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The Sharp Exponent for a Liouville-type Theorem for an Elliptic Inequality

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Summary. - *We determine the sharp exponent for a Liouville-type theorem for an elliptic inequality. This answers a question raised in [1] which is related to a conjecture by De Giorgi [5].*

1. Introduction

In a recent paper, Alberti-Ambrosio-Cabré [1] raised the following question:

Question (AAC): *Let $\varphi \in L^\infty_{loc}(\mathbf{R}^n)$ be a positive function and assume that $\sigma \in H^1_{loc}(\mathbf{R}^n)$ satisfies*

$$\sigma\ \mathrm{div}(\varphi^2 \nabla\sigma) \geq 0 \qquad \text{in } \mathbf{R}^n \tag{1}$$

in the distributional sense. Which is the optimal (largest) constant $\delta_n > 0$ such that the condition

$$\int_{B_R} (\varphi\sigma)^2\, dx \leq C R^{\delta_n} \qquad \forall R > 1$$

implies that σ is constant in $\mathbf{R}^n$? Here $B_R = \{|x| < R\}$ and $C > 0$ is independent of R.

Question (AAC) is related to a celebrated conjecture by De Giorgi [5]:

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Conjecture (DG): *Let* $u : \mathbf{R}^n \to (-1,1)$ *be a smooth entire solution of the equation* $\Delta u = u^3 - u$ *satisfying the monotonicity condition*

$$\frac{\partial u}{\partial x_n} > 0 \qquad \text{in } \mathbf{R}^n \ . \tag{2}$$

Then all the level sets $\{u = s\}$ *of* u *are hyperplanes, at least if* $n \le 8$.

This conjecture has been of large interest in recent years. Several weakened versions of it have also been considered. Here, we just deal with Conjecture (DG) in its original form. In this form it has been proved in [6] when $n = 2$ and in [2] when $n = 3$. Up to now, a proof for larger n is still missing. Also other nonlinearities $f(u)$ (replacing $u^3 - u$ in the equation) have been considered: a general statement merely requiring that $f \in C^1(\mathbf{R})$ has been proved in [1] when $n = 2, 3$. Moreover, in [1] a sharp growth estimate of $\int_{B_R} |\nabla u|^2$ (for varying R) is obtained for any solution u of the equation which satisfies (2). And this estimate implies that if $\delta_n \ge n - 1$ then Conjecture (DG) is true in dimension n. On the other hand, a Liouville-type Theorem by Berestycki-Caffarelli-Nirenberg [4] (see also [2, Proposition 2.1]) ensures that $\delta_n \ge 2$. Hence, the method of [1] immediately gives the proof of Conjecture (DG) (for general f) when $n = 2, 3$. This is the reason of Question (AAC).

In fact, the original question raised in [1] concerns the very same problem as Question (AAC) for the corresponding *equation*, namely the largest exponent γ_n when the *equality* holds in (1). Clearly, $\gamma_n \ge \delta_n$. However, the proof of the Liouville-type Theorem in [4] is based on Hölder inequality and hence there is no advantage in considering the equation instead of the inequality. We also refer to [7, 8] for an extensive (and sharp) use of the method of test functions for Liouville-type Theorems for general differential inequalities.

In [3] it is shown that $\gamma_n < n$ whenever $n \ge 3$, while a sharp choice of the exponents in the counterexamples in [6] shows that $\gamma_n < 2 + 2\sqrt{n-1}$ whenever $n \ge 7$, see also [1]. In this short note, we exhibit examples which prove that

$$\delta_n = 2 \qquad \forall n \ge 2 \ .$$

Therefore, in order to obtain a full proof of Conjecture (DG) a different method, other than Liouville-type Theorems for inequalities, seems to be necessary.

2. The counterexamples

In this section we prove

THEOREM 2.1. *Let $n \geq 2$. Then, for all $\delta > 2$ there exist*

$$\varphi \in L^\infty(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n)\ , \qquad \varphi > 0 \tag{3}$$

$$\sigma \in H^1_{loc}(\mathbf{R}^n)\ , \qquad \nabla\sigma \not\equiv 0\ , \tag{4}$$

such that

$$\exists C > 0 \quad \textit{such that} \quad \int_{B_R} (\varphi\sigma)^2 \leq CR^\delta \qquad \forall R > 1\ , \tag{5}$$

$$\sigma\ \mathrm{div}(\varphi^2 \nabla\sigma) \geq 0 \qquad \textit{in } \mathbf{R}^n \tag{6}$$

in the distributional sense.

Proof. Let $k = \delta - 2 > 0$ and let

$$\sigma(x) = |x|^k \qquad \varphi(x) = \frac{1}{(1 + |x|^2)^{\frac{n+k-2}{4}}}\ .$$

Under our assumptions, we clearly have

$$n + k - 2 > 0\ . \tag{7}$$

This immediately proves (3).

Moreover, $|\nabla\sigma(x)| = k|x|^{k-1}$ and therefore (7) implies $\nabla\sigma \in L^2_{loc}(\mathbf{R}^n)$, so that (4) follows.

After some calculations, for all $x \neq 0$ we find

$$\mathrm{div}(\varphi^2 \nabla\sigma) = k(n + k - 2)\frac{|x|^{k-2}}{(1 + |x|^2)^{\frac{n+k}{2}}}\ ;$$

therefore, by (7) (and $\sigma \geq 0$) we have $\sigma\,\mathrm{div}(\varphi^2\nabla\sigma) > 0$ in $\mathbf{R}^n \setminus \{0\}$. Moreover, using again (7), we see that $\mathrm{div}(\varphi^2\nabla\sigma) \in L^1_{loc}(\mathbf{R}^n)$ so that (6) is satisfied in distributional sense.

Finally, for all $R > 1$ we have

$$\int_{B_R} (\varphi\sigma)^2 = \omega_n \int_0^R r^{n-1}\ \frac{r^{2k}}{(1 + r^2)^{\frac{n+k-2}{2}}}\, dr = \omega_n \left(\int_0^1 + \int_1^R\right)$$

$$\leq \omega_n \left(C_1 + \int_1^R r^{k+1}\, dr \right) \leq \omega_n \left(C_1 + \frac{1}{k+2} \right) R^{k+2} = C\, R^\delta \ ,$$

which proves (5). □

About our choice of σ and φ several remarks are in order. First, note that $\sigma = \sigma_\delta$ converges uniformly on bounded sets to a constant as $\delta \to 2^+$. Next, note that in fact a *strict* inequality (in the distributional sense) holds in (6). Finally, if $\delta \leq n$ (which may occur only if $n \geq 3$ in view of $\delta > 2$) we have $(\sigma\varphi) \in L^\infty(\mathbf{R}^n)$ which is precisely the situation when applied to Conjecture (DG); therefore, even under the additional assumption that $(\sigma\varphi)$ is bounded, the sharp exponent is $\delta_n = 2$, at least if $n \geq 3$.

References

[1] G. Alberti, L. Ambrosio, and X. Cabré, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property*, Acta Appl. Math. **65** (2001), 9–33.

[2] L. Ambrosio and X. Cabré, *Entire solutions of semilinear ellitpic equations on* $\mathbf{r}^3$ *and a conjecture of De Giorgi*, J. Amer. Math. Soc. **13** (2000), 725–739.

[3] M.T. Barlow, *On the Liouville property for divergence form operators*, Canad. J. Math. **50** (1998), 487–496.

[4] H. Berestycki, L. Caffarelli, and L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains*, Ann. Sc. Norm. Sup. Pisa **25** (1997), 69–94.

[5] D. De Giorgi, *Convergence problems for functionals and operators*, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Roma, 1978) (Bologna), Pitagora, 1979, pp. 131–188.

[6] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. **311** (1998), 481–491.

[7] E. Mitidieri and S. Pohožaev, *Nonexistence of global positive solutions to quasilinear elliptic inequalities*, Dokl. Akad. Nauk. **359** (1998), 456–460.

[8] E. Mitidieri and S. Pohožaev, *Apriori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities*, Proc. Steklov Inst. Math. **234** (2001), 1–362, (translated from Russian).

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