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Symmetry and Monotonicity Results for Positive Solutions of p-Laplace Systems

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Summary. - *In this paper, we extend to a system of the type:*

$$\begin{cases} -\Delta_{p_1} u = f(v) \quad in\ \Omega, \quad u > 0 \quad in\ \Omega, \quad u = 0 \quad on\ \partial\Omega, \\ -\Delta_{p_2} v = g(u) \quad in\ \Omega, \quad v > 0 \quad in\ \Omega, \quad v = 0 \quad on\ \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is bounded, the monotonicity and symmetry results of Damascelli and Pacella obtained in [5] in the case of a scalar p-Laplace equation with $1 < p < 2$. For this purpose, we use the moving hyperplanes method and we suppose that $f, g : \mathbb{R} \to \mathbb{R}^+$ are increasing on $\mathbb{R}^+$ and locally Lipschitz continuous on $\mathbb{R}$ and $p_1, p_2 \in (1, 2)$ or $p_1 \in (1, \infty), p_2 = 2$.

1. Introduction and statement of the main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^1 boundary and let $f, g : \mathbb{R} \to \mathbb{R}^+$ be increasing on $\mathbb{R}^+$, locally Lipschitz continuous on $\mathbb{R}$ and such that $f(x) > 0, g(x) > 0$ for all $x > 0$. Let $(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be a weak solution of

$$\begin{cases} -\Delta_{p_1} u = f(v) \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \\ -\Delta_{p_2} v = g(u) \quad \text{in } \Omega, \quad v > 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

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The main goal of this paper is to use the moving hyperplanes method in view of extending to a system like (1) the monotonicity and symmetry results of Damascelli and Pacella contained in their very nice recent article [5].

We will consider separately the cases $p_1 \in (1,\infty), p_2 = 2$ (similarly $p_2 \in (1,\infty), p_1 = 2$) and $p_1, p_2 \in (1,2)$. The first case will be treated in a quite classical way, by using partly some comparison principles on small domains but also the Hopf Lemma and the strong maximum principle for the p-Laplacian (cf. [9]). On the other hand for the second case, we will establish some monotonicity results which will be variants of some earlier theorems of Damascelli and Pacella in [4, 5] by using the same ideas as in [5], but adapted in the case of a system.

Before stating the monotonicity results, we first introduce some notations used in [4, 5]. For any direction $\nu \in \mathbb{R}^N$, $|\nu| = 1$, we define

$$a(\nu) := \inf_{x\in\Omega} x.\nu,$$

and for all $\lambda \geq a(\nu)$,

$$\begin{aligned} \Omega_\lambda^\nu &:= \{x \in \Omega \,|\, x.\nu < \lambda\} (\neq \emptyset \text{ for } \lambda > a(\nu),\ \lambda - a(\nu) \text{ small}), \\ T_\lambda^\nu &:= \{x \in \Omega \,|\, x.\nu = \lambda\}. \end{aligned}$$

Let us denote by R_λ^ν the reflection with respect to the hyperplane T_λ^ν and by

$$\begin{aligned} x_\lambda^\nu &:= R_\lambda^\nu(x)\ \forall x \in \mathbb{R}^N, \\ (\Omega_\lambda^\nu)' &:= R_\lambda^\nu(\Omega_\lambda^\nu), \\ \Lambda_1(\nu) &:= \{\mu > a(\nu) \,|\, \forall \lambda \in (a(\nu), \mu),\ \text{we do have (2) and (3)}\}, \\ \lambda_1(\nu) &:= \sup \Lambda_1(\nu), \end{aligned}$$

where (2), (3) are defined as follows:

$$(\Omega_\lambda^\nu)' \text{ is not internally tangent to } \partial\Omega \text{ at some point } p \notin T_\lambda^\nu, \quad (2)$$

$$\nu(x).\nu \neq 0 \quad \text{for all } x \in \partial\Omega \cap T_\lambda^\nu, \quad (3)$$

where $\nu(x)$ denotes the inward unit normal to $\partial\Omega$ at x. Notice that since for $\lambda > a(\nu)$ and if λ is close to $a(\nu)$, (2) and (3) are satisfied and Ω is bounded, it follows that

$$\Lambda_1(\nu) \neq \emptyset \quad \text{and} \quad \lambda_1(\nu) < \infty. \tag{4}$$

Observe also that for all $\lambda > a(\nu)$, for all $c \in T_\lambda^{\nu(x)} \cap \Omega$ we have

$$dist(c, \partial\Omega) \leq \lambda - a(\nu). \tag{5}$$

We denote by $C_0^1(\overline{\Omega})$ the space $\{u \in C^1(\overline{\Omega}) \text{ s.t. } u = 0 \text{ on } \partial\Omega\}$.

The monotonicity results are the following:

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain satisfying the interior sphere condition and let $f, g : \mathbb{R} \to \mathbb{R}^+$ be nondecreasing on $\mathbb{R}^+$ and locally Lipschitz continuous on $\mathbb{R}$. Let $(u, v) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ be a weak solution of*

$$\begin{cases} -\Delta_{p_1} u &= f(v) \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \\ -\Delta v &= g(u) \quad \text{in } \Omega, \quad v > 0 \quad \text{in } \Omega, \end{cases}$$

where $1 < p_1$. Then, for any direction $\nu \in \mathbb{R}^N$ and for any λ in the interval $(a(\nu), \lambda_1(\nu)]$, we have

$$u(x) \leq u(x_\lambda^\nu) \quad \text{and} \quad v(x) \leq v(x_\lambda^\nu) \quad \forall x \in \Omega_\lambda^\nu.$$

Moreover

$$\frac{\partial v}{\partial \nu} > 0 \quad \text{in } \Omega_\lambda^\nu \quad \forall \lambda < \lambda_1(\nu). \tag{6}$$

The following result is the analogue of Theorem 1.1 from [5] for a system with increasing right-hand sides.

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^1 boundary and let $f, g : \mathbb{R} \to \mathbb{R}^+$ be strictly increasing on $\mathbb{R}^+$, locally Lipschitz continuous on $\mathbb{R}$ and such that $f(x) > 0, g(x) > 0$ for all $x > 0$. Let $(u, v) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ be a weak solution of*

$$\begin{cases} -\Delta_{p_1} u = f(v) \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \\ -\Delta_{p_2} v = g(u) \quad \text{in } \Omega, \quad v > 0 \quad \text{in } \Omega, \end{cases} \tag{7}$$

where $p_1, p_2 \in (1,2)$. Then we have

$$u(x) \leq u(x_\lambda^\nu) \quad \text{and} \quad v(x) \leq v(x_\lambda^\nu) \tag{8}$$

for all $x \in \Omega_\lambda^\nu$, for all $\nu \in \mathbb{R}^N$, for all $\lambda \in (a(\nu), \lambda_1(\nu)]$.

In Theorem 1.2, the restriction $p_1, p_2 \in (1,2)$ is due to the fact that if both p_1, p_2 are different from 2, we must use comparison principles, and these are less powerful if p_1 or p_2 is greater than 2. On the other hand, if p_1 (or p_2) is equal to 2, then, as already mentioned, we may partly use strong maximum principles, and this finally allows p_2 to take values greater or smaller than 2. Note that this restriction is also present in the monotonicity result of [5] in the case of a single equation. We emphasize that in Theorem 1.1, this condition is not needed if p_1 or p_2 is equal to two.

REMARK 1.3. *In [5], Damascelli and Pacella state Theorem 1.1 under the hypothesis that Ω is smooth. This condition is due to the fact that they use a sophisticated method consisting of moving hyperplanes perpendicularly to directions ν in a neighborhood of a fixed direction ν_0. To be efficient, this method require the continuity of $a(\nu)$ and the lower semicontinuity of $\lambda_1(\nu)$ with respect to ν, and to insure this continuity (and only for that reason), they assume Ω to be smooth. It appears (see [2]) that this continuity is guaranteed for a domain Ω of class C^1. To prove Theorem 1.2, we use the new technique of Damascelli and Pacella, and so we require Ω to be C^1. Observe that this condition does not appear if $p_2 = 2, p_1 > 1$. Indeed, in this case, we can use the classical moving plane procedure consisting in moving planes perpendicularly to a fixed direction ν_0.*

We obtain as a consequence of Theorems 1.2 and 1.1 the following symmetry result:

THEOREM 1.4. *Let $\nu \in \mathbb{R}^N$ and $\Omega \subset\in \mathbb{R}^N$ ($N \geq 2$) be a domain with C^1 boundary symmetric with respect to the hyperplane $T_0^\nu = \{x \in \mathbb{R}^N \mid x.\nu = 0\}$ and $\lambda_1(\nu) = \lambda_1(-\nu) = 0$. Assume that one of the following conditions holds:*

1. *$p_1, p_2 \in (1,2)$ and $f, g : \mathbb{R} \to \mathbb{R}^+$ are strictly increasing functions on $\mathbb{R}^+$ such that $f(x) > 0, g(x) > 0$ for all $x > 0$,*

2. $p_1 \in (1,\infty), p_2 = 2$ *and* $f, g : \mathbb{R} \to \mathbb{R}^+$ *are nondecreasing on* $\mathbb{R}^+$.

Moreover suppose that f *and* g *are locally Lipschitz continuous on* $\mathbb{R}$. *Then, if* $(u,v) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ *is a weak solution of (7) it follows that* u *and* v *are symmetric and decreasing. In particular, if* Ω *is the ball* $B_R(0)$ *in* $\mathbb{R}^N$ *with center at the origin and radius* R, *then* u, v *are radially symmetric. Moreover if* $f(x) > 0, g(x) > 0$ *for all* $x > 0$, *then* $u'(r), v'(r) < 0$ *for* $r \in (0,R)$, $r = |x|$.

Theorems 1.2 and 1.1 have also a relatively big impact in the study of p-Laplace systems since they are used in [1] to prove by blow-up some existence results and a-priori estimates for positive solutions of the system

$$\begin{cases} -\Delta_{p_1} u = f(|v|) & \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \\ -\Delta_{p_2} v = g(|u|) & \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $1 < p_1, p_2 < N$, Ω is convex, $f, g : \mathbb{R} \to \mathbb{R}^+$ are nondecreasing locally Lipschitz continuous on $(0,+\infty)$, continuous on $[0,+\infty)$ and satisfy

$$C_1|s|^{q_1} \le f(s) \le C_2|s|^{q_1}, \quad D_1|s|^{q_2} \le g(s) \le D_2|s|^{q_2} \quad \forall s \in \mathbb{R}^+ \tag{10}$$

for some positive constants C_1, C_2, D_1, D_2 and $q_1 q_2 > (p_1 - 1)(p_2 - 1)$.

This paper is organized as follows. In section 2, we recall some well known results concerning the p-Laplacian operator. In section 3, we prove some weak comparison principles on small domains which are some adaptations of Theorem 1.2 from [4] to systems. In section 3.1, we use these principles to prove the monotonicity results and finally, we prove as a corollary Theorem 1.4.

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2. Preliminaries

Suppose that f, g are given positive continuous functions as in the introduction.

DEFINITION 2.1. Let $t \geq 0$. A function $(u, v) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ is said a weak solution of (1) if for any function $\varphi \in C_c^\infty(\Omega)$ we have

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^{p_1-2}\nabla u.\nabla\varphi\, dx = \int_\Omega f(v)\varphi\, dx, \\ \displaystyle\int_\Omega |\nabla u|^{p_1-2}\nabla u.\nabla\varphi\, dx = \int_\Omega g(u)\varphi\, dx. \end{cases} \tag{1}$$

We are interested in monotonicity results for weak solutions of (1).

By the maximum principle and Hopf's lemma of [9] for the p-Laplacian, any weak solution (u, v) of (1) satisfies

$$u > 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} < 0 \quad \text{on } \partial\Omega,$$

$$v > 0 \quad \text{in } \Omega, \quad \frac{\partial v}{\partial \nu} < 0 \quad \text{on } \partial\Omega$$

where ν denotes the outward unit normal to $\partial\Omega$.

In the present section we recall some well known properties of the operator $-\Delta_p$. The following result is due to Damascelli ([4]).

LEMMA 2.2 (WEAK COMPARISON PRINCIPLE). *Let $p > 1$. If $u, v \in W^{1,\infty}(\Omega)$ are such that*

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi\, dx \leq \int_\Omega |\nabla v|^{p-2}\nabla v.\nabla\varphi\, dx \quad \forall\varphi \in C_c^\infty(\Omega), \varphi \geq 0 \tag{2}$$

and $u \leq v$ on $\partial\Omega$, then $u \leq v$ on Ω.

Next we state a strong comparison principle due to Damascelli in [4] (Theorem 1.4).

LEMMA 2.3 (STRONG COMPARISON PRINCIPLE). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $p > 1$. Let $u, v \in C^1(\Omega)$ satisfy*

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi\, dx \le \int_\Omega |\nabla v|^{p-2}\nabla v.\nabla\varphi\, dx, \\ u \le v \quad in \quad \Omega \end{cases}$$

for all $\varphi \in C_c^\infty(\Omega), \varphi \ge 0$, and define $Z := \{x \in \Omega \,|\, |\nabla u(x)| + |\nabla v(x)| = 0\}$ if $p \ne 2$, $Z := \emptyset$ if $p = 2$.

If $x_0 \in \Omega \setminus Z$ and $u(x_0) = v(x_0)$, then $u = v$ in the connected component of $\Omega \setminus Z$ containing x_0.

Finally we recall a lemma proved by Simon in [8] and Damascelli in [4] which will be used later.

LEMMA 2.4. *Let $p > 1$ and $N \in \mathbb{N}_0$. There exist some positive constants c_1, c_2 depending on p and N such that for all $\eta, \eta' \in \mathbb{R}^N$ with $|\eta| + |\eta'| > 0$*

$$||\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'| \le c_1(|\eta| + |\eta'|)^{p-2}|\eta - \eta'| \tag{3}$$

$$(|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta').(\eta - \eta') \ge c_2(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2. \tag{4}$$

3. Weak comparison principles

Let Ω be a bounded domain in $\mathbb{R}^N$ and let $(u, v), (\bar u, \bar v) \in \mathbb{R}^+ \times C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be solutions of

$$\begin{cases} -\Delta_{p_1} u = f(v) & \text{on } \Omega, \quad u \ge 0 \\ -\Delta_{p_2} v = g(u) & \text{on } \Omega, \quad v \ge 0 \end{cases} \tag{5}$$

$$\begin{cases} -\Delta_{p_1} \bar u = f(\bar v) & \text{on } \Omega, \quad \bar u \ge 0 \\ -\Delta_{p_2} \bar v = g(\bar u) & \text{on } \Omega, \quad \bar v \ge 0 \end{cases} \tag{6}$$

where $f, g : \mathbb{R} \to \mathbb{R}^+$ are locally Lipschitz continuous on $\mathbb{R}$ and nondecreasing on $\mathbb{R}^+$. As mentioned above, our first aim is to prove some comparison principles for solutions of (5), (6). We begin with a result in case $p_1, p_2 \in (1, 2)$ that will be an extension of Theorem 2.2 of [5] to systems with nondecreasing right hand side.

For any set $A \subseteq \Omega$, we define $M_A = M_A(u, \bar{u}) := \sup_A(|\nabla u| + |\nabla \bar{u}|)$ and $\tilde{M}_A = \tilde{M}_A(v, \bar{v}) := \sup_A(|\nabla v| + |\nabla \bar{v}|)$. We shall denote the measure of a measurable set B by $|B|$.

THEOREM 3.1. *Let Ω be a bounded domain contained in $\mathbb{R}^N$ and suppose that $1 < p_1, p_2 < 2$. Let $(u, v), (\bar{u}, \bar{v}) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be two solutions of (5), (6). Suppose that*

$$f, g : \mathbb{R} \to \mathbb{R}^+$$

are locally Lipschitz continuous on $\mathbb{R}$ and nondecreasing on $\mathbb{R}^+$. Then there exist $\alpha, M > 0$ depending on N, p_1, p_2, f, g, $|\Omega|$, M_Ω, $\tilde{M}_\Omega$ and the L^∞ norms of $u, \bar{u}, v, \bar{v}$ such that if $\Omega' \subset \Omega$ is an open set and if there exists measurable sets $A_i, \tilde{A}_i (i = 1, 2, 3)$ such that

$$\Omega' = A_1 \cup A_2 \cup A_3 = \tilde{A}_1 \cup \tilde{A}_2 \cup \tilde{A}_3$$

and

$$\begin{cases} A_1 \cup A_2, \tilde{A}_1 \cup \tilde{A}_2 \quad \text{are open} \\ A_i \cap A_j = \tilde{A}_i \cap \tilde{A}_j = \emptyset \quad \text{for all } i \neq j \end{cases}$$

and

$$\begin{cases} \max\{|A_1|, |\tilde{A}_1|\} < \alpha, \\ \max\{M_{A_2}, \tilde{M}_{\tilde{A}_2}\} < M, \\ u < \bar{u} \quad \text{on } A_3, \\ v < \bar{v} \quad \text{on } \tilde{A}_3, \end{cases}$$

then we have the implication

$$\begin{cases} u \leq \bar{u} \text{ on } \partial\Omega' \cup \partial(A_1 \cup A_2) \\ v \leq \bar{v} \text{ on } \partial\Omega' \cup \partial(\tilde{A}_1 \cup \tilde{A}_2) \end{cases} \Longrightarrow \begin{cases} u \leq \bar{u} \\ v \leq \bar{v} \end{cases} \quad \text{on } \Omega'.$$

Proof. By multiplying the first equations of (5), (6) by $(u - \bar{u})^+ \in W_0^{1,p_1}(\Omega')$ (cf. Theorem IX.17. and remark 20, p. 171-172 in [3]) and the second equations by $(v - \bar{v})^+ \in W_0^{1,p_2}(\Omega')$, and subtracting the resulting identities, we get

$$\begin{aligned} \int_{\Omega' \cap [u \geq \bar{u}]} & (|\nabla u|^{p_1 - 2} \nabla u - |\nabla \bar{u}|^{p_1 - 2} \nabla \bar{u}) . \nabla (u - \bar{u}) \, dx \\ & = \int_{\Omega' \cap [u \geq \bar{u}]} (f(v) - f(\bar{v}))(u - \bar{u}) \, dx, \end{aligned} \tag{7}$$

and

$$\int_{\Omega' \cap [v \geq \bar{v}]} (|\nabla v|^{p_2 - 2} \nabla v - |\nabla \bar{v}|^{p_2 - 2} \nabla \bar{v}) . \nabla (v - \bar{v}) \, dx = \int_{\Omega' \cap [v \geq \bar{v}]} (g(u) - g(\bar{u}))(v - \bar{v}) \, dx. \tag{8}$$

By Lemma 2.4, the left-hand sides of (7), (8) are respectively greater or equal to

$$c_2 M_{\Omega}^{p_1 - 2} \int_{A_1 \cap [u \geq \bar{u}]} |\nabla (u - \bar{u})|^2 \, dx + c_2 M_{A_2}^{p_1 - 2} \int_{A_2 \cap [u \geq \bar{u}]} |\nabla (u - \bar{u})|^2 \, dx \tag{9}$$

and

$$c_2 \tilde{M}_{\Omega}^{p_2 - 2} \int_{\tilde{A}_1 \cap [v \geq \bar{v}]} |\nabla (v - \bar{v})|^2 \, dx + c_2 \tilde{M}_{\tilde{A}_2}^{p_2 - 2} \int_{\tilde{A}_2 \cap [v \geq \bar{v}]} |\nabla (v - \bar{v})|^2 \, dx$$

where c_2 is a positive constant depending on p_1, p_2 and N. Since f and g are nondecreasing, the right-hand side of (7) can be further majorized with

$$\int_{\Omega' \cap [u \geq \bar{u}] \cap [v \geq \bar{v}]} (f(v) - f(\bar{v}))(u - \bar{u}) \, dx,$$

and by the local Lipschitz property of f, g, this latter quantity is smaller or equal to

$$\begin{aligned} &\Lambda \int_{\Omega' \cap [u \geq \bar{u}] \cap [v \geq \bar{v}]} (v - \bar{v})(u - \bar{u}) \\ &\leq \Lambda \| (v - \bar{v})^+ \|_{L^2((\tilde{A}_1 \cup \tilde{A}_2) \cap [u \geq \bar{u}])} \| (u - \bar{u})^+ \|_{L^2((A_1 \cup A_2) \cap [v \geq \bar{v}])} \\ &\leq \Lambda \| (v - \bar{v})^+ \|_{L^2((\tilde{A}_1 \cup \tilde{A}_2))} \| (u - \bar{u})^+ \|_{L^2((A_1 \cup A_2))}, \end{aligned}$$

for some constant $\Lambda > 0$ depending on f and the L^∞ norms of $v, \bar{v}$ (cf.. remark 2.1 in [5]). Using a version of Poincaré's inequality (see Lemma 2.2 of [4]), this last term is smaller than

$$\begin{aligned} &\Lambda w_N^{-2/N} |\Omega'|^{1/N} \\ &\left\{ |\tilde{A}_1|^{1/2N} \| \nabla (v - \bar{v}) \|_{L^2(\tilde{A}_1 \cap [v \geq \bar{v}])} + |\Omega|^{1/2N} \| \nabla (v - \bar{v}) \|_{L^2(\tilde{A}_2 \cap [v \geq \bar{v}])} \right\} \\ &\left\{ |A_1|^{1/2N} \| \nabla (u - \bar{u}) \|_{L^2(A_1 \cap [u \geq \bar{u}])} + |\Omega|^{1/2N} \| \nabla (u - \bar{u}) \|_{L^2(A_2 \cap [u \geq \bar{u}])} \right\}, \end{aligned}$$

where ω_N denotes the Lebesgue measure of the unit ball in $\mathbb{R}^N$. The same reasoning can be made with (8) (with another constant Λ'). Adding both inequalities, we obtain

$$\begin{aligned}
& c_2 M_\Omega^{p_1-2}\|\nabla(u-\bar{u})\|^2_{L^2(A_1\cap[u\geq\bar{u}])} \\
& +c_2 M_{A_2}^{p_1-2}\|\nabla(u-\bar{u})\|^2_{L^2(A_2\cap[u\geq\bar{u}])} \\
& +c_2 \tilde{M}_\Omega^{p_2-2}\|\nabla(v-\bar{v})\|^2_{L^2(\tilde{A}_1\cap[v\geq\bar{v}])} \\
& +c_2 \tilde{M}_{\tilde{A}_2}^{p_2-2}\|\nabla(v-\bar{v})\|^2_{L^2(\tilde{A}_2\cap[v\geq\bar{v}])} \\
\leq\ & 2\max\{\Lambda,\Lambda'\} w_N^{\frac{-2}{N}} |\Omega'|^{\frac{1}{N}} \\
& \Big\{ (|A_1||\tilde{A}_1|)^{\frac{1}{2N}} \|\nabla(v-\bar{v})\|_{L^2(\tilde{A}_1\cap[v\geq\bar{v}])} \|\nabla(u-\bar{u})\|_{L^2(A_1\cap[u\geq\bar{u}])} \\
& +|\Omega|^{\frac{1}{N}} \|\nabla(v-\bar{v})\|_{L^2(\tilde{A}_2\cap[v\geq\bar{v}])} \|\nabla(u-\bar{u})\|_{L^2(A_2\cap[u\geq\bar{u}])} \\
& +(|A_1||\Omega|)^{\frac{1}{2N}} \|\nabla(v-\bar{v})\|_{L^2(\tilde{A}_2\cap[v\geq\bar{v}])} \|\nabla(u-\bar{u})\|_{L^2(A_1\cap[u\geq\bar{u}])} \\
& +(|\Omega||\tilde{A}_1|)^{\frac{1}{2N}} \|\nabla(v-\bar{v})\|_{L^2(\tilde{A}_1\cap[v\geq\bar{v}])} \|\nabla(u-\bar{u})\|_{L^2(A_2\cap[u\geq\bar{u}])} \Big\}.
\end{aligned} \tag{10}$$

By Young inequality, the right-hand side of (10) is smaller or equal to

$$\begin{aligned}
& 2\max\{\Lambda,\Lambda'\} w_N^{\frac{-2}{N}} |\Omega'|^{\frac{1}{N}} \Big\{ |\tilde{A}_1|^{\frac{1}{N}} \|\nabla(v-\bar{v})\|^2_{L^2(\tilde{A}_1\cap[v\geq\bar{v}])} \\
& +|A_1|^{\frac{1}{N}} \|\nabla(u-\bar{u})\|^2_{L^2(A_1\cap[u\geq\bar{u}])} + |\Omega|^{\frac{1}{N}} \|\nabla(u-\bar{u})\|^2_{L^2(A_2\cap[u\geq\bar{u}])} \\
& +|\Omega|^{\frac{1}{N}} \|\nabla(v-\bar{v})\|^2_{L^2(\tilde{A}_2\cap[v\geq\bar{v}])} \Big\}.
\end{aligned}$$

From this we infer that if $|A_1|, |\tilde{A}_1|, M_{A_2}$ and $\tilde{M}_{\tilde{A}_2}$ are small enough, then

$$\begin{aligned}
& \|\nabla(v-\bar{v})\|_{L^2(\tilde{A}_1\cap[v\geq\bar{v}])} = \|\nabla(u-\bar{u})\|_{L^2(A_1\cap[u\geq\bar{u}])} \\
=\ & \|\nabla(u-\bar{u})\|_{L^2(A_2\cap[u\geq\bar{u}])} = \|\nabla(v-\bar{v})\|_{L^2(\tilde{A}_2\cap[v\geq\bar{v}])} \\
=\ & 0,
\end{aligned}$$

so that by Poincaré's inequality, $(u-\bar{u})^+ = (v-\bar{v})^+ = 0$ in respectively $A_1 \cup A_2$ and $\tilde{A}_1 \cup \tilde{A}_2$, and finally on Ω' by definition of $A_3, \tilde{A}_3$. □

We now give a weak comparison principle in the case $p_1 \in (1, +\infty)$, $p_2 = 2$ (or $p_1 = 2, p_2 \in (1, +\infty)$). Let $u, \bar{u}, v, \bar{v} \in C^1(\overline{\Omega})$ and $A \subset \Omega$. Set

$$m_A := \inf_A(|\nabla u| + |\nabla \bar{u}|), \quad \tilde{m}_A := \inf_A(|\nabla v| + |\nabla \bar{v}|).$$

THEOREM 3.2. *Let $m > 0$, $\Omega \subset \mathbb{R}^N$ be a bounded domain and $(u, v), (\bar{u}, \bar{v}) \in \mathbb{R}^+ \times C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be two solutions of (5), (6) where $f, g : \mathbb{R} \to \mathbb{R}^+$ are nondecreasing on $\mathbb{R}^+$ and locally Lipschitz continuous on $\mathbb{R}$.*

(i) If $p_1 \in (1, 2), p_2 = 2$, then there exists $\delta > 0$ depending on N, p_1, M_Ω, f, g and the L^∞-norms of $u, \bar{u}, v, \bar{v}$ such that for any open subset $\Omega' \subseteq \Omega$ with $|\Omega'| < \delta$, $u \leq \bar{u}$ and $v \leq \bar{v}$ on $\partial\Omega'$ implies $u \leq \bar{u}$ and $v \leq \bar{v}$ on Ω'.

(ii) If $p_1 > 2, p_2 = 2$, then there exists $\delta > 0$ depending on N, p_1, m, f, g and the L^∞ norms of $u, \bar{u}, v, \bar{v}$ such that if such that if $m_\Omega \geq m$, if $\Omega' \subseteq \Omega$ is an open subset with $|\Omega'| < \delta$, then $u \leq \bar{u}$ and $v \leq \bar{v}$ on $\partial\Omega'$ implies $u \leq \bar{u}$ and $v \leq \bar{v}$ on Ω'.

(iii) If $p_1 = p_2 = 2$, then there exists $\delta > 0$ depending on N, f, g and the L^∞-norms of $u, \bar{u}, v, \bar{v}$ such that for any open subset $\Omega' \subseteq \Omega$ with $|\Omega'| < \delta$, $u \leq \bar{u}$ and $v \leq \bar{v}$ on $\partial\Omega'$ implies $u \leq \bar{u}$ and $v \leq \bar{v}$ on Ω'.

Proof. Let us prove (i). As in the proof of Theorem 3.1, we first multiply the first equations of (5), (6) by $(u - \bar{u})^+ \in W_0^{1,p_1}(\Omega')$ and the second equations by $(v - \bar{v})^+ \in W_0^{1,2}(\Omega')$ and we subtract them. In this way we obtain (7), while (8) is replaced by

$$\int_{\Omega' \cap [v \geq \bar{v}]} |\nabla(v - \bar{v})|^2 \, dx = \int_{\Omega' \cap [v \geq \bar{v}]} (g(u) - g(\bar{u}))(v - \bar{v}) \, dx. \quad (11)$$

By Lemma 2.4, the left-hand side of (7) can be estimated with

$$c_2 M_\Omega^{p_1 - 2} \int_{\Omega' \cap [u \geq \bar{u}]} |\nabla(u - \bar{u})|^2 \, dx.$$

We then treat the right-hand sides of (7), (11) as in the proof of Theorem 3.1. For this purpose we use Lemma 2.2 from [4] with

$A_1 = \Omega', A_2 = \emptyset$ and we add the obtained inequalities to get

$$c_2 M_\Omega^{p_1-2} \|\nabla(u-\bar{u})^+\|^2_{L^2(\Omega')} + \|\nabla(v-\bar{v})^+\|^2_{L^2(\Omega')}$$
$$\leq 4\max\{\Lambda,\Lambda'\}\omega_N^{-\frac{2}{N}} |\Omega'|^{\frac{2}{N}} (\|\nabla(u-\bar{u})^+\|^2_{L^2(\Omega')} + \|\nabla(v-\bar{v})^+\|^2_{L^2(\Omega')}).$$

So, if $|\Omega'|$ is sufficiently small, then necessarily $\|\nabla(u-\bar{u})^+\|_{L^2(\Omega')} = \|\nabla(v-\bar{v})^+\|_{L^2(\Omega')} = 0$. By Poincaré's inequality this implies $(u-\bar{u})^+ = (v-\bar{v})^+ = 0$ in Ω', i.e. $u \leq \bar{u}$ and $v \leq \bar{v}$ in Ω'.

Let us now prove (ii). We again obtain (7), (11) and by Lemma 2.4, the right-hand side of (7) is greater or equal to

$$c_2 m_\Omega^{p_1-2} \int_{\Omega' \cap [u \geq \bar{u}]} |\nabla(u-\bar{u})|^2 \, dx.$$

Writing as above the same estimates of the left-hand sides of (7), (11), we get:

$$c_2 m_\Omega^{p_1-2} \|\nabla(u-\bar{u})^+\|^2_{L^2(\Omega')} + \|\nabla(v-\bar{v})^+\|^2_{L^2(\Omega')}$$
$$\leq 4\max\{\Lambda,\Lambda'\}\omega_N^{-\frac{2}{N}} |\Omega'|^{\frac{2}{N}} (\|\nabla(u-\bar{u})^+\|^2_{L^2(\Omega')} + \|\nabla(v-\bar{v})^+\|^2_{L^2(\Omega')}),$$

and we conclude as in the proof of (i).

Finally, we prove (iii). We get (11) while (7) is replaced by

$$\int_{\Omega' \cap [u \geq \bar{u}]} |\nabla(u-\bar{u})|^2 \, dx = \int_{\Omega' \cap [u \geq \bar{u}]} (f(v) - f(\bar{v}))(u-\bar{u}) \, dx. \quad (12)$$

We estimates as in the proof of (i) the left-hand sides of (12), (11) and we obtain

$$\|\nabla(u-\bar{u})^+\|^2_{L^2(\Omega')} + \|\nabla(v-\bar{v})^+\|^2_{L^2(\Omega')}$$
$$\leq 4\max\{\Lambda,\Lambda'\}\omega_N^{-\frac{2}{N}} |\Omega'|^{\frac{2}{N}} (\|\nabla(u-\bar{u})^+\|^2_{L^2(\Omega')} + \|\nabla(v-\bar{v})^+\|^2_{L^2(\Omega')}).$$

We conclude again as in the proof of (i). □

3.1. Proof of the monotonicity results

Let us introduce some more notations used in [5]. For any direction $\nu \in \mathbb{R}^N$, $|\nu| = 1$, we define

$$\Lambda_2(\nu) = \{\lambda > a(\nu) \,|\, (\Omega^\nu_\mu)' \subset \Omega \text{ for any } \mu \in (a(\nu), \lambda]\},$$

and, if $\Lambda_2(\nu) \neq \emptyset$,

$$\lambda_2(\nu) = \sup \Lambda_2(\nu).$$

If $a(\nu) < \lambda \leq \lambda_2(\nu)$, $x \in \Omega^\nu_\lambda$, $u, v \in C^1(\overline{\Omega})$, we set

$$u^\nu_\lambda(x) = u(x^\nu_\lambda), \quad v^\nu_\lambda(x) = v(x^\nu_\lambda),$$

$$Z^\nu_\lambda = Z^\nu_\lambda(u) = \{x \in \Omega^\nu_\lambda \,|\, \nabla u(x) = \nabla u^\nu_\lambda(x) = 0\},$$
$$\tilde{Z}^\nu_\lambda = \tilde{Z}^\nu_\lambda(v) = \{x \in \Omega^\nu_\lambda \,|\, \nabla v(x) = \nabla v^\nu_\lambda(x) = 0\}$$

and

$$Z = Z(u) = \{x \in \Omega \,|\, \nabla u(x) = 0\},$$
$$\tilde{Z} = \tilde{Z}(v) = \{x \in \Omega \,|\, \nabla v(x) = 0\}.$$

We also define

$$\Lambda_0(\nu) = \{\lambda \in (a(\nu), \lambda_2(\nu)] \,|\, u \leq u^\nu_\mu \text{ and } v \leq v^\nu_\mu \text{ in } \Omega^\nu_\mu \text{ for any } \mu \in (a(\nu), \lambda]\}$$

and if $\Lambda_0(\nu) \neq \emptyset$, we set

$$\lambda_0(\nu) = \sup \Lambda_0(\nu).$$

As remarked in [5], we obviously have $\lambda_0(\nu) \leq \lambda_1(\nu) \leq \lambda_2(\nu)$.

We begin now to prove Theorem 1.1. In the proof we shall use the weak comparison principles stated in Theorem 3.2 for the beginning of the moving plane procedure, but afterwards it becomes quite classical in the sense that it uses maximum principles and Hopf's lemma for the usual Laplacian. That's why the result is true for all $p_2 \in (0, +\infty)$, in opposition with Theorem 1.2.

Proof of Theorem 1.1. Let us fix a direction ν. If $\lambda \leq \lambda_1(\nu)$, we have $u \leq u_\lambda^\nu$, $v \leq v_\lambda^\nu$ on $\partial\Omega_\lambda^\nu$ since $u, v > 0$ on Ω, $u = v = 0$ on $\partial\Omega$. If $p_1 \leq 2$, there exists $\delta > 0$ such that Theorem 3.2 (i) or (iii) is applicable to the pairs (u, v), $(\bar{u}, \bar{v}) = (u_\lambda^\nu, v_\lambda^\nu)$ and $\Omega = \Omega' = \Omega_\lambda^\nu$ for all $\lambda \in (a(\nu), \lambda_2(\nu))$. Since for $\lambda > a(\nu)$, $\lambda - a(\nu)$ small enough, we have $|\Omega_\lambda^\nu| < \delta$, we get $u \leq u_\lambda^\nu$ and $v \leq v_\lambda^\nu$ on Ω_λ^ν for these values of λ. If $p_1 > 2$, then, by the Hopf's lemma (see [9]), there exists $\bar{\lambda} > a(\nu)$ and $m > 0$ such that $m_{\Omega_\lambda^\nu} \geq m$ for all $\lambda \in (a(\nu), \bar{\lambda})$. Moreover, as above, for $\lambda > a(\nu)$, $\lambda - a(\nu)$ small enough, it follows that $|\Omega_\lambda^\nu|$ is small. So for these values of λ, we can apply Theorem 3.2 (ii) with $u, \bar{u} := u_\lambda^\nu, v, \bar{v} := v_\lambda^\nu$, $\Omega = \Omega' = \Omega_\lambda^\nu$, to get $u \leq u_\lambda^\nu$ and $v \leq v_\lambda^\nu$ on Ω_λ^ν. This proves that $\Lambda_0(\nu) \neq \emptyset$ as soon as $\Lambda_2(\nu) \neq \emptyset$.

Suppose by contradiction that $\lambda_0(\nu) < \lambda_1(\nu)$. By the continuity of u, v it follows that $u \leq u_{\lambda_0(\nu)}^\nu$ and $v \leq v_{\lambda_0(\nu)}^\nu$ on $\Omega_{\lambda_0(\nu)}^\nu$. Thus, by Lemma 2.3, we have either $v < v_{\lambda_0(\nu)}^\nu$ or $v = v_{\lambda_0(\nu)}^\nu$ on $\Omega_{\lambda_0(\nu)}^\nu$. Since (2) holds with $\lambda = \lambda_0(\nu)$, we have $0 = v < v_{\lambda_0(\nu)}^\nu$ on $\partial\Omega_{\lambda_0(\nu)}^\nu \setminus T_{\lambda_0(\nu)}^\nu$, so that we are in the first case. Using the fact that g is nondecreasing, we obtain

$$\begin{cases} -\Delta(v - v_{\lambda_0(\nu)}^\nu) \leq 0 & \text{on } \Omega_{\lambda_0(\nu)}^\nu, \\ v - v_{\lambda_0(\nu)}^\nu < 0 & \text{on } \Omega_{\lambda_0(\nu)}^\nu, \\ v = v_{\lambda_0(\nu)}^\nu & \text{on } T_{\lambda_0(\nu)}^\nu. \end{cases}$$

Thus by Hopf's lemma we have

$$\frac{\partial(v - v_{\lambda_0(\nu)}^\nu)}{\partial\nu} > 0 \text{ on } T_{\lambda_0(\nu)}^\nu \cap \Omega.$$

Hence $\frac{\partial v}{\partial\nu} > 0$ on $T_{\lambda_0(\nu)}^\nu \cap \Omega$. Since $\lambda_0(\nu) < \lambda_1(\nu)$, we have $\nu(x).\nu > 0$ for all $x \in \partial\Omega \cap \partial\Omega_{\lambda_0(\nu)}^\nu$. By Hopf's lemma, for any $x \in \partial\Omega \cap \partial\Omega_{\lambda_0(\nu)}^\nu$, it holds that $\nabla v(x) = c(x)\nu(x)$ for some function $c(x) > 0$, so that $\frac{\partial v}{\partial\nu} > 0$ on $\partial\Omega \cap \partial\Omega_{\lambda_0(\nu)}^\nu$, and hence

$$\frac{\partial v}{\partial\nu} > 0 \text{ on } T_{\lambda_0(\nu)}^\nu \cap \overline{\Omega_{\lambda_0(\nu)}^\nu}. \tag{13}$$

Since we suppose by contradiction that $\lambda_0(\nu) < \lambda_1(\nu)$, by definition of $\lambda_0(\nu)$ we infer that there exists a sequence $(\lambda_n) \subset (\lambda_0(\nu), \lambda_1(\nu))$

such that $\lambda_n \to \lambda_0(\nu)$ and a sequence $(x_n) \subset \Omega^\nu_{\lambda_n}$ with the property

$$v(x_n) > v^\nu_{\lambda_n}(x_n) \quad \forall n \in \mathbb{N}. \tag{14}$$

Indeed, if $v \leq v^\nu_{\lambda_n}$ in $\Omega^\nu_{\lambda_n}$, then, due to the fact that f is nondecreasing, we would have $u \leq u^\nu_{\lambda_n}$ in $\Omega^\nu_{\lambda_n}$. However this contradicts $\lambda_n > \lambda_0(\nu)$. Since (x_n) is bounded, there exists $x \in \overline{\Omega^\nu_{\lambda_0(\nu)}}$ such that $x_n \to x$. Passing to the limit as $n \to \infty$ in (14) we obtain $v(x) \geq v^\nu_{\lambda_0(\nu)}(x)$. As a consequence it follows that $x \in T^\nu_{\lambda_0(\nu)}$. By (14), there exists a sequence $(y_n) \subset (x_n, (x_n)^\nu_{\lambda_n})$ (where (a,b) for $a, b \in \mathbb{R}^N$ denotes here the open segment of extremities a and b) such that $\frac{\partial v}{\partial \nu}(y_n) < 0$. Clearly $y_n \to x$ and then $\frac{\partial v}{\partial \nu}(x) \leq 0$, which contradicts (13).

To prove (6), it suffices to apply the same reasoning as above to the function $v - v^\nu_\lambda$ on Ω^ν_λ. Indeed we have $-\Delta(v - v^\nu_\lambda) \leq 0$, $v < v^\nu_\lambda$ on Ω^ν_λ and $v = v^\nu_\lambda$ on $T^\nu_\lambda \cap \Omega$. So we obtain $\frac{\partial (v - v^\nu_\lambda)}{\partial \nu} > 0$ on $T^\nu_\lambda \cap \Omega$ and finally $\frac{\partial v}{\partial \nu} > 0$ on $T^\nu_\lambda \cap \Omega$. Since $\Omega^\nu_{\lambda_0(\nu)} = \cup_{a(\nu) < \lambda < \lambda_0(\nu)} (T^\nu_\lambda \cap \Omega)$, we have (6). This completes the proof. □

Now we treat the case $p_1, p_2 \in (1,2)$. To prove Theorem 1.2, we will follow the same steps as in [5], but adapted to our case. For the sake of completeness and clarity, we will sometimes repeat some arguments from [5].

Since the proof is quite long, we would like to give the main ideas beyond it. We first prove Lemma 3.3, an extension to our system of Theorem 3.1 from [5] (see also Theorem 1.5 from [4]). It asserts that once we start the moving plane procedure along a direction ν, if $\lambda_0(\nu) < \lambda_2(\nu)$, then the set Z of critical points of u creates a connected component C of $\Omega \setminus Z$ symmetric with respect to $T^\nu_{\lambda_0(\nu)}$ and where $u = u^\nu_{\lambda_0(\nu)}$, and the same results holds for the function v with a component $\tilde{C}$ of $\Omega \setminus \tilde{Z}$. Hence our goal is to prove that such sets C or $\tilde{C}$ cannot exist.

A first step in that way is Lemma 3.4 which implies that if C is defined as above and if u is constant on a connected subset of

∂C whose projection on $T^\nu_{\lambda_0(\nu)}$ contains a relatively open nonempty subset, then such a set C cannot exists, and the analogue holds for function v with a component $\tilde{C}$.

In fact, Lemmas 3.3 and 3.4 imply that if (u, v) is solution of (7) and if $\lambda_0(\nu) < \lambda_2(\nu)$, then there exist a connected component C^ν of $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ and a component $\tilde{C}^\nu$ of $\Omega^\nu_{\lambda_0(\nu)} \setminus \tilde{Z}^\nu_{\lambda_0(\nu)}$ such that $u = u^\nu_{\lambda_0(\nu)}$ in C^ν and $v = v^\nu_{\lambda_0(\nu)}$ in $\tilde{C}^\nu$. So if we suppose moreover that either u or v is constant on each connected component of respectively Z or $\tilde{Z}$, then in the first case, ∂C^ν would contain a set Γ on which $\nabla u = 0$, u is constant and whose projection on $T^\nu_{\lambda_0(\nu)}$ contains an open subset of $T^\nu_{\lambda_0(\nu)}$, which will be impossible by Lemma 3.4.

At first sight, one could think that the assumption that u or v is constant on respectively Z or $\tilde{Z}$ is always satisfied by any C^1 function. In fact, this is not true and the question of finding sufficient condition on a connected set of critical points of a C^1 function ensuring that this function is constant there seems very complicated. Some counterexamples are cited in [5] (see [10] and [7]), and they show that this question is strictly related to Sard's lemma and the theory of fractal sets.

So, to prove that either u or v is constant on a connected set of respectively ∂C or $\partial\tilde{C}$ (where C and $\tilde{C}$ are introduced above) and that the projection of C on the hyperplane $T^\nu_{\lambda_0(\nu)}$ contains a nonempty open set, some extra work is needed. To do that, we use the new argument introduced by Damascelli and Pacella in [5] consisting in moving hyperplanes orthogonal to directions close to ν to prove that the set C (or $\tilde{C}$) is also symmetric with nearby hyperplanes and to show that on its boundary, there is at least one connected piece where u is constant, $\nabla u = 0$ and whose projection on $T^\nu_{\lambda_0(\nu)}$ contains a nonempty open set.

We first give the analogue for our system of Theorem 3.1 from [5].

LEMMA 3.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $(u, v) \in C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega})$ be a weak solution of (5) where $p_1, p_2 \in (1, 2)$, f, g are strictly increasing on $\mathbb{R}^+$, locally Lipschitz continuous on $\mathbb{R}$ and such that $f(x) > 0, g(x) > 0$ for all $x > 0$. For any direction ν such that $\Lambda_2(\nu) \neq \emptyset$ we have that $\Lambda_0(\nu) \neq \emptyset$. If moreover $\lambda_0(\nu) < \lambda_2(\nu)$,*

then there exist a connected component C^ν *of* $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ *and a connected component* $\tilde{C}^\nu$ *of* $\Omega^\nu_{\lambda_0(\nu)} \setminus \tilde{Z}^\nu_{\lambda_0(\nu)}$ *such that* $u \equiv u^\nu_{\lambda_0(\nu)}$ *in* C^ν *and* $v \equiv v^\nu_{\lambda_0(\nu)}$ *in* $\tilde{C}^\nu$. *For such components, we have*

$$\nabla u(x) \neq 0 \quad \forall x \in C^\nu, \quad \nabla v(x) \neq 0 \quad \forall x \in \tilde{C}^\nu, \tag{15}$$

$$\begin{cases} \nabla u(x) = 0 & \forall x \in \partial C^\nu \setminus (T^\nu_{\lambda_0(\nu)} \cup \partial\Omega), \\ \nabla v(x) = 0 & \forall x \in \partial \tilde{C}^\nu \setminus (T^\nu_{\lambda_0(\nu)} \cup \partial\Omega). \end{cases} \tag{16}$$

Moreover, for any $\lambda \in (a(\nu), \lambda_0(\nu))$, *we have*

$$u < u^\nu_\lambda \quad \text{in } \Omega^\nu_\lambda \setminus Z^\nu_\lambda, \quad v < v^\nu_\lambda \quad \text{in } \Omega^\nu_\lambda \setminus \tilde{Z}^\nu_\lambda, \tag{17}$$

and

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \forall x \in \Omega^\nu_{\lambda_0(\nu)} \setminus Z \quad \text{and} \quad \frac{\partial v}{\partial \nu}(x) > 0 \quad \forall x \in \Omega^\nu_{\lambda_0(\nu)} \setminus \tilde{Z}. \tag{18}$$

Proof. The proof will follow the same steps as in the proof of Theorem 3.1 in [5].

STEP 1: We take a direction ν such that $\Lambda_2(\nu) \neq \emptyset$ and we prove that $\Lambda_0(\nu) \neq \emptyset$. As in the proof of Theorem 3.1 in [5], we can prove that if $\lambda > a(\nu)$, $\lambda - a(\nu)$ is small, then $|\Omega^\nu_\lambda|$ is small and since $u \leq u^\nu_\lambda$, $v \leq v^\nu_\lambda$ on $\partial\Omega^\nu_\lambda$, by Theorem 3.1, we get $u \leq u^\nu_\lambda$, $v \leq v^\nu_\lambda$ on Ω^ν_λ. So $\Lambda_0(\nu) \neq \emptyset$. Here we have applied Theorem 3.1 to the pairs (u, v) and $(\bar{u}, \bar{v}) = (u^\nu_\lambda, v^\nu_\lambda)$ and with $A_1 = \tilde{A}_1 = \Omega^\nu_\lambda$ and $A_2 = \tilde{A}_2 = A_3 = \tilde{A}_3 = \emptyset$.

STEP 2: By continuity of u, v , the inequalities $u \leq u^\nu_{\lambda_0(\nu)}$ and $v \leq v^\nu_{\lambda_0(\nu)}$ hold in $\Omega^\nu_{\lambda_0(\nu)}$. Moreover, by Lemma 2.3, since f, g are nondecreasing on $\mathbb{R}^+$, we have that if C^ν and $\tilde{C}^\nu$ are connected components of respectively $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ and $\Omega^\nu_{\lambda_0(\nu)} \setminus \tilde{Z}^\nu_{\lambda_0(\nu)}$, then either $u < u^\nu_{\lambda_0(\nu)}$ or $u \equiv u^\nu_{\lambda_0(\nu)}$ in C^ν and either $v < v^\nu_{\lambda_0(\nu)}$ or $v \equiv v^\nu_{\lambda_0(\nu)}$ in $\tilde{C}^\nu$. Assume now that $\lambda_0(\nu) < \lambda_2(\nu)$, and by contradiction that $u < u^\nu_{\lambda_0(\nu)}$ in $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$. We first show that this implies $v < v^\nu_{\lambda_0(\nu)}$ on $\Omega^\nu_{\lambda_0(\nu)} \setminus \tilde{Z}^\nu_{\lambda_0(\nu)}$. Suppose by contradiction this is not the case and

that in a component $\tilde{C}^\nu$ of $\Omega^\nu_{\lambda_0(\nu)} \setminus \tilde{Z}^\nu_{\lambda_0(\nu)}$, we have $v \equiv v^\nu_{\lambda_0(\nu)}$. Since $\Omega^\nu_{\lambda_0(\nu)} \setminus \tilde{Z}^\nu_{\lambda_0(\nu)}$ is open, $\tilde{C}^\nu$ is also open. Moreover, $int(Z^\nu_{\lambda_0(\nu)}) = \emptyset$. Indeed, if $Z^\nu_{\lambda_0(\nu)}$ contains an open set Ω', we have

$$\int_{\Omega'} |\nabla u|^{p_1-2}\nabla u.\nabla\varphi\,dx = 0 = \int_{\Omega'} f(v)\varphi\,dx \quad \forall\varphi \in C^\infty_c(\Omega'), \qquad (19)$$

which is impossible since $v > 0$ on Ω and $f(x) > 0$ for all $x > 0$. So $int(Z^\nu_{\lambda_0(\nu)}) = \emptyset$ and a similar reasoning would show that $int(\tilde{Z}^\nu_{\lambda_0(\nu)}) = \emptyset$. Now if we assume by contradiction that in a component $\tilde{C}^\nu$ of $\Omega^\nu_{\lambda_0(\nu)} \setminus \tilde{Z}^\nu_{\lambda_0(\nu)}$, we have $v \equiv v^\nu_{\lambda_0(\nu)}$, then $u < u^\nu_{\lambda_0(\nu)}$ and $v = v^\nu_{\lambda_0(\nu)}$ in $\tilde{C}^\nu \cap (\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}) \neq \emptyset$. Denoting $\tilde{C}^\nu \cap (\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)})$ by A, this would imply

$$\int_A |\nabla v|^{p_2-2}\nabla v.\nabla\varphi\,dx = \int_A |\nabla v^\nu_{\lambda_0(\nu)}|^{p_2-2}\nabla v^\nu_{\lambda_0(\nu)}.\nabla\varphi\,dx$$

for all $\varphi \in C^\infty_c(A)$, and so

$$\int_A g(u)\varphi\,dx = \int_A g(u^\nu_{\lambda_0(\nu)})\varphi\,dx \quad \forall\varphi \in C^\infty_c(A),$$

which is impossible since g is strictly increasing. So we also have $v < v^\nu_{\lambda_0(\nu)}$ in $\Omega^\nu_{\lambda_0(\nu)} \setminus \tilde{Z}^\nu_{\lambda_0(\nu)}$.

As in [5], we can choose $\alpha, M > 0$ independent from $\lambda \in (a(\nu), \lambda_2(\nu)]$ so that Theorem 3.1 applies in Ω to the pairs (u, v) and $(\bar{u}, \bar{v}) = (u^\nu_\lambda, v^\nu_\lambda)$ and with $\Omega = \Omega' = \Omega^\nu_\lambda$. Following the same idea as in [5], we take two open sets $A, \tilde{A}$ with $Z^\nu_{\lambda_0(\nu)} \subset A \subset \Omega^\nu_{\lambda_0(\nu)}$ and $\tilde{Z}^\nu_{\lambda_0(\nu)} \subset \tilde{A} \subset \Omega^\nu_{\lambda_0(\nu)}$ such that $\sup_A(|\nabla u| + |\nabla u^\nu_{\lambda_0(\nu)}|) < \frac{M}{2}$, $\sup_{\tilde{A}}(|\nabla v| + |\nabla v^\nu_{\lambda_0(\nu)}|) < \frac{M}{2}$. We also fix a compact set $K \subset \Omega^\nu_{\lambda_0(\nu)}$ such that $|\Omega^\nu_{\lambda_0(\nu)} \setminus K| < \frac{\alpha}{2}$. If $K \setminus A \neq \emptyset$, $\min_{K\setminus A}(u^\nu_{\lambda_0(\nu)} - u) = m > 0$ and if $K \setminus \tilde{A} \neq \emptyset$, $\min_{K\setminus\tilde{A}}(v^\nu_{\lambda_0(\nu)} - v) = \tilde{m} > 0$. By continuity of u, v, there exists $\varepsilon > 0$ such that $\lambda_0(\nu) + \varepsilon < \lambda_2(\nu)$ and such that for all $\lambda \in (\lambda_0(\nu), \lambda_0(\nu) + \varepsilon)$,

$$|\Omega^\nu_\lambda \setminus K| < \alpha, \quad \sup_A(|\nabla u| + |\nabla u^\nu_\lambda|) < M, \quad \sup_{\tilde{A}}(|\nabla v| + |\nabla v^\nu_\lambda|) < M,$$

$$u^\nu_\lambda - u > \frac{m}{2} > 0 \quad \text{in } K \setminus A \quad \text{if} \quad K \setminus A \neq \emptyset,$$

and

$$v_\lambda^\nu - v > \frac{\tilde{m}}{2} > 0 \quad \text{in } K \setminus \tilde{A} \quad \text{if} \quad K \setminus \tilde{A} \neq \emptyset.$$

For such values of λ, we have that $u \leq u_\lambda^\nu$ and $v \leq v_\lambda^\nu$ on respectively $\partial(\Omega_\lambda^\nu \setminus (K \setminus A))$ and $\partial(\Omega_\lambda^\nu \setminus (K \setminus \tilde{A}))$. Indeed, if x_0 belongs to $\partial(\Omega_\lambda^\nu \setminus (K \setminus A))$, then either $x_0 \in \partial\Omega_\lambda^\nu$ where trivially $u \leq u_\lambda^\nu$, or $x_0 \in \partial(K \setminus A)$, where $u_\lambda^\nu - u$ is positive. The same kind of reasoning works in the case $x_0 \in \partial(\Omega_\lambda^\nu \setminus (K \setminus \tilde{A}))$.
So we can apply Theorem 3.1 to (u, v), $(\bar{u}, \bar{v}) = (u_\lambda^\nu, v_\lambda^\nu)$, by taking $\Omega' = \Omega_\lambda^\nu$, $A_1 = \tilde{A}_1 = \Omega_\lambda^\nu \setminus K$, $A_2 = K \cap A$, $\tilde{A}_2 = K \cap \tilde{A}$ and $A_3 = K \setminus A$, $\tilde{A}_3 = K \setminus \tilde{A}$. We verify easily that $A_1 \cup A_2 = \Omega_\lambda^\nu \setminus (K \setminus A)$ and $\tilde{A}_1 \cup \tilde{A}_2 = \Omega_\lambda^\nu \setminus (K \setminus \tilde{A})$ are open subsets contained in Ω_λ^ν. We finally conclude that $u \leq u_\lambda^\nu$ and $v \leq v_\lambda^\nu$ in Ω_λ^ν for $\lambda \in (\lambda_0(\nu), \lambda_0(\nu) + \varepsilon)$, which contradicts the definition of $\lambda_0(\nu)$.

We prove (15) and (16) exactly as in [5] (they are simple consequences of the definition of C^ν and $\tilde{C}^\nu$).

STEP 3: To prove (17), it suffices to prove

$$u < u_\lambda^\nu \quad \text{in } \Omega_\lambda^\nu \setminus Z \quad \text{if } \lambda \in (a(\nu), \lambda_0(\nu)), \tag{20}$$

and

$$v < v_\lambda^\nu \quad \text{in } \Omega_\lambda^\nu \setminus \tilde{Z} \quad \text{if } \lambda \in (a(\nu), \lambda_0(\nu)), \tag{21}$$

as in [5]. Indeed, If (17) is false, and if (for example) $u(x_0) = u_\lambda^\nu(x_0)$ for some point $x_0 \in \Omega_\lambda^\nu \setminus Z_\lambda^\nu$, then by Lemma 2.3, $u = u_\lambda^\nu$ in the component of $\Omega_\lambda^\nu \setminus Z_\lambda^\nu$ to which x_0 belongs, and this implies that both $|\nabla u(x_0)|$ and $|\nabla u_\lambda^\nu(x_0)|$ are not zero, i.e. $x_0 \in \Omega_\lambda^\nu \setminus Z$, so that (20) does not hold. The same reasoning also works for v.

Let us prove (20). The argument is the same as in [5]. We recall it here for the sake of completeness. For simplicity of notations, we assume that $\nu = e_1 = (1, 0, \ldots, 0)$ and we omit the superscript e_1 in $\Omega_\lambda^{e_1}, \lambda_0(e_1), a(e_1), u_\lambda^{e_1}, \ldots$ We write coordinates in $\mathbb{R}^N$ as $x = (y, z)$ with $y \in \mathbb{R}$, $z \in \mathbb{R}^{N-1}$.

Arguing by contradiction, we assume that there exists μ with $a < \mu < \lambda_0$ and $x_0 = (y_0, z_0) \in \Omega_\mu \setminus Z$ such that $u(x_0) = u_\mu(x_0)$. Since $v \leq v_\mu$ on Ω_μ and f is nondecreasing, Lemma 2.3 implies $u = u_\mu$ in the component C of $\Omega_\mu \setminus Z_\mu$ to which x_0 belongs. If $\lambda > \mu$

and $\lambda - \mu$ is small, $(x_0)_\lambda = x_\mu$ for some point $x = (y, z_0) \in C$ with $y < y_0$. Since for $\lambda \in (\mu, \lambda_0]$, $u \leq u_\lambda$ in Ω_λ, we have $u(y, z_0) = u(x) = u(x_\mu) = u((x_0)_\lambda) \geq u(x_0) = u(y_0, z_0)$. This implies that $u(y, z_0) = u(y_0, z_0)$ since by definition of λ_0, u is nondecreasing in the e_1-direction in Ω_{λ_0}. Therefore the set

$$U := \{y < y_0 \,|\, (y, z_0) \in \Omega \text{ and } u(y, z_0) = u(y_0, z_0)\},$$

is not empty. If we set $y_1 := \inf U$, we show that $x_1 := (y_1, z_0) \in \partial\Omega$. Assume by contradiction that $x_1 \in \Omega$ and put $\lambda_1 := \frac{y_1 + y_0}{2}$. By continuity of u, $u(x_1) = u(x_0)$ and since $(x_1)_{\lambda_1} = x_0$ and $\nabla u(x_0) \neq 0$, we have $x_1 \in \Omega_{\lambda_1} \setminus Z_{\lambda_1}$. By Lemma 2.3, $u = u_{\lambda_1}$ in the component of $\Omega_{\lambda_1} \setminus Z_{\lambda_1}$ to which x_1 belongs, which implies that $\nabla u(x_1) \neq 0$ as above. Repeating the previous arguments with μ, x_0 substituted by λ_1, x_1, we obtain that $u(y, z_0) = u(y_0, z_0)$ for some $y < y_1$, $y_1 - y$ small, and this contradicts the definition of y_1. So $x_1 \in \partial\Omega$ and $u(x_1) = 0 = u(x_0) > 0$, a contradiction. This proves (20) and hence (17) for u. The proof of (21) is similar.

The proof of (18) can be made as in [5] using the usual Hopf's lemma. □

The following result is the analogue of Proposition 3.1 in [5].

LEMMA 3.4. *Suppose that $(u, v) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ is a weak solution of (7) where $p_1, p_2 \in (1, 2)$ and f, g satisfy the hypotheses of Lemma 3.3. Then for any direction ν, the set $\Omega^\nu_{\lambda_0(\nu)}$ does not contain any subset Γ of Z on which u is constant and whose projection on the hyperplane $T^\nu_{\lambda_0(\nu)}$ contains a non empty open subset of $T^\nu_{\lambda_0(\nu)}$ relatively to the induced topology. Similarly, for any direction ν, the set $\Omega^\nu_{\lambda_0(\nu)}$ does not contain any subset $\tilde{\Gamma}$ of $\tilde{Z}$ on which v is constant and whose projection on the hyperplane $T^\nu_{\lambda_0(\nu)}$ contains a non empty open subset of $T^\nu_{\lambda_0(\nu)}$.*

Proof. The proof is identical to that of Proposition 3.1 of [5], case 2. We give it here for the sake of completeness.

For simplicity of notations, we take $\nu = e_1 = (1, 0, \ldots, 0)$ and denote a point $x \in \mathbb{R}^N$ as $x = (y, z)$ with $y \in \mathbb{R}, z \in \mathbb{R}^{N-1}$. We omit the superscript $\nu = e_1$ in $\Omega_\lambda^\nu, u_\lambda^\nu, \ldots$

Arguing by contradiction, we assume that Ω_{λ_0} contains a set Γ with the properties:

(i) there exists $\gamma > 0$ and $z_0 \in \mathbb{R}^{N-1}$ such that for each $(\lambda_0, z) \in T_{\lambda_0}$ with $|z - z_0| < \gamma$, there exists $y < \lambda_0$ with $(y, z) \in \Gamma$,

(ii) $\nabla u(x) = 0$ for all $x \in \Gamma$,

(iii) $u(x) = m > 0$ for all $x \in \Gamma$.

Observe that $\bar{\Gamma}$ satisfies the same properties as Γ and that by (iii) (or (ii)), $\bar{\Gamma} \cap \partial\Omega = \emptyset$. Let $\omega = \omega_\gamma$ be the $(N-1)$-dimensional ball centered at z_0 with radius γ. We consider the cylinder $\mathbb{R} \times \omega$ and denote by Σ the intersection $(\mathbb{R} \times \omega) \cap \Omega_{\lambda_0}$. We then define the "right part of Σ with respect to Γ"

$$\Sigma_r := \{(y, z) \in \Sigma \,|\, z \in \omega, \sigma(z) < y < \lambda_0\},$$

where

$$\sigma(z) := \sup\{y \in \mathbb{R} |\, (y', z) \notin \bar{\Gamma} \text{ for all } y' < y\}.$$

One can see that Σ_r is open and well defined and by the monotonicity of u in Ω_{λ_0}, we have $u(x) \geq m$ for all $x \in \Sigma_r$. We claim that $u \neq m$ in Σ_r. Indeed if this is not the case i.e. $u = m$ in Σ_r, then $\nabla u = 0$ in Σ_r which is impossible since $f(x) > 0$ for all $x > 0$ and $v > 0$ in Σ_r. Furthermore,

$$-\Delta_{p_1}(u - m) = -\Delta_{p_1} u > 0 \quad \text{on } \Sigma_r.$$

One can see as in [5] that there exists x' on $\partial\Sigma_r \cap \bar{\Gamma}$ at which the interior sphere condition is satisfied. At such a point, we have that $u(x') = m = \min_{\overline{\Sigma_r}} u$. By the Hopf's lemma, we conclude that $\frac{\partial u}{\partial s}(x') > 0$ for an interior directional derivative, which contradicts (ii) for $\bar{\Gamma}$.

The same arguments can be used to prove the nonexistence of a set $\tilde{\Gamma}$. □

Before giving the proof of Theorem 1.2, we introduce some more notations. We denote by $\mathcal{F}_\nu$ (resp. $\tilde{\mathcal{F}}_\nu$) the collection of the connected components C^ν of $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ (resp. $\tilde{C}^\nu$ of $\Omega^\nu_{\lambda_0(\nu)} \setminus \tilde{Z}^\nu_{\lambda_0(\nu)}$) such that $u = u^\nu_{\lambda_0(\nu)}$ in C^ν, $\nabla u \neq 0$ in C^ν and $\nabla u = 0$ on $\partial C^\nu \setminus (T^\nu_{\lambda_0(\nu)} \cup \partial\Omega)$ (resp. $v = v^\nu_{\lambda_0(\nu)}$ in $\tilde{C}^\nu$, $\nabla v \neq 0$ in $\tilde{C}^\nu$ and $\nabla v = 0$ on $\partial\tilde{C}^\nu \setminus (T^\nu_{\lambda_0(\nu)} \cup \partial\Omega)$). Then we define

$$C'^\nu := C^\nu, \tag{22}$$

if $\nabla u(x) = 0$ for all $x \in \partial C^\nu$, or, if there are some points $x \in \partial C^\nu \cap T^\nu_{\lambda_0(\nu)}$ such that $\nabla u(x) \neq 0$, we define

$$C'^\nu := C^\nu \cup C^\nu_1 \cup C^\nu_2, \tag{23}$$

where $C^\nu_1 = R_{\lambda_0(\nu)}(C^\nu)$, $C^\nu_2 = \{x \in \partial C^\nu \cap T_{\lambda_0(\nu)} \,|\, \nabla u(x) \neq 0\}$. As in [5], we can check that C'^ν is open and connected, with

$$\nabla u \neq 0 \quad \text{in } C'^\nu, \qquad \nabla u = 0 \quad \text{on } \partial C'^\nu. \tag{24}$$

We define in the same way $\tilde{C}'^\nu$ with u replaced by v. Finally we define $\mathcal{F}'_\nu := \{C'^\nu \,|\, C^\nu \in \mathcal{F}_\nu\}$ and $\tilde{\mathcal{F}}'_\nu := \{\tilde{C}'^\nu \,|\, \tilde{C}^\nu \in \tilde{\mathcal{F}}_\nu\}$.

REMARK 3.5. We have the analogue of remark 4.1 from [5] for u and v: if ν_1, ν_2 are two directions, then either $C'^{\nu_1} = C'^{\nu_2}$ or $C'^{\nu_1} \cap C'^{\nu_2} = \emptyset$. Indeed, if $C^{\nu_1} \in \mathcal{F}_{\nu_1}, C^{\nu_2} \in \mathcal{F}_{\nu_2}$, if $C'^{\nu_1} \cap C'^{\nu_2} \neq \emptyset$ and $C'^{\nu_1} \neq C'^{\nu_2}$, then $\partial C'^{\nu_1} \cap C'^{\nu_2} \neq \emptyset$ or $C'^{\nu_1} \cap \partial C'^{\nu_2} \neq \emptyset$ by Corollary 4.1 of [5], which is impossible since $\nabla u \neq 0$ in C'^{ν_i}, $\nabla u = 0$ on $\partial C'^{\nu_i}$ for $i = 1, 2$. So necessarily, either $C'^{\nu_1} = C'^{\nu_2}$ or $C'^{\nu_1} \cap C'^{\nu_2} = \emptyset$. In a similar way, if $\tilde{C}^{\nu_1} \in \tilde{\mathcal{F}}_{\nu_1}, \tilde{C}^{\nu_2} \in \tilde{\mathcal{F}}_{\nu_2}$, then either $\tilde{C}'^{\nu_1} = \tilde{C}'^{\nu_2}$ or $\tilde{C}'^{\nu_1} \cap \tilde{C}'^{\nu_2} = \emptyset$.

Proof of Theorem 1.2. The proof will follow the steps of that of Theorem 1.1 from [5].
If ν is a direction and $\delta > 0$, we denote by $I_\delta(\nu)$ the set

$$I_\delta(\nu) := \{\mu \in \mathbb{R}^N \,|\, |\mu| = 1, |\mu - \nu| < \delta\}.$$

As in the proof of Lemma 3.3, we can fix $\alpha, M > 0$ such that Theorem 3.1 applies to any direction ν and any set $\Omega' \subset \Omega_\lambda^\nu$ for all $\lambda \in (a(\nu), \lambda_1(\nu)]$.
Suppose by contradiction that ν_0 is a direction such that $\lambda_0(\nu_0) < \lambda_1(\nu_0)$. Since $\lambda_1(\nu_0) \leq \lambda_2(\nu_0)$, then $\lambda_0(\nu_0) < \lambda_2(\nu_0)$ and by Lemma 3.3, $\mathcal{F}_{\nu_0}$ and $\tilde{\mathcal{F}}_{\nu_0}$ are non empty. Since $\mathbb{R}^N$ is separable and since each component of $\mathcal{F}_{\nu_0}$ and $\tilde{\mathcal{F}}_{\nu_0}$ is open, $\mathcal{F}_{\nu_0}$ and $\tilde{\mathcal{F}}_{\nu_0}$ contain at most countable many sets, so $\mathcal{F}_{\nu_0} = \{C_i^{\nu_0} \,|\, i \in I \subseteq \mathbb{N}\}$ and $\tilde{\mathcal{F}}_{\nu_0} = \{\tilde{C}_i^{\nu_0} \,|\, i \in \tilde{I} \subseteq \mathbb{N}\}$. In case I is infinite, since the components are disjoint, we have that $\sum_{i=1}^{\infty} |C_i^{\nu_0}| \leq |\Omega|$, so we can choose $n_0 \geq 1$ for which

$$\sum_{i=n_0+1}^{\infty} |C_i^{\nu_0}| < \frac{\alpha}{6},$$

and the same remark holds for $\tilde{I}$ with a number $\tilde{n}_0$. If I and $\tilde{I}$ are finite, let n_0 and $\tilde{n}_0$ be their cardinality. We then choose two compacts $K_0 \subset (\Omega_{\lambda_0(\nu_0)}^{\nu_0} \setminus \cup_{i\in I} C_i^{\nu_0})$ and $\tilde{K}_0 \subset (\Omega_{\lambda_0(\nu_0)}^{\nu_0} \setminus \cup_{i\in \tilde{I}} \tilde{C}_i^{\nu_0})$ such that

$$|(\Omega_{\lambda_0(\nu_0)}^{\nu_0} \setminus \cup_{i\in I} C_i^{\nu_0}) \setminus K_0| < \frac{\alpha}{6} \quad \text{and} \quad |(\Omega_{\lambda_0(\nu_0)}^{\nu_0} \setminus \cup_{i\in \tilde{I}} \tilde{C}_i^{\nu_0}) \setminus \tilde{K}_0| < \frac{\alpha}{6}.$$

Then we take some compact sets $K_i \subset C_i^{\nu_0}, i = 1, \ldots, n_0$ and $\tilde{K}_i \subset \tilde{C}_i^{\nu_0}, i = 1, \ldots, \tilde{n}_0$, such that

$$|C_i^{\nu_0} \setminus K_i| < \frac{\alpha}{6n_0} \quad i = 1, \ldots, n_0$$

and

$$|\tilde{C}_i^{\nu_0} \setminus \tilde{K}_i| < \frac{\alpha}{6\tilde{n}_0} \quad i = 1, \ldots, \tilde{n}_0.$$

So we have decomposed $\Omega_{\lambda_0(\nu_0)}^{\nu_0}$ in the sets $K_0, K_1, \ldots, K_{n_0}$ and in a remaining part with measure less than $\frac{\alpha}{2}$, and the same remark holds with the sets $\tilde{K}_i$.
We define then $A := \{x \in \Omega_{\lambda_0(\nu_0)}^{\nu_0} \,|\, |\nabla u(x)| + |\nabla u_{\lambda_0(\nu_0)}^{\nu_0}(x)| < \frac{M}{2}\}$ and $\tilde{A} := \{x \in \Omega_{\lambda_0(\nu_0)}^{\nu_0} \,|\, |\nabla v(x)| + |\nabla v_{\lambda_0(\nu_0)}^{\nu_0}(x)| < \frac{M}{2}\}$. Clearly, the sets $K_0 \setminus A$ and $\tilde{K}_0 \setminus \tilde{A}$ are compact. By Lemma 2.3 and since $K_0 \subset (\Omega_{\lambda_0(\nu_0)}^{\nu_0} \setminus \cup_{i\in I} C_i^{\nu_0})$ and $\tilde{K}_0 \subset (\Omega_{\lambda_0(\nu_0)}^{\nu_0} \setminus \cup_{i\in \tilde{I}} \tilde{C}_i^{\nu_0})$, we have that

$u < u^{\nu_0}_{\lambda_0(\nu_0)}$ in $K_0 \setminus A$ if $K_0 \setminus A \neq \emptyset$ and $v < v^{\nu_0}_{\lambda_0(\nu_0)}$ in $\tilde{K}_0 \setminus \tilde{A}$ if $\tilde{K}_0 \setminus \tilde{A} \neq \emptyset$. So there exists $m > 0$ such that

$$u^{\nu_0}_{\lambda_0(\nu_0)} - u \geq m > 0 \quad \text{in } K_0 \setminus A \quad \text{if} \quad K_0 \setminus A \neq \emptyset,$$

and

$$v^{\nu_0}_{\lambda_0(\nu_0)} - v \geq m > 0 \quad \text{in } \tilde{K}_0 \setminus \tilde{A} \quad \text{if} \quad K_0 \setminus \tilde{A} \neq \emptyset.$$

Since Ω is of class C^1, $a(\nu)$ is continuous and $\lambda_1(\nu)$ is lower semi-continuous (see [2]). By continuity, there exists $\varepsilon_0, \delta_0 > 0$ such that if $|\lambda - \lambda_0(\nu_0)| \leq \varepsilon_0$ and $|\nu - \nu_0| \leq \delta_0$, then

$$\lambda_1(\nu) > \lambda_0(\nu_0) + \varepsilon_0, \tag{25}$$

$$|\nabla u| + |\nabla u^\nu_\lambda| < M \text{ on } A \quad \text{and} \quad |\nabla v| + |\nabla v^\nu_\lambda| < M \text{ on } \tilde{A}, \tag{26}$$

$$K_i \subset \Omega^\nu_\lambda \quad \text{for } i = 0, \ldots, n_0,$$

$$\tilde{K}_i \subset \Omega^\nu_\lambda \quad \text{for } i = 0, \ldots, \tilde{n}_0,$$

$$R^\nu_\lambda(K_i) \subset R^\nu_\lambda(C^{\nu_0}_i) \quad \text{for } i = 1, \ldots, n_0,$$

$$R^\nu_\lambda(\tilde{K}_i) \subset R^\nu_\lambda(\tilde{C}^{\nu_0}_i) \quad \text{for } i = 1, \ldots, \tilde{n}_0,$$

$$|\Omega^\nu_\lambda \setminus \cup_{i=0}^{n_0} K_i| < \alpha \quad \text{and} \quad |\Omega^\nu_\lambda \setminus \cup_{i=0}^{\tilde{n}_0} \tilde{K}_i| < \alpha,$$

and finally

$$u^\nu_\lambda - u \geq \frac{m}{2} \quad \text{in } K_0 \setminus A \quad \text{and} \quad v^\nu_\lambda - v \geq \frac{m}{2} \quad \text{in } \tilde{K}_0 \setminus \tilde{A}. \tag{27}$$

We now proceed in several steps in order to show that there exists $i_1 \in \{1, \ldots, n_0\}$ or $j_1 \in \{1, \ldots, \tilde{n}_0\}$ and a direction $\nu_1 \in I_{\delta_0}$ such that $C'^{\nu_0}_{i_1} \in \mathcal{F}'_\nu$ for any direction ν in a suitable neighborhood $I_\delta(\nu_1)$ of ν_1 and $\partial C^{\nu_0}_{i_1}$ contains a set Γ as in Lemma 3.4 (with respect to the direction ν_1) or such that $\tilde{C}'^{\nu_0}_{j_1} \in \tilde{\mathcal{F}}'_\nu$ for any direction ν in a suitable neighborhood $I_\delta(\nu_1)$ of ν_1 and $\partial \tilde{C}^{\nu_0}_{j_1}$ contains a set $\tilde{\Gamma}$ as in Lemma 3.4. This together with Lemma 3.4 would lead to a contradiction and end the proof.

STEP 1: We show that $\lambda_0(\nu)$ is continuous at ν_0 with respect to ν

i.e. for all $\varepsilon \in (0, \varepsilon_0)$, there exists $\delta \in (0, \delta_0)$ such that if $\nu \in I_\delta(\nu_0)$, then

$$(i) \quad \lambda_0(\nu_0) - \varepsilon < \lambda_0(\nu) < \lambda_0(\nu_0) + \varepsilon.$$

Moreover, for all $\nu \in I_{\delta(\varepsilon_0)}(\nu_0)$, we have

$$(ii) \quad (\exists i \in \{1, \dots, n_0\} \text{ such that } C_i'^{\nu_0} \in \mathcal{F}'_\nu)$$
$$\text{or}$$
$$(\exists i \in \{1, \dots, \tilde{n}_0\} \text{ such that } \tilde{C}_i'^{\nu_0} \in \tilde{\mathcal{F}}'_\nu).$$

PROOF OF STEP 1: Let $0 < \varepsilon \leq \varepsilon_0$ be fixed. By definition of $\lambda_0(\nu_0)$, there exists $\lambda \in (\lambda_0(\nu_0), \lambda_0(\nu_0) + \varepsilon)$ and $x \in \Omega_\lambda^{\nu_0}$ such that $u(x) > u_\lambda^{\nu_0}(x)$ or $\tilde{x} \in \Omega_\lambda^{\nu_0}$ such that $v(\tilde{x}) > v_\lambda^{\nu_0}(\tilde{x})$. Suppose we are in the first case. Then by continuity, there exists $\delta_1 \in (0, \delta_0]$ such that for all $\nu \in I_{\delta_1}(\nu_0)$, x belongs to Ω_λ^ν and $u(x) > u_\lambda^\nu(x)$. This implies that for all $\nu \in I_{\delta_1}(\nu_0)$, we have $\lambda_0(\nu) < \lambda < \lambda_0(\nu_0) + \varepsilon$. We conclude in the same way if we are in the second case.

We then show that there exists $\delta_2 \in (0, \delta_0]$ such that $\lambda_0(\nu) > \lambda_0(\nu_0) - \varepsilon$ for all $\nu \in I_{\delta_2}(\nu_0)$. Suppose that this is false. Thus we can find a sequence $\nu_n \to \nu_0$ with $\lambda_0(\nu_n) \leq \lambda_0(\nu_0) - \varepsilon$ for all $n \in \mathbb{N}_0$. Passing to a subsequence still denoted by ν_n, we have $\lambda_0(\nu_n) \to \bar{\lambda} \leq \lambda_0(\nu_0) - \varepsilon$. Since $\lambda_0(\nu_n) > a(\nu_n)$ and $a(\nu_n) \to a(\nu_0)$ by the smoothness of $\partial\Omega$ (see [2]), we also have $a(\nu_0) \leq \bar{\lambda}$. In fact, this inequality is strict since by Theorem 3.1 and by definition of $\lambda_0(\nu_n)$, $|\Omega_{\lambda_0(\nu_n)}^{\nu_n}| \geq \alpha$ thus we also have $|\Omega_{\bar{\lambda}}^{\nu_0}| > 0$ which implies $\bar{\lambda} > a(\nu_0)$. Since $\bar{\lambda} < \lambda_0(\nu_0)$, by (17) of Lemma 3.3, we have

$$u < u_{\bar{\lambda}}^{\nu_0} \quad \text{in} \quad \Omega_{\bar{\lambda}}^{\nu_0} \setminus Z_{\bar{\lambda}}^{\nu_0}, \tag{28}$$
$$v < v_{\bar{\lambda}}^{\nu_0} \quad \text{in} \quad \Omega_{\bar{\lambda}}^{\nu_0} \setminus \tilde{Z}_{\bar{\lambda}}^{\nu_0}. \tag{29}$$

Now we make the same reasoning as in step 2 of the proof of Lemma 3.3. We can construct open sets $A, \tilde{A} \subset \Omega_{\bar{\lambda}}^{\nu_0}$ and a compact set $K \subset \Omega_{\bar{\lambda}}^{\nu_0}$ such that

$$Z_{\bar{\lambda}}^{\nu_0} \subset A, \quad \sup_A(|\nabla u| + |\nabla u_{\bar{\lambda}}^{\nu_0}|) < \frac{M}{2}, \quad |\Omega_{\bar{\lambda}}^{\nu_0} \setminus K| < \frac{\alpha}{2},$$

$$\tilde{Z}_{\bar{\lambda}}^{\nu_0} \subset \tilde{A}, \quad \sup_{\tilde{A}}(|\nabla v| + |\nabla v_{\bar{\lambda}}^{\nu_0}|) < \frac{M}{2},$$

and

$$u_{\bar\lambda}^{\nu_0} - u \geq \gamma > 0 \quad \text{in} \quad K \setminus A \quad \text{if} \quad K \setminus A \neq \emptyset,$$
$$v_{\bar\lambda}^{\nu_0} - v \geq \gamma > 0 \quad \text{in} \quad K \setminus \tilde A \quad \text{if} \quad K \setminus \tilde A \neq \emptyset,$$

for some $\gamma > 0$. By continuity of u, v and their gradients, there exist $r, \delta > 0$ such that

$$\sup_A(|\nabla u| + |\nabla u_\lambda^\nu|) < M, \quad |\Omega_\lambda^\nu \setminus K| < \alpha, \quad u_\lambda^\nu - u \geq \frac{\gamma}{2} > 0 \quad \text{in } K \setminus A,$$

$$\sup_{\tilde A}(|\nabla v| + |\nabla v_\lambda^\nu|) < M, \quad v_\lambda^\nu - v \geq \frac{\gamma}{2} > 0 \quad \text{in } K \setminus \tilde A$$

for all $\nu \in I_r(\nu_0)$ and $\lambda \in (\bar\lambda - \delta, \bar\lambda + \delta)$. We then apply Theorem 3.1 to the pairs (u, v) and $(\bar u, \bar v) = (u_\lambda^\nu, v_\lambda^\nu)$, with $\Omega' = \Omega_\lambda^\nu, A_1 = \tilde A_1 = \Omega_\lambda^\nu \setminus K$, $A_2 = K \cap A, \tilde A_2 = K \cap \tilde A$, $A_3 = K \setminus A, \tilde A_3 = K \setminus \tilde A$ and we obtain

$$u \leq u_\lambda^\nu \quad \text{and} \quad v \leq v_\lambda^\nu \quad \text{on } \Omega_\lambda^\nu \quad \forall \nu \in I_r(\nu_0), \forall \lambda \in (\bar\lambda - \delta, \bar\lambda + \delta).$$

Taking $\nu = \nu_n, \lambda = \lambda_0(\nu_n) + \eta$ for n large and $\eta > 0$ small, this contradicts the definition of $\lambda_0(\nu_n)$ and proves (i). Notice that everything above has some sense since $\lambda < \lambda_1(\nu)$ for λ close to $\bar\lambda$ and ν close to ν_0 by the lower semicontinuity of $\lambda_1(\nu)$ and since $\bar\lambda < \lambda_1(\nu_0) - \varepsilon$. Remark also that the proof of (i) does not use the fact that $\lambda_0(\nu_0) < \lambda_2(\nu_0)$, even when we use Lemma 3.3, so that the continuity of $\lambda_0(\nu)$ is in fact insured at all ν not necessarily equal to ν_0.

Observe that since $\varepsilon \leq \varepsilon_0$ and $\delta \leq \delta_0$, by (i) and (25), we have $\lambda_1(\nu) > \lambda_0(\nu)$ for all $\nu \in I_{\delta(\varepsilon_0)}(\nu_0)$ and (26)–(27) are true for $\nu \in I_{\delta(\varepsilon_0)}(\nu_0)$, $\lambda = \lambda_0(\nu)$.

Let us now prove (ii). We fix a direction $\nu \in I_{\delta(\varepsilon_0)}(\nu_0)$. Suppose that there exist $i \in \{1, \dots, n_0\}$ and a point $x_i \in K_i$ such that $u(x_i) = u_{\lambda_0(\nu)}^\nu(x_i)$. Since $\nabla u(x_i) \neq 0$, by Lemma 2.3, we have $u = u_{\lambda_0(\nu)}^\nu$ in the component C^ν of $\mathcal{F}_\nu$ to which x_i belongs. Since $K_i \subset C_i^{\nu_0} \subset C_i'^{\nu_0}$ it follows that $x_i \in C'^\nu \cap C_i'^{\nu_0}$, hence by the analogue of remark 4.1 of [5], it holds that $C'^\nu = C'^{\nu_0}$. As a consequence,

we conclude that (ii) holds with an index $i \in \{1, \dots, n_0\}$. The same argument works for v: if there exists $\tilde{x}_i \in \tilde{K}_i$ for some $i \in \{1, \dots, \tilde{n}_0\}$ such that $v(\tilde{x}_i) = v^\nu_{\lambda_0(\nu)}(\tilde{x}_i)$, then $\tilde{C}'^{\nu_0}_i \in \tilde{\mathcal{F}}'_\nu$.

Next we analyze the case when $u < u^\nu_{\lambda_0(\nu)}$ on $\cup_{i=1}^{n_0} K_i$ and $v < v^\nu_{\lambda_0(\nu)}$ on $\cup_{i=1}^{n_0} \tilde{K}_i$. If we take $\lambda > \lambda_0(\nu)$, $|\lambda - \lambda_0(\nu)|$ small, then by (26)–(27), we also have $u < u^\nu_\lambda$ on $\cup_{i=1}^{n_0} K_i \cup (K_0 \setminus A)$, $v < v^\nu_\lambda$ on $\cup_{i=1}^{\tilde{n}_0} \tilde{K}_i \cup (\tilde{K}_0 \setminus \tilde{A})$, $\sup_{K_0 \cap A}(|\nabla u| + |\nabla u^\nu_\lambda|) < M$, $\sup_{\tilde{K}_0 \cap \tilde{A}}(|\nabla v| + |\nabla v^\nu_\lambda|) < M$, and finally $|\Omega^\nu_\lambda \setminus \cup_{i=0}^{n_0} K_i| < \alpha$ and $|\Omega^\nu_\lambda \setminus \cup_{i=0}^{n_0} \tilde{K}_i| < \alpha$. Applying Theorem 3.1 to the pairs (u, v), $(\bar{u}, \bar{v}) = (u^\nu_\lambda, v^\nu_\lambda)$, with $\Omega' = \Omega^\nu_\lambda$ and

$$A_1 = \Omega^\nu_\lambda \setminus \cup_{i=0}^{n_0} K_i, \quad \tilde{A}_1 = \Omega^\nu_\lambda \setminus \cup_{i=0}^{\tilde{n}_0} \tilde{K}_i,$$

$$A_2 = K_0 \cap A, \quad \tilde{A}_2 = \tilde{K}_0 \cap \tilde{A},$$

$$A_3 = \cup_{i=1}^{n_0} K_i \cup (K_0 \setminus A), \quad \tilde{A}_3 = \cup_{i=1}^{\tilde{n}_0} \tilde{K}_i \cup (\tilde{K}_0 \setminus \tilde{A}),$$

we get $u \le u^\nu_\lambda$ and $v \le v^\nu_\lambda$ on Ω^ν_λ, which contradicts the definition of $\lambda_0(\nu)$. So we conclude that the case $u < u^\nu_{\lambda_0(\nu)}$ on $\cup_{i=1}^{n_0} K_i$ and $v < v^\nu_{\lambda_0(\nu)}$ on $\cup_{i=1}^{n_0} \tilde{K}_i$ is impossible. This proves (ii).

STEP 2: We prove that there exist $\nu_1 \in I_{\delta(\varepsilon_0)}(\nu_0)$ and a neighborhood $I_{\delta_1}(\nu_1)$ such that either there exists an index $i_1 \in \{1, \dots, n_0\}$ such that

$$C'^{\nu_0}_{i_1} \in \mathcal{F}'_\nu \quad \forall \nu \in I_{\delta_1}(\nu_1),$$

or an index $j_1 \in \{1, \dots, \tilde{n}_0\}$ such that

$$\tilde{C}'^{\nu_0}_{j_1} \in \tilde{\mathcal{F}}'_\nu \quad \forall \nu \in I_{\delta_1}(\nu_1).$$

Observe that in the proof of (ii) of step 1, we have seen that if $\nu \in I_{\delta(\varepsilon_0)}(\nu_0)$ and if there exists $x_i \in K_i$ for some $i \in \{1, \dots, n_0\}$ such that $u(x_i) = u^{\nu_0}_{\lambda_0(\nu_0)}(x_i)$, then $u \equiv u^{\nu_0}_{\lambda_0(\nu_0)} \equiv u^\nu_{\lambda_0(\nu)}$ in K_i and $C'^{\nu_0}_i \in \mathcal{F}'_\nu$. So for any $\nu \in I_{\delta(\varepsilon_0)}(\nu_0)$ and each $i \in \{1, \dots, n_0\}$, we have the alternative: either $u \equiv u^\nu_{\lambda_0(\nu)}$ in K_i and $C'^{\nu_0}_i \in \mathcal{F}'_\nu$, or $u < u^\nu_{\lambda_0(\nu)}$ in K_i (by Lemma 2.3, using the monotonicity of f and the definition of $\lambda_0(\nu)$). In the latter case, since K_i is compact, $u^\nu_{\lambda_0(\nu)} - u \ge m > 0$ in K_i for some m, and since the function $\nu \mapsto \lambda_0(\nu)$ is continuous

by step 1, we get that the inequality $u < u^{\mu}_{\lambda_0(\mu)}$ holds in K_i for all μ in a suitable neighborhood $I(\nu)$ of ν and so $C_i'^{\nu_0} \notin \mathcal{F}'_{\mu}$ for all $\mu \in I(\nu)$. The same remark holds for v and any component $\tilde{C}_j'^{\nu_0}$ for $j \in \{1, \ldots, \tilde{n}_0\}$.

We start by taking two sets in $\mathcal{F}'_{\nu_0}$ and $\tilde{\mathcal{F}}'_{\nu_0}$, say $C_1'^{\nu_0}$ and $\tilde{C}_1'^{\nu_0}$ and we argue as follows. If $C_1'^{\nu_0} \in \mathcal{F}'_{\nu}$ for any $\nu \in I_{\delta(\varepsilon_0)}(\nu_0)$ or $\tilde{C}_1'^{\nu_0} \in \tilde{\mathcal{F}}'_{\nu}$ for any $\nu \in I_{\delta(\varepsilon_0)}(\nu_0)$, then the assertion is proved. Otherwise, for what we explained above, for some $\mu_1 \in I_{\delta(\varepsilon_0)}(\nu_0)$, we have $C_1'^{\nu_0} \notin \mathcal{F}'_{\mu_1}$ and $u < u^{\mu_1}_{\lambda_0(\mu_1)}$ in K_1, and so $C_1'^{\nu_0} \notin \mathcal{F}'_{\mu}$ for any μ in a suitable neighborhood $I_{\delta_1}(\mu_1)$ of μ_1 which can be chosen such that $I_{\delta_1}(\mu_1) \subset I_{\delta(\varepsilon_0)}(\nu_0)$. If $n_0 = \tilde{n}_0 = 1$, then by (ii) of step 1, $\tilde{C}_1'^{\nu_0} \in \tilde{\mathcal{F}}'_{\mu}$ for any μ in $I_{\delta_1}(\mu_1)$ and the assertion is proved. So we can suppose $n_0 + \tilde{n}_0 \geq 3$. Now, by (ii) of step 1, there exists $i \in \{2, \ldots, n_0\}$ such that $C_i'^{\nu_0} \in \mathcal{F}'_{\mu_1}$ or $j \in \{1, \ldots, \tilde{n}_0\}$ such that $\tilde{C}_j'^{\nu_0} \in \tilde{\mathcal{F}}'_{\mu_1}$. Let us denote by A_2 this set $C_i'^{\nu_0}$ or $\tilde{C}_j'^{\nu_0}$ and suppose $A_2 \in \mathcal{F}'_{\mu_1}$ (the argument is similar in the other case $A_2 \in \tilde{\mathcal{F}}'_{\mu_1}$). If $A_2 \in \mathcal{F}'_{\mu}$ for all $\mu \in I_{\delta_1}(\mu_1)$, then the assertion is proved. Otherwise, there exists $\mu_2 \in I_{\delta_1}(\mu_1)$ such that $A_2 \notin \mathcal{F}'_{\mu_2}$ and as above, this implies that $A_2 \notin \mathcal{F}'_{\mu}$ for all μ in a suitable neighborhood $I_{\delta_2}(\mu_2)$ which can be chosen such that $I_{\delta_2}(\mu_2) \subset I_{\delta_1}(\mu_1)$. If $n_0 + \tilde{n}_0 = 3$, by (ii) of step 1, this implies that there exists a set A_3 in $\mathcal{F}'_{\nu_0}$ or $\tilde{\mathcal{F}}'_{\nu_0}$ ($A_3 = \tilde{C}_1'^{\nu_0}$ if $A_2 \in \mathcal{F}'_{\nu_0}$) with $A_3 \in \mathcal{F}'_{\mu}$ for all μ in $I_{\delta_2}(\mu_2)$ (or $A_3 \in \tilde{\mathcal{F}}'_{\mu}$ for all μ in $I_{\delta_2}(\mu_2)$) and the assertion is proved. If $n_0 + \tilde{n}_0 > 3$, we proceed as before taking a set A_3 in $\mathcal{F}'_{\nu_0}$ (or $\tilde{\mathcal{F}}'_{\nu_0}$) such that $A_3 \in \mathcal{F}'_{\mu_2}$ (or in $\tilde{\mathcal{F}}'_{\mu_2}$). Arguing as we did before, after $k < n_0 + \tilde{n}_0$ steps we get a set A_k in $\mathcal{F}'_{\nu_0}$ (or $\tilde{\mathcal{F}}'_{\nu_0}$) with $A_k \in \mathcal{F}'_{\mu}$ for all μ in $I_{\delta_{k-1}}(\mu_{k-1})$ (or $A_k \in \tilde{\mathcal{F}}'_{\mu}$ for all μ in $I_{\delta_{k-1}}(\mu_{k-1})$) proving the assertion, or after $n_0 + \tilde{n}_0$ steps, we get a direction $\mu_{n_0+\tilde{n}_0} \in I_{\delta(\varepsilon_0)}(\nu_0)$ such that $C_i'^{\nu_0} \notin \mathcal{F}'_{\mu_{n_0+\tilde{n}_0}}$ for all $i \in \{1, \ldots, n_0\}$ and $\tilde{C}_j'^{\nu_0} \notin \tilde{\mathcal{F}}'_{\mu_{n_0+\tilde{n}_0}}$ for all $j \in \{1, \ldots, \tilde{n}_0\}$, a contradiction with (ii) of step 1.

STEP 3: Here we prove that if step 2 is satisfied with a set $C_1'^{\nu_0} \in \mathcal{F}'_{\nu_0}$, then the set $\Omega^{\nu}_{\lambda_0(\nu)}$ contains a subset Γ of Z on which u is constant and whose projection on the hyperplane $T^{\nu}_{\lambda_0(\nu)}$ contains an open subset of $T^{\nu}_{\lambda_0(\nu)}$ relatively to the induced topology. In the same way, if

step 2 is satisfied with a set $\tilde{C}_i'^{\nu_0} \in \tilde{\mathcal{F}}'_{\nu_0}$, then $\Omega^{\nu}_{\lambda_0(\nu)}$ contains a subset $\tilde{\Gamma}$ of $\tilde{Z}$ on which v is constant and whose projection on the hyperplane $T^{\nu}_{\lambda_0(\nu)}$ contains an open subset of $T^{\nu}_{\lambda_0(\nu)}$. In both cases, we arrive to a contradiction with Lemma 3.4 and Theorem 1.2 is proved. We shall proceed exactly as in step 3 of the proof of Theorem 1.1 from [5]. We recall the argument for the sake of completeness.

Suppose step 2 is satisfied with some set $C_i'^{\nu_0} \in \mathcal{F}'_{\nu_0}$ (the same reasoning can be made in the other case, replacing u by v in what follows). Let us assume that ν_1, i_1, δ_1 are as in step 2 and denote by C the set $C^{\nu_0}_{i_1}$ from step 2. We assume moreover for simplicity that $\nu_1 = e_1 = (1, 0, \dots, 0)$ and we omit ν_1 in $\lambda_0(\nu_1), \Omega^{\nu_1}_{\lambda_0(\nu_1)}, T^{\nu_1}_{\lambda_0(\nu_1)}, \dots$

By step 2 and definition of C' (see (22,23)), we have that for each $\nu \in I_{\delta_1}(e_1)$,

$$u \equiv u^{\nu}_{\lambda_0(\nu)} \quad \text{in } C' \quad \text{and} \quad C' \in \mathcal{F}'_{\nu}. \tag{30}$$

Since C' is open (see the definition of C'), $R^{\nu}_{\lambda_0(\nu)}(C') \cap R_{\lambda_0}(C') \neq \emptyset$ if ν is sufficiently close to e_1. Moreover, by (24), $\nabla u = 0$ on $\partial C'$ which implies by (30) that

$$\nabla u = 0 \text{ on } \partial R^{\nu}_{\lambda_0(\nu)}(C') \quad \text{for all } \nu \in I_{\delta_1}(e_1). \tag{31}$$

Arguing as in Remark 3.5, if $R^{\nu}_{\lambda_0(\nu)}(C') \cap R_{\lambda_0}(C') \neq \emptyset$ and $R^{\nu}_{\lambda_0(\nu)}(C') \neq R_{\lambda_0}(C')$, then by Corollary 4.1 from [5], $\partial(R^{\nu}_{\lambda_0(\nu)}(C')) \cap R_{\lambda_0}(C') \neq \emptyset$ or $R^{\nu}_{\lambda_0(\nu)}(C') \cap \partial(R_{\lambda_0}(C')) \neq \emptyset$, which is impossible by (24), (30) and (31). Thus, we get that

$$R^{\nu}_{\lambda_0(\nu)}(C') \equiv R_{\lambda_0}(C')$$

for ν sufficiently close to e_1, say for $\nu \in I_{\delta_2}(e_1)$ for some $0 < \delta_2 \leq \delta_1$. Then we take a point $\bar{x} = (\bar{y}, \bar{z})$ in $\partial C \cap \Omega_{\lambda_0}$ and consider $\bar{x}' := (2\lambda_0 - \bar{y}, \bar{z})$, the symmetric of $\bar{x}$ with respect to the hyperplane T_{λ_0}. By reflecting $\bar{x}'$ through the hyperplanes $T^{\nu}_{\lambda_0(\nu)}$ for $\nu \in I_{\delta_2}(e_1)$, we obtain the points

$$A(\nu) = (y(\nu), z(\nu)) = \bar{x}' + 2(\lambda_0(\nu) - \bar{x}'.\nu)\nu,$$

which, for what we remarked before, belong to ∂C . Indeed, $\bar{x} \in \partial C \cap \Omega_{\lambda_0}$, so $\bar{x} \in \partial C'$, $\bar{x}' \in \partial(R_{\lambda_0}(C')) = \partial(R^{\nu}_{\lambda_0(\nu)}(C'))$ and so

$A(\nu) \in \partial C'$. Since $\bar{x} \in \Omega_{\lambda_0}$, we can suppose, taking δ_2 smaller if necessary, that for each $\nu \in I_{\delta_2}(e_1)$, the point $A(\nu)$ belongs to Ω_{λ_0} and that $\lambda_0(\nu) - \bar{x}'.\nu < 0$ (since $\bar{x}'.e_1 > \lambda_0$ and $\nu \mapsto \lambda_0(\nu)$ is continuous). So $A(\nu) \in \partial C$ for $\nu \in I_{\delta_2}(e_1)$ (see the definition of C and C'). Observe that by the continuity of $\nu \mapsto \lambda_0(\nu)$, the function $\nu \mapsto A(\nu)$ is also continuous and it is injective as it is easy to see.

By (30), $u(\bar{x}) = u(\bar{x}') = u(A(\nu))$ for each $\nu \in I_{\delta_2}(e_1)$, so that the function u is constant on the set $\Gamma := \{A(\nu) \,|\, \nu \in I_{\delta_2}(e_1)\}$. Since $A(\nu) \in \partial C \cap \Omega_{\lambda_0} \subset \partial C'$ for all $\nu \in I_{\delta_2}(e_1)$, we have by (24) $\nabla u = 0$ on Γ. We will prove that the projection of Γ on the hyperplane T_{λ_0} contains an open subset of T_{λ_0}, this will lead to a contradiction with Lemma 3.4 and the proof of Theorem 1.2 will be concluded.

Let us now write the generic direction $\nu \in S^{N-1} := \{v \in \mathbb{R}^N \,|\, |v| = 1\}$ as $\nu = (\nu_y, \nu_z)$, with $\nu_y \in \mathbb{R}$, $\nu_z \in \mathbb{R}^{N-1}$. If ν is close to e_1, then $\nu_y = \sqrt{1 - |\nu_z|^2}$.

We take now $\beta > 0$ small and consider the set

$$K := \left\{ \nu = (\nu_y, \nu_z) \,|\, \nu_z \in \bar{B}_\beta, \nu_y = \sqrt{1 - |\nu_z|^2} \right\},$$

where $\bar{B}_\beta = \left\{ z \in \mathbb{R}^{N-1} \,|\, |z| \leq \beta \right\}$ is the closed ball in $\mathbb{R}^{N-1}$ centered at the origin with radius β.

By construction, K is a compact neighborhood of e_1 in the metric space S^{N-1}, and if β is small enough, then K is contained in $I_{\delta_2}(e_1)$. We will show that if $A(\nu) = (y(\nu), z(\nu))$ then the set $\{z(\nu) \,|\, \nu \in K\}$ contains an open set in $\mathbb{R}^{N-1}$.

By definition of $A(\nu)$, $z(\nu) = \bar{z}' + 2(\lambda_0(\nu) - \bar{x}'.\nu)\nu_z$, where $\nu = (\sqrt{1 - |\nu_z|^2}, \nu_z) \in K$, $\nu_z \in \bar{B}_\beta$. We will prove that the image of the function

$$F(\nu_z) := 2(\lambda_0(\nu) - \bar{x}'.\nu)\nu_z, \quad \nu_z \in \bar{B}_\beta, \quad \nu = (\sqrt{1 - |\nu_z|^2}, \nu_z)$$

contains a $(N-1)$-dimensional ball centered at the origin. This will imply that $\{z(\nu) \,|\, \nu \in K\}$ contains an open set.

Let us consider a point $l \in S^{N-1} := \{z \in \mathbb{R}^{N-1} \,|\, |z| = 1\}$ and the segment $S_l := \{tl \,|\, |t| \leq \beta\} \subset \bar{B}_\beta$. By definition of F and since F is continuous, the image $F(S_l)$ is a segment contained in the line passing through the origin with direction l in R^{N-1}. Moreover, since $\lambda_0(\nu) - \bar{x}'.\nu < 0$ for all $\nu \in K$ and since S_l contains points $\nu_z = tl$

with t both positive and negative, the origin is an interior point of $F(S_l)$. Hence we can write

$$F(S_l) = \{tl \,|\, t \in [d_1(l), d_2(l)]\} \qquad d_1(l) < 0 < d_2(l),$$

for all $l \in S^{N-1}$ and for some functions $d_1(l), d_2(l)$. By the compactness of K and the continuity of $\lambda_0(\nu)$ with respect to ν, there exists $\delta \in (0,1)$ such that $\lambda_0(\nu) - \bar{x}'.\nu < -\delta < 0$ for all $\nu \in K$, so that $d_1(l) \leq -\beta\delta$ and $d_2(l) \geq \beta\delta$ for all $l \in S^{N-2}$. Thus the set

$$\{z \in \mathbb{R}^{N-1} \,|\, |z| \leq \beta\delta\} = \bar{B}_{\beta\delta} \subseteq F(\bar{B}_\beta),$$

which ends the proof. □

A direct consequence of Theorem 1.2 is the following symmetry result (Theorem 1.4 of the introduction).

COROLLARY 3.6. *Let $\nu \in \mathbb{R}^N$ and $\Omega \subset \mathbb{R}^N$ be a domain with C^1 boundary symmetric with respect to the hyperplane*

$$T_0^\nu = \{x \in \mathbb{R}^N \,|\, x.\nu = 0\}$$

and $\lambda_1(\nu) = \lambda_1(-\nu) = 0$. Assume that one of the following conditions holds

1. *$p_1, p_2 \in (1,2)$ and $f, g : \mathbb{R} \to \mathbb{R}^+$ are strictly increasing functions on $\mathbb{R}^+$ such that $f(x) > 0, g(x) > 0$ for all $x > 0$,*

2. *$p_1 \in (1,\infty), p_2 = 2$ and $f, g : \mathbb{R} \to \mathbb{R}^+$ are nondecreasing on $\mathbb{R}^+$.*

Moreover suppose that f and g are locally Lipschitz continuous on $\mathbb{R}$. Let $(u,v) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ is a weak solution of (7), then u and v are symmetric and decreasing in the ν direction in Ω_0^ν. In particular, if Ω is the ball $B_R(0)$ in $\mathbb{R}^N$ with center at the origin and radius R, then u, v are radially symmetric. If moreover $f(x) > 0, g(x) > 0$ for all $x > 0$, then $u'(r), v'(r) < 0$ for $0 < r < R$, $r = |x|$.

Proof. The symmetry of u, v is a direct consequence of Theorems 1.1 and 1.2. Suppose that Ω is a ball. If $r \in (0, R)$ and $G := B_R \setminus \bar{B}_r$, then $m := u(r)$ is the maximum of u in $\bar{G}$ and the minimum of u in $\bar{B}_r$. Since $f(x) > 0, g(x) > 0$ for all $x > 0$,

$$-\Delta_{p_1}(u - m) = -\Delta_{p_1} u > 0 \quad \text{in } B_r,$$

and by Hopf's lemma, $u'(r) < 0$. A similar result shows that $v'(r) < 0$ in B_r. □

References

[1] C. Azizieh, P. Clément, and E. Mitidieri, *Existence and a priori estimates for positive solutions of p-Laplace systems*, J. Differential Equations **184** (2002), 422–442.

[2] C. Azizieh and L. Lemaire, *A note on the moving hyperplanes method*, Electron. J. Differential Equations, Conf. **8** (2002), 1–6.

[3] H. Brezis, *Analyse fonctionnelle, théorie et applications*, Masson, Paris, 1983.

[4] L. Damascelli, *Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results*, Ann. Inst. H. Poincaré Anal. Non Linéaire **15** (1998), 493–516.

[5] L. Damascelli and F. Pacella, *Monotonicity and symmetry of solutions of p−Laplace equations, $1 < p < 2$, via the moving plane method*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. **IV** (1998), 689–707.

[6] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, second edition ed., Springer-Verlag, 1983.

[7] A. Norton, *A critical set with nonnull image has a large Hausdorff dimension*, Trans. Amer. Math. Soc. **296** (1986), 126–150.

[8] J. Simon, *Régularité de la solution d'un problème aux limites non linéaires*, Ann. Fac. Sci. Toulouse Math. **III** (1981), no. 6, 247–274.

[9] J.L. Vasquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.

[10] H. Whitney, *A function not constant on a connected set of critical points*, Duke Math. J. **1** (1935), 514–517.

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