{f+g}{f}\overline{\mathcal{K}}f+\frac{f+g}{g}\underline{\mathcal{K}}g-2fg\left[\frac{D_1f}{f}-\frac{D_1g}{g}\right]^2\le\overline{\mathcal{K}}(f+g)$

$$\le\frac{f+g}{f}\overline{\mathcal{K}}f+\frac{f+g}{g}\overline{\mathcal{K}}g-2fg\left[\frac{D_1f}{f}-\frac{D_1g}{g}\right]^2.$$

6. $\frac{f-g}{f}\overline{\mathcal{K}}f-\frac{f-g}{g}\overline{\mathcal{K}}g+2fg\left[\frac{D_1f}{f}-\frac{D_1g}{g}\right]^2\le\overline{\mathcal{K}}(f-g)$

$$\le\frac{f-g}{f}\overline{\mathcal{K}}f-\frac{f-g}{g}\underline{\mathcal{K}}g+2fg\left[\frac{D_1f}{f}-\frac{D_1g}{g}\right]^2.$$

Proof. The first set of identities is trivial. Assuming the limits exist, we have from Lemma 3.2.5

$$\mathcal{K}(f+g)=\lim_{h\to0+}\frac{Q_h(f+g)(\varphi)}{h^3}=\frac{f+g}{f}\mathcal{K}f+\frac{f+g}{g}\mathcal{K}g$$

$$-2\lim_{h\to0+}\frac{1}{h^2}\frac{[f(\varphi-h)g(\varphi+h)-f(\varphi+h)g(\varphi-h)]^2}{(f(\varphi-h)+f(\varphi+h))(g(\varphi-h)+g(\varphi+h))}.$$

The remaining limit is equal to

$$\frac{fg}{4}\lim_{h\to0+}\frac{1}{h^2}\left[\frac{f(\varphi-h)}{f(\varphi+h)}-\frac{g(\varphi-h)}{g(\varphi+h)}\right]^2$$

$$=fg\lim_{h\to0+}\left[\frac{f(\varphi-h)}{f(\varphi+h)g(\varphi+h)}\frac{g(\varphi+h)-g(\varphi-h)}{2h}\right.$$

$$\left.-\frac{g(\varphi-h)}{f(\varphi+h)g(\varphi+h)}\frac{f(\varphi+h)-f(\varphi-h)}{2h}\right]^2=fg\left[\frac{D_1f}{f}-\frac{D_1g}{g}\right]^2.$$

This completes the proof for the operator $\mathcal{K}$. The inequalities in Lemma 3.8.3-6 are obtained in a similar way with the aid of the identity $\underline{\lim}(-f)=-\overline{\lim}f$ and the chain of inequalities $\underline{\lim}f+\underline{\lim}g\le\underline{\lim}(f+g)\le\overline{\lim}f+\underline{\lim}g\le\overline{\lim}(f+g)\le\overline{\lim}f+\overline{\lim}g$, which are valid provided no sum is of the form $\infty-\infty$. ☐

LEMMA 3.9. *Assume that $f>0$ is continuous and has a finite left and right derivative at φ. Then*

1. $\underline{\mathcal{K}}f(\varphi) = \underline{\lim}_{h\to 0+}\frac{Q_h f(\varphi)}{h^3} = f^2 + 2(D_1 f)^2 - f\overline{D}_2 f$

$$= f^3(\frac{1}{f} + \underline{D}_2\frac{1}{f}), \text{ and}$$

2. $\overline{\mathcal{K}}f(\varphi) = \overline{\lim}_{h\to 0+}\frac{Q_h f(\varphi)}{h^3} = f^2 + 2(D_1 f)^2 - f\underline{D}_2 f$

$$= f^3(\frac{1}{f} + \overline{D}_2\frac{1}{f}).$$

Proof. Since the proofs are symmetric we only prove 1. From the definition of Q_h and the Taylor expansion of the cosine function we have

$$\frac{Q_h f}{h^3} =$$

$$\frac{f(\varphi-h)f(\varphi)\sin h + f(\varphi)f(\varphi+h)\sin h - f(\varphi-h)f(\varphi+h)\sin 2h}{h^3}$$

$$= \frac{\sin h}{h}\frac{f(\varphi-h)f(\varphi) + f(\varphi)f(\varphi+h) - 2f(\varphi-h)f(\varphi+h)\cos h}{h^2}$$

$$= f(\varphi-h)f(\varphi+h)$$

$$+\frac{\sin h}{h}\frac{f(\varphi-h)f(\varphi) + f(\varphi)f(\varphi+h) - 2f(\varphi-h)f(\varphi+h)}{h^2} + O(h),$$

where $O(h)$ represents a function that goes to 0 as h goes to 0. Now write

$$\begin{aligned}
&f(\varphi-h)f(\varphi) + f(\varphi)f(\varphi+h) - 2f(\varphi-h)f(\varphi+h)\\
&= -f(\varphi-h)(f(\varphi+h) - f(\varphi)) - f(\varphi+h)(f(\varphi-h) - f(\varphi))\\
&= -f(\varphi-h)(f(\varphi-h) + f(\varphi+h) - 2f(\varphi))\\
&\quad +2(f(\varphi) - f(\varphi-h))\frac{f(\varphi+h) - f(\varphi-h)}{2}.
\end{aligned}$$

In a similar fashion one may also write

$$\begin{aligned}
&f(\varphi-h)f(\varphi) + f(\varphi)f(\varphi+h) - 2f(\varphi-h)f(\varphi+h)\\
&= -f(\varphi+h)(f(\varphi-h) + f(\varphi+h) - 2f(\varphi))
\end{aligned}$$

$$+2(f(\varphi+h)-f(\varphi))\frac{f(\varphi+h)-f(\varphi-h)}{2}.$$

Substituting these expressions we obtain the two formulae

$$\frac{Q_h f}{h^3} = O(h) + f(\varphi-h)f(\varphi+h) + \frac{\sin h}{h}\times$$
$$\left[-f(\varphi-h)\frac{f(\varphi-h)+f(\varphi+h)-2f(\varphi)}{h^2}\right.$$
$$\left.+2\frac{f(\varphi)-f(\varphi-h)}{h}\frac{f(\varphi+h)-f(\varphi-h)}{2h}\right]$$

and

$$\frac{Q_h f}{h^3} = O(h) + f(\varphi-h)f(\varphi+h + \frac{\sin h}{h}\times$$
$$\left[-f(\varphi+h)\frac{f(\varphi-h)+f(\varphi+h)-2f(\varphi)}{h^2}\right.$$
$$\left.+2\frac{f(\varphi+h)-f(\varphi)}{h}\frac{f(\varphi+h)-f(\varphi-h)}{2h}\right].$$

Taking the average of these two expressions gives

$$\frac{Q_h f}{h^3} = O(h) + f(\varphi-h)f(\varphi+h)$$
$$+\frac{\sin h}{h}\left[2\left(\frac{f(\varphi+h)-f(\varphi-h)}{2h}\right)^2\right.$$
$$\left.-\frac{f(\varphi+h)+f(\varphi-h)}{2}\frac{f(\varphi-h)+f(\varphi+h)-2f(\varphi)}{h^2}\right].$$

Taking the $\underline{\lim}$ of both sides of the above expression gives the first identity in 3.9.1. The derivation of the second is similar. □

REMARK 3.10. *In light of Remark 3.7, notice that if f is concave toward or away from the origin, $\underline{\mathcal{K}}f = \overline{\mathcal{K}}f = \mathcal{K}f$ almost everywhere. If f is concave toward the origin, $\underline{\mathcal{K}}f = +\infty$ at nonsmooth points, and if f is concave away from the origin $\overline{\mathcal{K}}f = -\infty$ at nonsmooth points.*

The formula for the signed curvature at a point $(r(\varphi), \varphi)$ on the graph of a C^2 polar curve r is

$$\kappa_r(\varphi) = \frac{r^2 + 2(r')^2 - rr''}{(r^2 + (r')^2)^{3/2}}.$$

Thus $\mathcal{K}r = \kappa_r(\varphi)(r^2 + (r')^2)^{3/2}$. Lemma 3.8.2 can be rewritten as a relation between geometric quantities related to a convex body K and its directed X-ray. Indeed, suppose that $K \subset \mathcal{H}$ is a convex body with near side r, far side R and directed X-ray $X = R - r$. Letting $\varphi,\ \psi,\ \omega,\ \xi$ be as shown in Figure 1, we obtain from (1)

$$\frac{R'}{R} - \frac{r'}{r} = \cot(\omega - \varphi) - \cot(\psi - \varphi) = \frac{\sin(\psi - \omega)}{\sin(\psi - \varphi)\sin(\omega - \varphi)}.$$

Writing $\mathcal{K}X = \mathcal{K}(R - r)$, one obtains the formula

$$\frac{X^3 \kappa_X(\varphi)}{|\sin^3(\xi - \varphi)|} = \frac{XR^2\kappa_R(\varphi)}{|\sin^3(\omega - \varphi)|} - \frac{Xr^2\kappa_r(\varphi)}{|\sin^3(\psi - \varphi)|} + 2rR\frac{\sin^2(\psi - \omega)}{\sin^2(\psi - \varphi)\sin^2(\omega - \varphi)}. \tag{4}$$

The formula is rotation invariant and all terms on the right are non-negative. ($\kappa_r \le 0$ since it denotes the signed curvature.) The formula also shows the special geometric relation that must hold when there is a point of zero curvatrue on the graph of X. We explore this further in the next section.

THEOREM 3.11. *Let r and R be positive, continuous functions on (α, β). Then r is concave away from the origin (R is concave toward the origin) if and only if $\overline{\mathcal{K}}r \le 0$ ($\underline{\mathcal{K}}R \ge 0$) on (α, β).*

Proof. If r is concave away from the origin, then from Lemma 3.2.2, $Q_h r \le 0$ and from the definition of $\overline{\mathcal{K}}$, $\overline{\mathcal{K}}r \le 0$. To prove the converse note that Lemma 3.2.2 and the definition of limit guarantees that r is concave away from the origin if $\overline{\mathcal{K}}r < 0$ for all $\varphi \in (\alpha, \beta)$. Suppose then that $\overline{\mathcal{K}}r \le 0$. Choose a C^2 function s which is concave away from the origin on $[\alpha, \beta]$ and satisfies $\mathcal{K}s < b < 0$ on $[\alpha, \beta]$. (For example $s = \csc(a\varphi)$ for $a > 1$ and sufficiently close to 1 will do. See (3).) Using Lemma 3.8.1 and Lemma 3.8.5 we find that for all $t > 0$

$$\overline{\mathcal{K}}(r+ts) \le t\frac{r+ts}{s}\mathcal{K}s < bt\frac{r+ts}{s} < 0.$$

Hence $r+ts$ is concave away from the origin for all $t>0$. This implies $Q_h(r+ts) \le 0$ for all $t>0$. If we let $t \to 0$, we see by continuity in t, that $Q_h(r) \le 0$. Lemma 3.2.2 now shows that r is concave away from the origin.

When R is concave toward the origin the proof is similar except one applies the left hand side of Lemma 3.8.6 to $R-ts$ with s concave away from the origin and $R-ts>0$. □

4. Points of Zero Curvature

To avoid repeating the same hypotheses throughout sections 4, 5 and much of section 6, until further notice we assume that $K \subset \mathcal{H}$ is a convex body with near side r, far side R and directed X-ray $X = R-r$. We turn our attention to points of zero curvature on the graph of X, roughly points where $\underline{\mathcal{K}}X = 0$. The special roll played by these points is developed in the following theorem.

Theorem 4.1.

1. *If* $\alpha < \varphi_0 < \beta$ *and* $\underline{\mathcal{K}}X(\varphi_0) = 0$*, then* $\underline{\mathcal{K}}R(\varphi_0) = 0$*,* $\overline{\mathcal{K}}r(\varphi_0) = 0$ *and the tangent lines to the graphs of* r, R *and* X *at* φ_0 *are parallel.*
2. *If* $\alpha \le \varphi_0 < \beta$ *and* $\underline{\lim}_{\eta\to\varphi_0+}\underline{\mathcal{K}}X(\eta) = 0$*, then* $\underline{\lim}_{\eta\to\varphi_0+}\underline{\mathcal{K}}R(\eta) = 0$*,* $\overline{\lim}_{\eta\to\varphi_0+}\overline{\mathcal{K}}r(\eta) = 0$ *and the left hand tangent lines to the graphs of* r, R *and* X *at* φ_0 *are parallel.*
3. *If* $\alpha < \varphi_0 \le \beta$ *and* $\underline{\lim}_{\eta\to\varphi_0-}\underline{\mathcal{K}}X(\eta) = 0$*, then* $\underline{\lim}_{\eta\to\varphi_0-}\underline{\mathcal{K}}R(\eta) = 0$*,* $\overline{\lim}_{\eta\to\varphi_0-}\overline{\mathcal{K}}r(\eta) = 0$ *and the right hand tangent lines to the graphs of* r, R *and* X *at* φ_0 *are parallel.*

Proof. Suppose that $\underline{\mathcal{K}}X(\varphi_0) = 0$. Since $\underline{\mathcal{K}}X = +\infty$ at nonsmooth points of X, Lemma 3.8.4 shows that the graphs of r, R and X are smooth at φ_0, that is each has a unique tangent line. Writing $X = R-r$ and applying Lemma 3.8.4 gives at the point φ_0

$$0 \geq \frac{X}{R}\underline{\mathcal{K}}R - \frac{X}{r}\overline{\mathcal{K}}r + 2rR\left[\frac{R'}{R} - \frac{r'}{r}\right]^2.$$

Since all terms on the right hand side are nonnegative, all must vanish. Recall $\psi = \psi(\varphi)$ and $\omega = \omega(\varphi)$ are the angles of inclination of the tangent lines to the graphs of r, R along the ray φ. We have as computed in (4)

$$\frac{R'}{R} - \frac{r'}{r} = \frac{\sin(\psi - \omega)}{\sin(\omega - \varphi)\sin(\psi - \varphi)}.$$

The vanishing of this expression at φ_0 implies the tangent lines are parallel. This proves the first statement in the theorem.

The proofs of the second and third statements are similar, but with a few subtleties. Consider the second statement. Applying $\underline{\lim}$ in Lemma 3.8.6 gives

$$\underline{\lim}_{\eta\to\varphi_0+}\left[\frac{D_1R(\eta)}{R(\eta)} - \frac{D_1r(\eta)}{r(\eta)}\right]^2 = 0,$$

and $\underline{\lim}_{\eta\to\varphi_0+}\underline{\mathcal{K}}R(\eta) = 0$, $\overline{\lim}_{\eta\to\varphi_0+}\overline{\mathcal{K}}r(\eta) = 0$ provided $X(\alpha) \neq 0$ in the case $\varphi_0 = \alpha$. But $D_1R(\eta) = \frac{R'_+(\eta)+R'_-(\eta)}{2} \to R'_+(\varphi_0)$ as $\eta \to \varphi_0+$ by Lemma 2.1.2 and 2.1.3. Similarly, $D_1r(\eta) \to r'_+(\varphi_0)$ as $\eta \to \varphi_0+$. In any case the left hand tangent lines to the graphs of r, R, X are parallel at φ_0. This is clearly impossible at α if $X(\alpha) = 0$. □

Definition 4.2.

1. *If $\underline{\mathcal{K}}f(\varphi) = 0$, then $(f(\varphi), \varphi)$ is called a smooth point of lower curvature zero.*
2. *If $\underline{\lim}_{\eta\to\varphi+}\underline{\mathcal{K}}f(\eta) = 0$, then $(f(\varphi), \varphi)$ is called a point of lower left curvature zero.*
3. *If $\underline{\lim}_{\eta\to\varphi-}\underline{\mathcal{K}}f(\eta) = 0$, then $(f(\varphi), \varphi)$ is called a point of lower right curvature zero.*

We call a point that satisfies any of the above conditions a point of lower curvature zero and define

$$Z = Z(f) = \{\varphi \in [\alpha, \beta] : (f(\varphi), \varphi) \text{ is a point of lower curvature 0}\}.$$

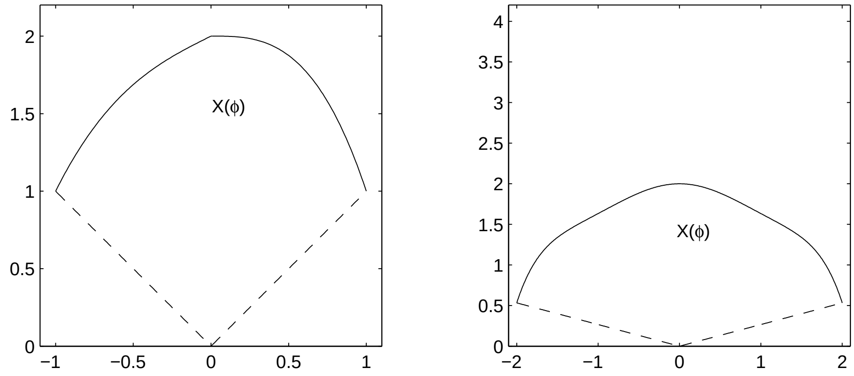

Figure 2: Two functions concave toward the origin which are not directed X-rays of a convex body in $\mathcal{H}$.

Points of upper curvature zero are defined in an analogous way using the operator $\overline{\mathcal{K}}$.

A *wedge* is a quadralateral (or triangle) with vertices on rays $\varphi = \gamma$ and $\varphi = \delta$, where $0 < \delta - \gamma < \pi$. A *parallel wedge* is a trapezoid. If A is a set of angles, we let $\mathcal{C}(A)$ be the cone $\{(r, \varphi) : r \geq 0,\ \varphi \in A\}$. The next result details the structure of convex bodies whose directed X-rays have points of lower curvature zero on their graphs.

THEOREM 4.3. *If* $Z \neq \emptyset$, *then* $Z = [\gamma, \delta]$ *is a closed interval (possibly a single point). If* φ *is an interior point of* Z, *then* $(X(\varphi), \varphi)$ *is a smooth point. If* $(X(\gamma), \gamma)$ *is not a smooth point, then it is a point of lower left curvature zero, but not a point of lower right curvature zero. If* $(X(\delta), \delta)$ *is not a smooth point, then it is a point of lower right curvature zero, but not a point of lower left curvature zero. In particular if* Z *has nonempty interior, then* $\{(X(\varphi), \varphi) : \varphi \in Z\}$ *is a (nontrivial) line segment and* $K \cap \mathcal{C}(Z)$ *is a parallel wedge.*

Proof. First we show that Z is connected. Let $W = \{\varphi : \underline{\mathcal{K}}X(\varphi) = 0\}$. Suppose that $W \neq \emptyset$ and $\varphi_1, \varphi_2 \in W$. By Theorem 4.1.1 the tangent lines to the graphs of r and R at $(r(\varphi_1), \varphi_1)$ and $(R(\varphi_1), \varphi_1)$ must be parallel. The same holds for the tangent lines to the graph of $r,\ R$ along the ray φ_2. Since $r,\ R$ are the near and far side of a convex body, simple geometric considerations show this is impossible unless the tangent lines at $(r(\varphi_1), \varphi_1)$ and $(r(\varphi_2), \varphi_2)$ are coincident. Similarly with the two tangent lines to the graphs of R and X at their intersections with the rays $\varphi = \varphi_1$ and $\varphi = \varphi_2$,. Thus $\{(X(\varphi), \varphi) : \varphi \in [\varphi_1, \varphi_2]\}$ is a line segment. Hence W is an interval. Let $\gamma = \inf W$. Suppose that $\gamma \notin W$. From Definition 4.1.2 $(X(\gamma)!, \gamma)$ is a point of lower left curvature zero. Moreover, Lemma 2.1 gives that $X'(\varphi) \to X'_+(\gamma)$ as $\varphi \to \gamma+$, so the left hand tangent line at $(X(\gamma), \gamma)$ is coincident with the common tangent line to points in$W$. The same argument concerning parallel tangent lines shows that a nonsmooth point cannot be both a point of lower left and lower right curvature zero. In a similar way, we define $\delta = \sup W$. The same argument applies to δ. Thus $Z \supset [\gamma, \delta]$. But the parallel tangent argument now shows that $Z = [\gamma, \delta]$.

We have shown that $\{(X(\varphi), \varphi) : \varphi \in Z\}$ is a line segment. Theorem 4.1 now gives that the sets $\{(R(\varphi), \varphi) : \varphi \in Z\}$ and $\{(r(\varphi), \varphi) : \varphi \in Z\}$ are also line segemts that are parallel to this one. Hence $K \cap \mathcal{C}(Z)$ is a parallel wedge. □

From Theorem 4.3 the set of points of zero curvature on the graph of the directed X-ray data must have a specific structure. Figure 2 gives two examples where this structure is not present, and hence these functions are not directed X-rays. In rectangular coordinates the function on the left is the graph of $y = 2 - x^3,\ 0 \le x \le 1$ and $y = (4 + x + x^3)/2,\ \ -1 \le x \le 0$. The point $(0, 2)$ on the graph is a nonsmooth point of zero left and right curvature. Again in rectangular coordinates, on the right in Figure 2 is the graph of $y = 2 - x^6/30 + x^4/6 - x^2/2,\ \ -2 \le x \le 2$. It is easy to check that the only points of zero curvature are $(1, 49/30)$ and $(-1, 49/30)$. Theorem 4.2 shows that neither function can be the directed X-ray of a convex body $K \subset \mathcal{H}$.

5. Convex Bodies with a Common Directed X-ray

In this section we use the results of the previous sections to construct convex bodies with a common directed X-ray. Geometric and topological properties of the family of all convex bodies with a common directed X-ray are explored in [2, 3]. We will use the quadratic form Q. In a manner similar to the proof of Lemma 3.2.5 one can derive

$$Q(r+s) = \frac{(r_1+s_1)\sin(\varphi_2-\varphi_1)+(r_3+s_3)\sin(\varphi_3-\varphi_2)}{r_1\sin(\varphi_2-\varphi_1)+r_3\sin(\varphi_3-\varphi_2)}Q(r)$$

$$+\frac{(r_1+s_1)\sin(\varphi_2-\varphi_1)+(r_3+s_3)\sin(\varphi_3-\varphi_2)}{s_1\sin(\varphi_2-\varphi_1)+s_3\sin(\varphi_3-\varphi_2)}Q(s)- \tag{5}$$

$$\frac{\sin(\varphi_2-\varphi_1)\sin(\varphi_3-\varphi_2)\sin(\varphi_3-\varphi_1)[r_1s_3-r_3s_1]^2}{d(\varphi_1,\varphi_2,\varphi_3)}, \text{ where}$$

$d(\varphi_1,\varphi_2,\varphi_3) = (r_1\sin(\varphi_2-\varphi_1)+r_3\sin(\varphi_3-\varphi_2))(s_1\sin(\varphi_2-\varphi_1)+s_3\sin(\varphi_3-\varphi_2))$.

THEOREM 5.1. *For $t \geq 0$, let K_t be the body with near side tr and far side $tr+X$.*

1. *There exist $t^* \geq 1$ such that K_t is convex body for $0 \leq t \leq t^*$.*
2. *If K is a wedge but not a parallel wedge, then $t^* = 1$.*
3. *K_t is convex for all $t \geq 0$ if and only if r is a line segment and $Q(X) \leq Q(R)$ for all $\varphi_1 \leq \varphi_2 \leq \varphi_3$.*

Proof. The bodies K_t clearly have directed X-ray X. Only the convexity is in question. Since $Q(tr) = t^2Q(r) \leq 0$, tr is concave away from the origin. It remains to look at the far side. From (5), $q(t) = Q(tr+X)$ is quadratic in t and the coefficient of t^2 is $Q(r) \leq 0$. If $Q(r) < 0$, the graph of q is a parabola opening downwards. If $Q(r) = 0$, the graph is a straight line. Since $q(0) = Q(X) \geq 0$ and $q(1) = Q(r+X) \geq 0$, in either case $q(t) \geq 0$ for $0 \leq t \leq 1$. Let $t^* = \sup\{t : Q(tr+X) \geq 0 \text{ for all ordered triples } \varphi_1, \varphi_2, \varphi_3\}$. Then $t^* \geq 1$ and by continuity the body with near side t^*r and far side $X + t^*r$ is convex.

If K is a wedge but not a parallel wedge, the graph of q is a line with $q(0) = Q(X) > 0$ and $q(1) = Q(R) = 0$. The line has negative slope and hence $q(t) < 0$, when $t > 1$.

Recall that when $Q(r) < 0$, the graph of $q(t)$ is a parabola opening downward. Hence $q(t) < 0$ for t sufficiently large. Thus in order that K_t be convex for all $t \geq 0$, we must have $Q(r) = 0$, that is the near side must be a line segment. In this case $q(t)$ is linear with $q(0) = Q(X) \geq 0$ and $q(1) = Q(R) \geq 0$. So, K_t will be convex for all $t \geq 0$ if and only if $Q(X) \leq Q(R)$. $\square$

REMARK 5.2. *To get an idea of the geometry of the body corresponding to $t = t^*$, replace the quadratic form Q by $\underline{K}$. From continuity in t it follows that either $\underline{K}(X + t^*r)(\varphi_0) = 0$ for some φ_0 (in which case K_{t^*} has a point of zero curvature on the far side boundary along that ray), or $X + tr$ is not concave toward the origin at a supporting ray when $t > t^*$.*

Next we consider convex bodies obtained from K by dilating the far side.

THEOREM 5.3. *For $t \geq 1$ let K^t be the body with far side tR and near side $tR - X$.*

1. *K^t is a convex body for all $t \geq 1$ if and only if $Q(R) = 0$, that is R is a line segment.*
2. *Otherwise there exists $t^* \geq 1$ such that K^t is a convex body for all $1 \leq t \leq t^*$.*

Proof. The proof is almost identical to that of Theorem 5.1. This time we need to determine values of t for which $Q(tR - X) \leq 0$. Now $q(t) = Q(tR - X)$ is quadratic in t with leading coefficient $Q(R) \geq 0$. If $Q(R) > 0$ for some triple of angles, then $q(t) > 0$ for the same triple when t is sufficiently large. Thus the condition $Q(R) = 0$ is necessary. On the other hand if $Q(R) = 0$, then $q(t)$ is linear with $q(1) = Q(r) \leq 0$ and $q(0) = Q(-X) = Q(X) \geq 0$. Consequently $q(t) \leq 0,\ t \geq 1$. $\square$

The same reasoning as in Remark 5.2 shows that the body K^{t^*} either has a point of zero upper curvature on the near side boundary, or concavity away from the origin will be lost at a supporting ray for $t > t^*$.

REMARK 5.4. *Suppose K is not a parallel wedge. Combining the constructions in Theorems 5.1 and 5.3 we may construct a two parameter family of convex bodies with the same directed X-ray as K.*

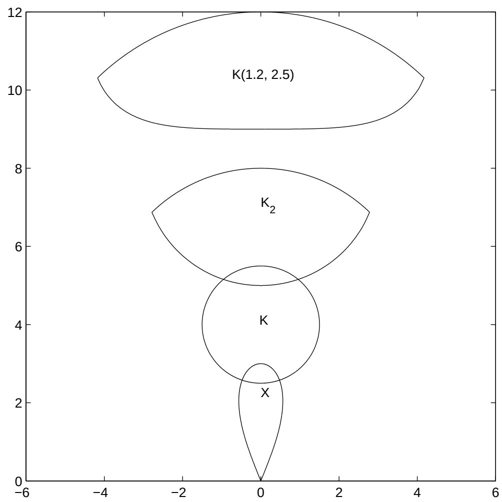

Figure 3: K, disc with center $(0, 4)$, radius 1.5, directed X-ray X. Body K_2 of Theorem 5.1 and body $K(1.2, 2.5)$ of Remark 5.4.

Explicitly, we first form the bodies with near side $sR - X$ and far side sR, and then for each real number s form the body with near side $t(sR - X)$ and far side $t(sR - X) + X$. This leads to a two parameter family of bodies $K(s, t)$ with near side $t(s-1)R + tr$ and far side $(t(s-1)+1)R + (t-1)r$. See Figure 3. It is not hard to see that for most convex bodies K the family of convex bodies with the same directed X-ray as K is infinite dimensional [2]. One exception occurs when K is a parallel wedge.

Figure 3 contains graphs of a convex body K, the convex body K_2 of Theorem 5.2 and the convex body $K(1.2, 2.5)$ of the previous remark. The latter body has a point of zero curvature on the near side at (0, 9).

6. Functions that are directed X-rays - $\inf \underline{\mathcal{K}} X > 0$

In this section we give necessary and sufficient conditions that a function X be the directed X-ray of a convex body $K \subset \mathcal{H}$. Corollary 3.4 (Longinetti's Theorem [10]) states that the graph of X must be concave toward the origin. This is not sufficient in general. In addition to the concavity condition, in most situations X must satisfy certain estimates near the supporting rays and near points of lower curvature zero. First some preliminary results on the behavior of the directed X-ray of a convex body near the supporting rays. We continue to use $K,\ r,\ R,\ X$ according to the conventions of the last two sections. Recall that $\varphi = \alpha$ and $\varphi = \beta$ are the supporting rays.

LEMMA 6.1. *If $X(\alpha) = 0$, then for $\varphi > \alpha$ and sufficiently close to α $X'_+(\varphi) \geq X'_+(\alpha)/2 > 0$. Similarly, if $X(\beta) = 0$, then for $\varphi < \beta$ and sufficiently close to β $X'_-(\varphi) \leq X'_-(\beta)/2 < 0$.*

Proof. We only treat the supporting ray $\varphi = \alpha$, the case $\varphi = \beta$ being almost identical. By Lemma 2.1.3 $X'_+(\varphi) \to X'_+(\alpha)$ as $\varphi \to \alpha_+$. Hence we need only show that $X'_+(\alpha) > 0$. Since $X(\alpha) = 0$ and $X > 0$ on (α, β) it is clear that $X'_+(\alpha) \geq 0$. If $X'_+(\alpha) = 0$, then Lemma 2.1.5 ensures that $R'_+(\alpha)$ and $r'_+(\alpha)$ are finite. Since $X(\alpha) = 0,\ R(\alpha) = r(\alpha)$. From (1)

$$\cot(\omega - \alpha) = \frac{R'(\alpha+)}{R(\alpha)} = \frac{r'(\alpha+)}{r(\alpha)} = \cot(\psi - \alpha),$$

where $\omega,\ \psi$ are the angles of inclination of the tangent lines to the graphs or R, r respectively at α. But this implies that the two tangent lines are coincident and hence that K has a smooth point there. Finally Lemma 2.1.5 gives that $R'_+(\alpha) = +\infty$ and $r'(\alpha) = -\infty$ which is a contradiction. Hence $X'_+(\alpha) > 0$. $\square$

LEMMA 6.2.

1. *$\underline{K}X \geq \frac{2rX^2}{R}\left[\frac{D_1X}{X} - \frac{D_1r}{r}\right]^2 \geq \frac{2r}{R}(D_1X)^2 - \frac{4X}{R}D_1rD_1X.$*
2. *$\overline{D}_2X \leq X + \frac{2}{R}(D_1X)^2 + \frac{4}{R}D_1rD_1X.$*

Proof. From Lemma 3.8.4 and the fact that $X = R - r$ we have

$$\underline{K}X \geq 2rR\left[\frac{D_1R}{R} - \frac{D_1r}{r}\right]^2 = 2rR\left[\frac{D_1r + D_1X}{r + X} - \frac{D_1r}{r}\right]^2$$

$$= 2rR\left[\frac{rD_1X - XD_1r}{r(r + X)}\right]^2 = \frac{2rX^2}{R}\left[\frac{D_1X}{X} - \frac{D_1r}{r}\right]^2.$$

This proves the first part of Lemma 6.2.1. The second follows from basic algebra and the trivial inequality $(a-b)^2 \geq a^2 - 2ab$.

To obtain the estimate in the second part, note that from the first part

$$X^2 - X\overline{D}_2 X = \underline{K}X - 2(D_1X)^2 \geq \frac{2(r-R)}{R}(D_1X)^2 - 4\frac{XD_1rD_1X}{R}$$
$$= -\frac{2X}{R}(D_1X)^2 - 4\frac{XD_1rD_1X}{R}.$$

Dividing by X gives

$$X - \overline{D}_2 X \geq -\frac{2(D_1X)^2}{R} - 4\frac{D_1rD_1X}{R}.$$

This inequality is clearly equivalent to 6.2.2. □

The following consequence of Lemma 6.2 gives the a priori estimates for $\underline{K}X$ and $\overline{D}_2X$ near the supporting rays.

Lemma 6.3. *Suppose that $\frac{X'_+(\alpha)}{X(\alpha)}$ (or $\frac{X'_-(\beta)}{X(\beta)}$) is infinite. Then there exist constants $C_1,\ C_2 > 0$ such that for all φ sufficiently close to α (or sufficiently close to β)*

1. $\underline{K}X \geq C_1(D_1X)^2 > 0$;
2. $\overline{D}_2X \leq C_2(D_1X)^2$.

Proof. Suppose that $\frac{X'_+(\alpha)}{X(\alpha)}$ is infinite. There are two cases to consider: (1) when $X(\alpha) = 0$, and (2) when $X(\alpha) > 0$ and $X'_+(\alpha) = +\infty$. Let's consider the second case first. Lemma 6.2.1 gives that

$$\underline{K}X \geq \frac{2r(D_1X)^2}{R} - \frac{4XD_1rD_1X}{R}.$$

If $r'_+(\alpha) = -\infty$, then the second term can be discarded since it will be positive for φ sufficiently close to α. Otherwise, Lemma 2.1.5 ensures that D_1r is bounded near α and for φ sufficiently close to α the term involving $(D_1X)^2$ dominates the term involving D_1X. The second summand can be discarded at the expense of decreasing the coefficient $(2r/R > 0)$ of the first. Under the hypothesis $X(\alpha) > 0$

and $X'_+(\alpha) = +\infty$, Lemma 6.3.2 is trivial. Indeed, the inequality $\underline{\mathcal{K}}X \geq 0$ can be rewritten as $\overline{D}_2 X \leq X + \frac{2(D_1X)^2}{X}$ which trivially implies the desired estimate by the continuity of X.

Now suppose that $X(\alpha) = 0$. If $X'_+(\alpha) = +\infty$, then 6.3.1 follows via the same argument, while 6.3.2 follows from 6.2.2 and a similar domination argument.

If $X'_+(\alpha)$ is finite, Lemma 6.1 and Lemma 2.1.5 show that there exist constants m, M such that $0 < m < X'_+(\varphi) < M < \infty$ for φ sufficiently close to α. In this case, both inequalities follow immediately from Lemma 6.2. $\square$

The theorem characterizing functions that are directed X-rays of a convex body in $\mathcal{H}$ will require different hypotheses on the behavior of X near the supporting ray. Therefore, we construct a definition describing the different situations.

Definition 6.4. *Suppose that $X > 0$ on (α, β) and is concave toward the origin.*

1. *The data X is of class W, or wedgelike, at α (or β) if $X(\alpha) > 0$ and $X'_+(\alpha) < \infty$ (or $X(\beta) > 0$ and $X'_-(\beta) > -\infty$).*

2. *The data X is of class C, or cornerlike, at α (or β) if $X(\alpha) = 0$ and $X'_+(\alpha) < \infty$, (or $X(\beta) = 0$ and $X'_-(\beta) > -\infty$).*

3. *The data X is of class S or smooth at α (or β) if $X(\alpha) = 0$ and $X'_+(\alpha) = +\infty$ (or $X(\beta) = 0$ and $X'_-(\beta) = -\infty$.)*

4. *The data X is of class SW or that of a smooth wedge at α (or β) if $X(\alpha) > 0$ and $X'_+(\alpha) = +\infty$ (or $X(\beta) > 0$ and $X'_-(\beta) = -\infty$.)*

Note that a convex body K which is not smooth at a supporting ray may give rise to data of type S or SW. However, either the near side or the far side must be smooth at the supporting ray. This is demonstrated by the convex bodies and directed X-ray data in Figures 3 and 4. The convex body and directed X-ray data on the left in Figure 4 exhibit behavior of type C at the supporting ray $\varphi = \alpha$ and type W at $\varphi = \beta$. On the right in Figure 4 is an example of behavior of type SW at $\varphi = \alpha$ and type C at $\varphi = \beta$.

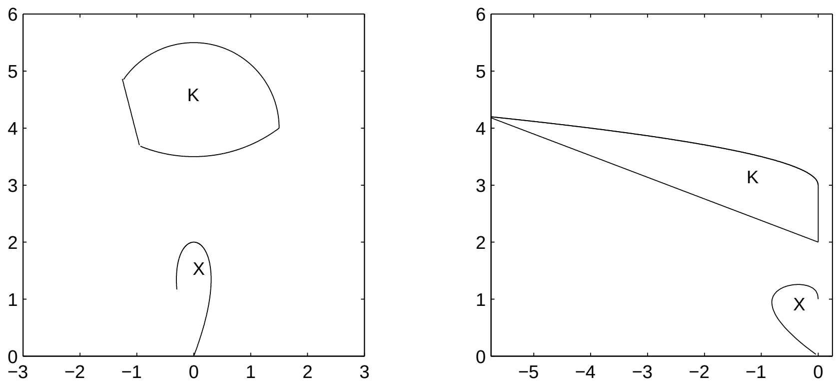

Figure 4: Convex bodies and directed X-rays exhibiting different behavior types near supporting rays.

We now give the first theorem describing functions that are directed X-rays. For the time being we ignore points of lower curvature zero. We no longer assume that X is the directed X-ray of a convex body, but rather wish to come to that conclusion.

THEOREM 6.5. *Let $X \in C[\alpha, \beta]$, $0 < \alpha < \beta < \pi$ be positive on (α, β). Suppose that $\inf_{\varphi \in (\alpha,\beta)} \underline{\mathcal{K}} X(\varphi) > 0$. Suppose further that:*

1. *If X is of type C or S at a supporing ray, then X'_+ (or X'_-) is nonzero there, and there exists a constant $C > 0$ such that $\overline{D}_2 X(\varphi) \leq C(D_1 X(\varphi))^2$ for φ sufficiently close to the supporting ray.*

2. *If X is of type SW at a supporting ray, then there exists a constant $C > 0$ such that $\underline{\mathcal{K}} X(\varphi) \geq C(D_1 X(\varphi))^2$ for φ sufficiently close to the supporting ray.*

Then there exists a convex body K in $\mathcal{H}$ such that X is the directed X-ray of K.

Proof. From Theorem 3.11 it suffices to show that there exists a line $\ell = \ell(\varphi) > 0$, $\varphi \in [\alpha, \beta]$, and a $t > 0$ such that $\underline{\mathcal{K}}(X + t\ell) \geq 0$. In fact, we will show that any line which intersects the supporting rays a positive distance from the origin will do. Choose and fixsuch a line. Since $\inf_{\varphi\in(\alpha,\beta)} \underline{\mathcal{K}}X(\varphi) > 0$, there exists an $\epsilon > 0$ such that $\underline{\mathcal{K}}X(\varphi) > \epsilon$ for all $\varphi \in (\alpha,\beta)$. Since $\mathcal{K}\ell = 0$, Lemma 3.8.3 gives for all $t > 0$

$$\underline{\mathcal{K}}(X+t\ell) \geq \underline{\mathcal{K}}X + t\ell X\left(\frac{\underline{\mathcal{K}}X}{X^2} - 2\left[\frac{D_1X}{X} - \frac{\ell'}{\ell}\right]^2\right). \tag{6}$$

The second summand in the above inequality is bounded below on compact subsets of (α,β). Hence for $t > 0$ sufficiently small $\underline{\mathcal{K}}(X+t\ell) > 0$ on compact subsets of (α,β). Thus we need only show that this is also true near the supporting rays. If the data is of type W, then $\frac{D_1X}{X}$ is bounded and boundedness near the supporting rays is trivial since $\frac{\ell'}{\ell}$ is also bounded.

Suppose now that the data is of type C or S at a supporting ray. Using Lemma 3.9.1 and expanding the last term in (6) we obtain

$$\underline{\mathcal{K}}(X+t\ell) \geq \underline{\mathcal{K}}X + t\ell\left[X - \overline{D}_2X + 4t\ell' D_1X - 2t\ell X\left(\frac{\ell'}{\ell}\right)^2\right]. \tag{7}$$

Applying hypothesis 6.5.1 gives

$$\begin{aligned}\underline{\mathcal{K}}X &= X^2 + 2(D_1X)^2 - X\overline{D}_2X\\ &\geq X^2 + 2(D_1X)^2 - CX(D_1X)^2) \geq (2-CX)(D_1X)^2.\end{aligned}$$

Using this inequality and 6.5.1 again in (7) gives

$$\underline{\mathcal{K}}(X+t\ell) \geq (2 - CX - t\ell C)(D_1X)^2 + 4t\ell' D_1X - 2t\ell X\left(\frac{\ell'}{\ell}\right)^2.$$

Since $X = 0$ at the supporting ray, we may choose $t > 0$ so that $2 - CX - t\ell C > 0$ when φ is sufficiently close to that ray. The assumption that $X'_+ \neq 0$ (or $X'_- \neq 0$) at the supporting ray, Lemma 2.1.2 and 2.1.3 give that D_1X is bounded away from zero near the supporting ray. Thus, the last expression is clearly positive for t sufficiently small and φ sufficiently close to the supporting ray.

We now deal with the case where the data is of type SW near a supporting ray, that is $X > 0$ and D_1X unbounded. Now the condition $\underline{\mathcal{K}}X > 0$ implies that $\overline{D}_2 X < X + \frac{(D_1 X)^2}{X}$. Proceeding as before, applying this inequality and hypothesis 6.5.2 we obtain

$$\underline{\mathcal{K}}(X + t\ell) = \underline{\mathcal{K}}X + t\ell\left(X - \overline{D}_2 X + 4tD_1 X + \ell' - 2t\ell X\left(\frac{\ell'}{\ell}\right)^2\right)$$

$$\geq (C - \frac{t\ell}{X})(D_1 X)^2 + 4t\ell' D_1 X - 2t\ell X\left(\frac{\ell'}{\ell}\right)^2.$$

Since $X > 0$ near the supporting ray, we may choose $t > 0$ sufficiently small so that $C - \frac{t\ell}{X} > 0$. The fact that $X'_+(\alpha) = \infty$ (or $X'_-(\beta) = -\infty$) and ℓ' is bounded give that for $t > 0$ sufficiently small and φ sufficiently close to α, $\underline{\mathcal{K}}(X + t\ell) > 0$. □

Before considering points of zero curvature on the graph of X, we present some examples which show the necessity of the hypotheses on the behavior of X near supporting rays. Lemmas 6.1 and 6.3 show that the hypotheses 6.5.1 and 6.5.2 of Theorem 6.5 are necessary, but one might ask whether they are implied by the condition $\inf \underline{\mathcal{K}}X > 0$. We give examples showing that Theorem 6.5 fails if these hypotheses are omitted.

EXAMPLE 6.6. *Data of type C. Let $X(\varphi) = -\cos 2\varphi + \sqrt{(-\cos 2\varphi)^3}$, $\pi/4 \leq \varphi \leq 3\pi/4$. One computes $X' = 2\sin 2\varphi + 3\sin 2\varphi\sqrt{-\cos 2\varphi}$ and $X'' = 4\cos 2\varphi + 6\cos 2\varphi\sqrt{-\cos 2\varphi} + \frac{3\sin^2 2\varphi}{\sqrt{-\cos(2\varphi)}}$. Also $X(\pi/4) = 0, X'_+(\pi/4) = 2$, and $X''(\pi/4) = +\infty$. Consequently, the inequality $X'' \leq C(X')^2$ fails near the supporting ray $\varphi = \pi/4$. On the other hand one can show that $\mathcal{K}X \geq 8$, when $\varphi \in [\pi/4, 3\pi/4]$. Lemma 6.3 shows that X is not the directed X-ray of a convex body in $\mathcal{H}$.*

An example showing the necessity of 6.5.1 for data of type S is more difficult to construct. To see why, suppose that X were C^2 near α. The mean value theorem then implies that there is a sequence $\varphi_k \to \alpha+$ such that $X''(\varphi_k) \to -\infty$. But in order that the inequality $X'' \leq C(X')^2$ fail near α, we must also have a sequence $\eta_k \to \alpha_+$ such that $X''(\eta_k) \to +\infty$. Thus in such an example X' must have wild oscillations as $\varphi \to \alpha+$. Nevertheless, such an example exists.

Example 6.7. *Data of type S. For $\varphi \in [\pi/4, \pi/2]$ let*

$$X'(\varphi) = \frac{1 + \sin^2(\ln(-\cos 2\varphi))}{\sqrt{-\cos 2\varphi}}, \quad X(\varphi) = \int_{\pi/4}^{\varphi} X'(\eta) d\eta.$$

A straightforward computation gives

$$X''(\varphi) = \frac{-\sin 2\varphi}{(-\cos(2\varphi))^{3/2}} \left[1 + \sin^2(\ln(-\cos 2\varphi)) + 2\sin(2\ln(-\cos 2\varphi))\right].$$

It is not difficult to see that the inequality $X'' \le C(X')^2$ fails near $\pi/4$. Simply choose $\varphi_k \to \pi/4+$ so that $-\cos(2\varphi_k) = e^{-(2k+1)\pi/2}$. It remains to show that X is concave toward the origin. To do this we show that $\mathcal{K}X > 0$ on $[\pi/4, \pi/2]$. Note that

$$\begin{aligned} X &\le 2\int_{\pi/4}^{\varphi} \frac{d\eta}{\sqrt{-\cos 2\eta}} = 2\int_{\pi/4}^{\varphi} \frac{\sin 2\eta d\eta}{\sin 2\eta \sqrt{-\cos 2\eta}} \\ &\le \frac{2}{\sin 2\varphi} \int_{\pi/4}^{\varphi} \frac{\sin 2\eta d\eta}{\sqrt{-\cos 2\eta}} = \frac{2\sqrt{-2\cos\varphi}}{\sin 2\varphi}. \end{aligned}$$

We only need be concerned where $X'' > 0$. Replacing X by the upper bound above we have

$$\begin{aligned} \mathcal{K}X &\ge 2(X')^2 - XX'' \\ &\ge \frac{2\left[1 + \sin^2(\ln(-\cos 2\varphi)) + \sin(2\ln(-\cos 2\varphi))\right]}{-\cos 2\varphi}, \end{aligned}$$

when $X'' > 0$. The above estimate clearly shows that $\mathcal{K}X > 0$ on $(\pi/4, \pi/2)$.

Example 6.8. *Data of type SW. We seek an X that is concave toward the origin, satisfies $X(\alpha) > 0$, $X'_+(\alpha) = \infty$, while the inequality $\underline{\mathcal{K}}X \ge C(D_1 X)^2$ does not hold for any $C > 0$. We assume that X is C^2 on (α, β). It is convenient to rotate coordinates so that $\alpha = 0$ and locally the graph of X is the graph of a convex function f (in rectangular coordinates) such that $a = X(0) > 0$, $f(a) = 0$, $f'_+(a) = 0$. To translate the inequality involving $\mathcal{K}X$ into an equivalent inequality involving f note that if $x = X(\varphi)\cos(\varphi)$, then*

$$\mathcal{K}X(\varphi) = X^2 + 2(X')^2 - XX'' = f''(x)\left(\frac{dx}{d\varphi}\right)^3 = \frac{X^3\cos^3(\xi)}{\sin^3(\xi-\varphi)} f''(x),$$

where ξ is the angle of inclination of the tangent line to the graph of X at φ. Now $(X')^2 = X^2\left(\frac{X'}{X}\right)^2 = X^2\cot^2(\xi-\varphi)$ and $0 < \varphi < \xi$ when $\varphi > 0$ and near 0. Also the law of sines shows that $\sin(\xi - \varphi)/\sin(\xi) \to 1$ as $\varphi \to 0+$. Since $f'(x) = \tan(\xi)$, the inequality of hypothesis 6.5.2 becomes for $x > a$

$$f''(x) \geq \frac{C}{X}\frac{\cos^2(\xi-\varphi)}{\cos^3\xi}\sin(\xi-\varphi) \geq C\sin(\xi-\varphi) \geq Cf'(x) > 0, \quad (8)$$

where $C > 0$ may be a new constant at each step. Now define for $x > a$

$$g''(x) = 1 + \sin\left(\frac{1}{x-a}\right),\ g'(x) = \int_a^x g''(t)dt,\ g(x) = a + \int_a^x g'(t)dt.$$

g is convex since $g'' \geq 0$ but there are points arbitrarily close to a at which $g'' = 0$ so (8) is not satisfied. The desired function f is obtained by repeating the above construction, but beginning with $f''(x) = g''(x) + (g'(x))^2$.

7. Functions that are directed X-rays - $\inf \underline{\mathcal{K}}X = 0$

We now allow points of zero lower curvature on the graph of X. To derive the necessary a priori estimate, again suppose that $K \subset \mathcal{H}$ is a convex body with near side r, far side R and directed X-ray X.

LEMMA 7.1. *Let*

$$Z = \{\varphi \in [\alpha,\beta] : (X(\varphi),\varphi) \text{ is a point of lower curvature zero}\}.$$

Then there exists a constant $C > 0$ such that in some open subset of $[\alpha,\beta]$ containing Z

$$\underline{\mathcal{K}}X(\varphi) \geq C\left[\frac{D_1X}{X} - \frac{\ell'}{\ell}\right]^2,$$

where ℓ is a line whose angle of inclination is uniquely determined by Z.

Proof. Suppose at first Z consists of a single point φ_0 and that $(X(\varphi_0), \varphi_0)$ is a smooth point of lower curvature zero. From Theorem 4.1 we know that $(r(\varphi_0), \varphi_0)$ is a smooth point of upper curvature zero and the tangent lines to both these points have the same angle of inclination, i.e $\psi_0 = \xi_0$. This is the angle of inclination determined by Z that we choose for ℓ. From Lemma 6.2.1

$$\underline{\mathcal{K}}X \geq \frac{2rX^2}{R}\left[\frac{D_1X}{X} - \frac{D_1r}{r}\right]^2 = \frac{2rX^2}{R}\left[\frac{D_1X}{X} - \frac{\ell'}{\ell} + \frac{\ell'}{\ell} - \frac{D_1r}{r}\right]^2.$$

The inequality will follow provided we show that

$$\left(\frac{D_1X}{X} - \frac{\ell'}{\ell}\right)\left(\frac{\ell'}{\ell} - \frac{D_1r}{r}\right) \geq 0$$

in a neighborhood of φ_0. Since $\xi_0 = \psi_0$ and this is the angle of inclination of ℓ, (1) implies that

$$\left(\frac{D_1X}{X} - \frac{\ell'}{\ell}\right)\left(\frac{\ell'}{\ell} - \frac{D_1r}{r}\right) = \frac{\sin(\xi_0 - \xi)\sin(\psi - \psi_0)}{\sin^2(\xi_0 - \varphi)\sin(\xi - \varphi)\sin(\psi - \varphi)}.$$

Since ξ is a nondecreasing function of φ and ψ is a nonincreasing function of φ, and $\xi_0 = \xi(\varphi_0)$, $\psi(\varphi_0) = \psi_0$, the numerator is nonnegative in a neghborhood of φ_0. Similarly, $\sin(\xi_0 - \varphi)\sin(\psi_0 - \varphi) > 0$ and Lemma 2.1 gives that this expression is also positive in a neighbor hood of φ_0. This completes the proof when $(X(\varphi_0), \varphi_0)$ is a smooth point of lower curvature zero. If it is a point of lower left or lower right curvature zero, the proof is similar. Note that the inequality trivially holds at nonsmooth points since there the left hand side is $+\infty$ while the right hand side is finite. If Z is an interval, then a similar argument works at the end points of Z. □

THEOREM 7.2. *Let $X \in C[\alpha, \beta]$ With $X > 0$ on (α, β). Let*

$$Z = \{\varphi \in [\alpha, \beta] : (X(\varphi), \varphi) \textit{ is a point of lower curvature zero}\}.$$

Necessary and sufficient conditions that X is the directed X-ray of a convex body $K \subset \mathcal{H}$ are the following.

1. *X is concave toward the origin.*
2. *X satisfies hypotheses 1 and 2 of Theorem 6.5.*
3. *If $Z \neq \emptyset$, then $\{(X(\varphi), \varphi) : \varphi \in Z\}$ is a line segment (possibly a single point), and no nonsmooth point is a point of both lower left and lower right curvature zero.*
4. *$\underline{\mathcal{K}} X(\varphi) \geq C \left[\frac{D_1 X}{X} - \frac{\ell'}{\ell}\right]^2$ in an open subset of $[\alpha, \beta]$ containing Z, where $C > 0$ is a constant and ℓ is a line whose angle of inclination is uniquely determined by Z.*

Proof. The necessity of the conditions was established in Corollary 3.4, Theorem 4.3, Lemma 6.3, and Lemma 7.1. When $Z = \emptyset$ this result reduces to Theorem 6.5. Assume for now that Z consists of a single point φ_0 and that $(X(\varphi_0), \varphi_0)$ is a smooth point of lower curvature zero. This implies $\alpha < \varphi_0 < \beta$. Let ξ_0 be the angle of inclination of the tangent line to the graph of X at $(X(\varphi_0), \varphi_0)$, and ℓ be a fixed line with angle of inclination ξ_0 such that $\ell(\varphi) > 0$, $\varphi \in [\alpha, \beta]$. As in the proof of Theorem 6.5, we will show that $\underline{\mathcal{K}}(X + t\ell) \geq 0$ for $t > 0$ sufficiently small. The proof of Theorem 6.5 shows that this inequality holds near the supporting rays and also holds outside any open set containing Z. Thus, it only remains to establish the inequality in a neighborhood of φ_0. From ! Lemma 3.8.4, Lemma 7.1 and the fact that $X(\varphi_0) > 0$ we have

$$\begin{aligned} \underline{\mathcal{K}}(X + t\ell) &\geq \frac{X + t\ell}{X} \underline{\mathcal{K}} X - 2t\ell X \left[\frac{X'}{X} - \frac{\ell'}{\ell}\right]^2 \\ &\geq \frac{C(X + t\ell) - 2t\ell X^2}{X} \left[\frac{X'}{X} - \frac{\ell'}{\ell}\right]^2 \geq 0 \end{aligned}$$

in a neighborhhod of φ_0 when $t > 0$ is sufficiently small. This completes the proof when Z consists of a single point which gives rise to a smooth point of lower curvature zero. If Z consists of a single point of lower left or lower right curvature zero, the proof is almost identical. The difference is that one chooses ξ_0 to be the angle of inclination of the left hand tangent line if φ_0 has lower left curvature zero. The opposite choice is made if φ_0 is a point of lower right curvature zero. (From hypothesis 6.5.2 the point α (β) can be a point of lower right (left) curvature zero only when the data is of type SW there, that is $X(\alpha) > 0$ ($X(\beta) > 0$).) If Z has nonempty interior, then one applies the above argument at the endpoints and may choose ξ_0 to be the angle of inclination of the tangent line to any point on the graph of X corresponding to an interior point of Z. □

It is natural to ask whether hypothesis 4 of Theorem 7.2 is needed in addition to hypothesis 3. Using the mean value theorem one can show that if X'' is monotone to the right and left of the points of zero curvature, then hypothesis 3 does imply hypothesis 4. However, this is not true in general. As in Example 6.8 it suffices to deal with the situation where X is C^2 and locally the graph of X is also the graph of a C^2 function f in rectangular coordinates. The a priori estimate 4 of Theorem 7.2 translates to

$$f''(x)\frac{X^3\cos^3\xi}{\sin^3(\xi-\varphi)} \geq C\frac{\sin^2(\xi-\xi_0)}{\sin^2(\xi-\varphi)\sin^2(\xi_0-\varphi)}.$$

We take $\xi_0 = \pi$, $\xi \to \pi_-$ and $\varphi \to \pi/4-$. Then $\cos\xi < 0$ and the inequality becomes

$$f''(x) \leq \frac{C\sin(\xi-\varphi)}{X^3\cos\xi\sin^2\varphi}\tan^2\xi \leq -C_1(f'(x))^2, \tag{9}$$

for some new constant $C_1 > 0$ since $f'(x) = \tan\xi$. We need to produce a continuous function f so that $f(a) = a > 0$, $f'(a) = 0$, $f''(a) = 0$, $f''(x) < 0$ in a punctured neighborhood of a, while the inequality (9) fails to hold in a neighborhood of a. Such a function can be constructed in a manner similar to that used in Example 6.8. Define $g''(x) = -|x-a|(1+\sin(\frac{1}{x-a}))$, $g'(x) = \int_a^x f''(t)dt$, and then

$$f''(x) = g''(x) - (g'(x))^4, f'(x) = \int_a^x f''(t)dt,\ f(t) = a + \int_a^x f'(t)dt.$$

The conditions characterizing functions that are directed X-rays are fairly complicated. We state a geometric form of the result that is a consequence of the proof.

Theorem 7.3. *Let X be continuous on $[\alpha,\beta]$. If X is the directed X-ray of a convex body K in $\mathcal{H}$, then there exists a line $\ell = \ell(\varphi) >$*

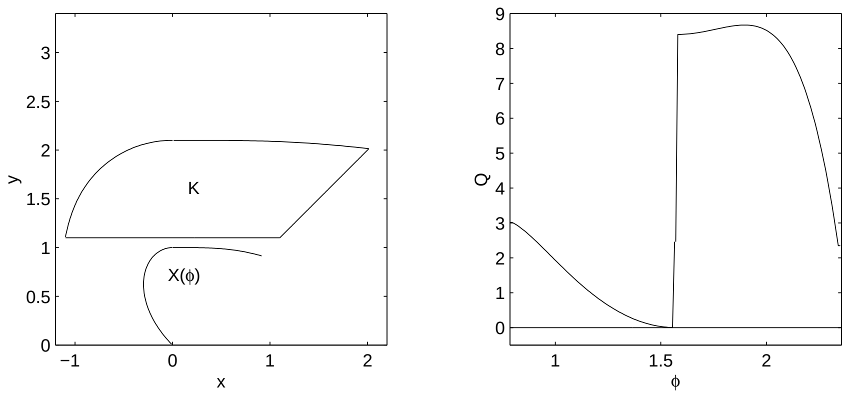

Figure 5: Left: $X(\phi)$ directed X-ray of K, body with near side $r = 1.1\csc(\phi)$, far side $R = r + X$. Right: $Q_h(R)$, $h = 0.01$

0, $\varphi \in [\alpha, \beta]$ *and a real number* $t_\ell > 0$ *such that for all* t, $0 < t < t_\ell$, *the body with near side* $t\ell$ *and far side* $X + t\ell$ *is a convex body with directed X-ray* X.

1. *If* $\inf_{\varphi\in(\alpha,\beta)} \underline{\mathcal{K}}X(\varphi) > 0$, *then* ℓ *may be any line which intersects the supporting rays a positive distance from the origin.*

2. *If* $\inf_{\varphi\in(\alpha,\beta)} \underline{\mathcal{K}}X(\varphi) = 0$, *then the angle of inclination of* ℓ *is unique and* ℓ *must be parallel to the (right or left) tangent line to the graph of* X *at a point of zero curvature.*

We end with an example. On the left in Figure 5 the directed X-ray data is given by $X = 2 - \sin\varphi$, $\pi/4 \le \varphi \le \pi/2$, and $X = -\cos(2\varphi)(2 - \sin\varphi)$, $\pi/2 \le \varphi \le 3\pi/4$. One readily checks that $(1, \pi/2)$ is a smooth point with positive left curvature and zero right curvature. X is the directed X-ray of the convex body in the figure. The body on the left in Figure 5 has near side $r = 1.1\csc\varphi$ and far side $X + r$. On the right side of Figure 5 is the graph of $Q_h(X + 1.1\csc\varphi)$, $h = 0.01$. Note the graph is everywhere nonnegative and vanishes as $\varphi \to \pi/2-$. The large jump as $\varphi \to \pi/2+$ is due to the discontinuity in the second derivative of X.

References

[1] W. Black and D. Koop, *Properties of planar convex bodies from one directed X-ray*, Proceedings of the REU Program in Mathematics (2000), 1–19.

[2] D. Butcher and A. Medin, *Infinite dimensionality and unboundedness for sets of planar convex bodies with a common directed x-ray*, Proceedings of the REU Program in Mathematics (2002), 25–52.

[3] D. Butcher, A. Medin and D. C. Solmon, *Planar convex bodies with a common directed X-ray*, in preparation.

[4] K. J. Falconer, *X-ray problems for point sources*, Proc. London Math. Soc. (3) **46** (1983), 241–262.

[5] R. J. Gardner, *Symmetrals and X-rays of planar convex bodies*, Arch. Math. **4** (1983), 183–189.

[6] R. J. Gardner, *Geometric tomography*, Encyclopedia of Mathematics and its Applications **58** (1995).

[7] P. C. Hammer, *Problem 2. in convexity*, Proceedings of Symposia in Pure Mathematics **VII** (1963), 498–499, Providence, RI.

[8] J. Kimble, *Functions that are directed X-rays of planar convex bodies*, Proceedings of the REU Program in Mathematics (2001), 83–98.

[9] D. Lam and D. C. Solmon, *Reconstructing convex polygons in the plane from one directed x-ray*, Discrete and Comput. Geom. **26** (2001), 105–146.

[10] M. Longinetti, *Some questions of stability in the reconstruction of plane convex bodies from projections*, Inverse Problems **1** (1985), 87–97.

[11] H. L. Royden, *Real analysis*, Macmillan, New York, 1988.

[12] A. Volčič, *A three point solution to Hammer's X-ray problem*, J. London Math. Soc. **40** (1989), no. 2, 171–178.

Received January 13, 2003.