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Singular Semilinear Elliptic Equations in the Half-space

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Summary. - *We show that equation $x_N^q \Delta u + u^{p-1} = 0$ on the half-space $Y = \mathbf{R}^{N-1} \times (0,\infty)$ and on some of its subsets has a ground state solution for $q = N - \frac{p(N-2)}{2}$, $p \in (2, 2^*)$. For $N \geq 3$ the end point cases $p = 2$ and $p = 2^*$ correspond to the Hardy inequality and the limit exponent Sobolev inequality respectively. For $N = 2$ the problem can be interpreted in terms of Laplace-Beltrami operator on the hyperbolic half-plane.*

1. Introduction

In this paper we consider the equation

$$x_N^q \Delta u + u^{p-1} = 0,\ u(x) > 0,\ x \in \Omega, \tag{1}$$

where $\Omega \subset Y := \mathbf{R}^{N-1} \times (0, \infty)$, $N \geq 2$ and $2^* = \frac{2N}{N-2}$ for $N > 2$, $2^* = \infty$ for $n = 2$.

Solutions to semilinear elliptic equations on unbounded domains often fail to exist. In the problem (1) with $\Omega = Y$, any other value of q but

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$$p_* := N - \frac{p(N-2)}{2} \tag{2}$$

leads to non-existence for the homogeneity reason: if u were a solution, changing variables x to tx' and letting $t \to \infty$ or $t \to 0$ when the left hand and the right hand side of the equation have different order of homogeneity, yields $u = 0$. Calculation of p_* that equalizes dilation homogeneity in both sides of the equation is elementary and is left to the reader. This is the same homogeneity argument that is used to show that one does not have non-trivial solutions for $\Delta u + u^p = 0$ in $\mathbf{R}^N$ with $p \neq 2^* - 1$, which connects the solution of (1) with the Talenti solution for $\Delta u + u^{2^*-1} = 0$. In some other problems non-existence can be proved by using parallel translations. For example, an equation $\Delta u - \lambda u + u^p = 0$, $\lambda > 0$, considered with the Dirichlet boundary condition on a proper open subset $\Omega \subset \mathbf{R}^N$, has no ground state solution when Ω contains balls of every size (cf. [3]): if w is a minimizer for the problem on $\mathbf{R}^N$, a minimzing sequence u_k for the subset Ω can be given as $\chi_\Omega w(x - \alpha_k)$, with a smooth cut-off function χ_Ω and $\alpha_k \in \Omega$ chosen so that $d(\alpha_k, \partial\Omega) \to \infty$. This yields that the value of the infimum for Ω is same as for $\mathbf{R}^N$. Then any minimizer for Ω would be a minimizer for $\mathbf{R}^N$ contrary to the strong maximum principle.

When invariance of equation with respect to actions of a non-compact group becomes a necessary condition for solvability, it also sets up the problem as inherently non-compact, necessitating a concentration-compactness argument using the same group. In application to singular equations, this approach has been carried out in [9] for elliptic equations with a singularity at the origin. In general, when it is impossible to draw on a compact case (as in [1]), the concentration compactness technique seems to be the natural choice ([2, 5]). The invariant problem serves, at least implicitly, as a sort of reference point for comparison with non-invariant modifications of the problem, such as restriction of the domain or introduction into the equation of lower order terms or penalizing varaible coefficients.

The present paper addresses a model case not covered by the references above, using an abstract version of concentration compactness from [6] which can be applied to a range of degenerate problems without compactness (see, e.g., an application to the Heisenberg group [8]).

The equation (1) is equivalent to the Euler-Lagrange equation for the minimization problem

$$c(\Omega, p, q) := \inf_{u \in \mathcal{D}^{1,2}(\Omega): \int_\Omega |u|^p x_N^{-q} dx = 1} \int_\Omega |\nabla u|^2 dx \tag{3}$$

Homogeneity with respect to dilations in (3) with $\Omega = Y$ yields $c(Y, p, q) = 0$ unless $q = p_*$. (Obviously, this does not imply $c(\Omega, p, q) = 0$ for any $\Omega \subset Y$: by compactness in the Sobolev imbedding, the infimum is positive whenever $\overline{\Omega}$ is a compact subset of Y. Nonetheless we restrict our study here to the case $q = p_*$.)

Let us show now that the infimum in (3) for $q = p_*$ is positive, in other words, that for every $p \in (2, 2^*)$ there is a $C(p) > 0$ such that

$$\|u\|_{L^p(Y, x_N^{-p_*})} \leq C(p) \|u\|_{\mathcal{D}^{1,2}(Y)}. \tag{4}$$

When $N \geq 3$, the relation (4) is an iterpolation between the endpoint cases $p = 2$ and $p = 2^*$, which follows easily from the Hölder inequality. In the left endpoint case one has $p_* = 2$ so that (4) is the Hardy inequality, in the right endpoint case $p_* = 0$ and (4) is the Sobolev inequality with the limit exponent. When $N = 2$, one has $2^* = \infty$, the interpolation argument of higher dimension case cannot be applied, and a separate proof of (4) for $N = 2$ is left for the Appendix. For any $N \geq 2$, the infimum in (3) only increases if one replaces Y by its subset.

We will now define a class of Ω for which we will formulate the existence result. The variable $x \in Y$ we will represent as $(\bar{x}, x_N)$. Let

$$d := \{\eta_{\alpha,j} : (\bar{x}, x_N) \mapsto (2^{-j}\bar{x} - \alpha, 2^{-j} x_N),\ j \in \mathbf{Z}, \alpha \in \mathbf{Z}^{N-1}\}. \tag{5}$$

We will also use the notation

$$d^* := \{\eta_{\alpha,j}^{-1} : (\bar{x}, x_N) \mapsto (2^j(\bar{x} + \alpha), 2^j x_N),\ j \in \mathbf{Z}, \alpha \in \mathbf{Z}^{N-1}\} \tag{6}$$

Let us define the following set of unitary operators on $\mathcal{D}^{1,2}(Y)$ (they preserve $L^p(Y, x_N^{-p_*})$-norms as well):

$$D := \{g_{\alpha,j} : u \mapsto 2^{-j(N-2)/2} u(\eta_{\alpha,j} \cdot), \eta_{\alpha,j} \in d\}. \quad (7)$$

Throughout this paper we shall consider the spaces $\mathcal{D}^{1,2}(\Omega)$ with open $\Omega \subset Y$ as subspaces of $\mathcal{D}^{1,2}(Y)$.

We can now give the definition of asymptotically contractive domains:

DEFINITION 1.1. *An open set $\Omega \subset Y$ will be called asymptotically contractive (with respect to D) if for every sequence $g_k \in D$ such that $g_k u_k$ converges weakly in $\mathcal{D}^{1,2}(Y)$, there is an $\alpha \in \mathbf{R}^{N-1}$ and a $t > 0$ such that w-lim $g_k u_k \in \mathcal{D}^{1,2}(\eta\Omega)$, where $\eta(x) = (t\bar{x} + \alpha, tx_N)$.*

Geometric characterization of asymptotic contractivity will be given in Section 2. It is immediate that Y itself is asymptotically contractive. The main result of the paper is the statement:

THEOREM 1.2. *Let $\Omega \subset Y$ be an asymptotically contractive domain with respect to D and let $p \in (2, 2^*)$, $q = p_*$. Then the equation (1) possesses a solution that (up to a scalar multiple) minimizes (3).*

Section 2 of the paper deals with concentration compactness with respect to the dislocation set D, with structure of minimizing sequences and with analytic interpretation of concentrated convergence. In Section 3 we prove Theorem 1.2.

2. Concentration compactness in half-space.

DEFINITION 2.1. *Let H be a separable Hilbert space. We say that a set D of unitary operators on H is a dislocation set if $id \in D$ and*

$$\begin{aligned} &g_k, h_k \in D, h_k^{-1} g_k \not\rightharpoonup 0 \Rightarrow \\ &\exists \{k_j\} \subset \mathbf{N}, h_{k_j}^{-1} g_{k_j} \textit{ is strongly convergent.} \end{aligned} \quad (8)$$

Note that operators $h_k^{-1} g_k$ are not required to be elements of D.

DEFINITION 2.2. *Let $u, u_k \in H$. We will say that u_k converges to u* weakly with concentration *(under dislocations D), which we will denote as*

$$u_k \overset{D}{\rightharpoonup} u,$$

if for all $\varphi \in H$,

$$\lim_{k\to\infty} \sup_{g\in D}(g(u_k - u), \varphi) = 0. \tag{9}$$

THEOREM 2.3 ([6]). *Let* $u_k \in H$ *be a bounded sequence. Then there exists* $w^{(n)} \in H$, $g_k^{(n)} \in D$, $k, n \in \mathbf{N}$ *such that for a renumbered subsequence*

$$\begin{aligned} g_k^{(1)} &= id,\ {g_k^{(n)}}^{-1} g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m, && (10)\\ w^{(n)} &= w\text{-}lim\ {g_k^{(n)}}^{-1} u_k && (11)\\ \sum_{n\in\mathbf{N}} \|w^{(n)}\|^2 &\leq \limsup \|u_k\|^2 && (12)\\ u_k - \sum_{n\in\mathbf{N}} g_k^{(n)} w^{(n)} &\overset{D}{\rightharpoonup} 0. && (13) \end{aligned}$$

We now return to the case $H = \mathcal{D}^{1,2}(Y)$. To use Theorem 2.3 with the set D defined in the previous section we have to show that D is a set of dislocations. We begin with

LEMMA 2.4. *A sequence* $(\alpha_k, j_k) \in \mathbf{R}^{N-1} \times \mathbf{Z}$ *has a bounded subsequence if and only if for every* $u \in H \setminus \{0\}$, $2^{j_k(N-2)/2}u(2^{j_k}\bar{x} + \alpha_k, 2^{j_k}x_N) \not\rightharpoonup 0$.

Proof. If on a renamed subsequence $j_k = j$ and $\alpha_k \to \alpha$, then obviously $2^{j_k(N-2)/2}u(2^{j_k}\bar{x}+\alpha_k, 2^{j_k}x_N) \rightharpoonup 2^{j(N-2)/2}u(2^j\bar{x}+\alpha, 2^j x_N) \neq 0$. Conversely, if $2^{j_k(N-2)/2}u(2^{j_k}\bar{x}+\alpha_k, 2^{j_k}x_N) \not\rightharpoonup 0$, assume without loss of generality that $u \in C_0^\infty(Y)$. Then there exists a $v \in C_0^\infty(\Omega)$ such that $2^{j_k(N-2)/2}(u(2^{j_k}\bar{x}, 2^{j_k}x_N), v(\bar{x} - \alpha_k, x_N)) \not\to 0$. If $j_k \to +\infty$, then we have integration restricted to a shrinking neighborhood of $(\alpha_k, 0)$ which yields zero limit, a contradiction. If $j_k \to -\infty$, the argument can be repeated by rewriting the scalar product above in the form $2^{-j_k(N-2)/2}(v(2^{-j_k}\bar{x}, 2^{-j_k}x_N), u(\bar{x} - \alpha_k', x_N))$. Finally, if on an appropriate renamed subsequence, $j_k = j$, but $|\alpha_k| \to \infty$, then the supports of $2^{j(N-2)/2}u(2^j\bar{x}, 2^j x_N)$ and $v(\bar{x} - \alpha_k, x_N)$ become disjoint for k sufficiently large, which implies

$$\begin{aligned} 2^{j_k(N-2)/2}(u(2^{j_k}\bar{x} + \alpha_k, 2^{j_k}x_N), v) &= \\ 2^{j(N-2)/2}(u(2^j\bar{x}, 2^j x_N), v(\bar{x} - \alpha_k, x_N)) &= 0, \end{aligned}$$

i.e.

$$2^{j_k(N-2)/2}u(2^{j_k}\bar{x}+\alpha_k, 2^{j_k}x_N) \to 0,$$

a contradiction that yields α_k bounded. □

LEMMA 2.5. *The set D is a set of dislocations in $\mathcal{D}^{1,2}(Y)$.*

Proof. It suffices to verify (8). Assume that $h_k^{-1}g_k \not\rightharpoonup 0$. If $g_k u(x) = 2^{-j_k(N-2)/2}u(2^{-j_k}\bar{x}-\alpha_k, 2^{-j_k}x_N)$ and $h_k u(x) = 2^{-j'_k(N-2)/2}u(2^{-j'_k}\bar{x}-\alpha'_k, 2^{-j'_k})$ then by Lemma 2.4, on a renamed subsequence, $j'_k - j_k = j$ and $2^j\alpha_k - \alpha'_k = \alpha$ with some $\alpha \in \mathbf{Q}^{N-1}$ and $j \in Z$. Then for any $u \in H$,

$$h_k^{-1}g_k u = 2^{j(N-2)/2}u(2^j\bar{x}+\alpha, 2^j x_N). \tag{14}$$

□

LEMMA 2.6. *Let $p \in (2, 2^*)$, and let $u_k \in \mathcal{D}^{1,2}(Y)$ be a bounded sequence. Then*

$$u_k \overset{D}{\rightharpoonup} 0 \Leftrightarrow u_k \to 0 \text{ in } L^p(Y, x_N^{-p_*}). \tag{15}$$

Proof. First, assume that $u_k \to 0$ in $L^p(Y, x_N^{-p_*})$. Then for every sequence $g_k \in D$, $g_k u_k \rightharpoonup 0$ in $L^p(Y, x_N^{-p_*})$. However, since u_k is bounded in $\mathcal{D}^{1,2}$-norm, $g_k u_k \rightharpoonup 0$ in $\mathcal{D}^{1,2}$ and therefore, $u_k \overset{D}{\rightharpoonup} 0$.

Assume now that $u_k \overset{D}{\rightharpoonup} 0$. Let

$$B = (0,1)^{N-1} \times (1,2), \tag{16}$$

and let

$$B_{\alpha,j} = \eta_{\alpha,j}^{-1}B, \; \alpha \in \mathbf{Z}^{N-1}, \; j \in \mathbf{Z}, \tag{17}$$

in other words, $B_{\alpha,j} = \{(2^j\alpha_1, 2^j(\alpha_1+1)) \times \ldots (2^j\alpha_{N-1}, 2^j(\alpha_{N-1}+1)) \times (2^j, 2^{j+1})\}$, $\alpha \in \mathbf{Z}^{N-1}, j \in \mathbf{Z}$. Note that $\bigcup_{\alpha\in\mathbf{Z}^{N-1}, j\in\mathbf{Z}} B_{\alpha,j} = Y$ up to a set of measure zero.

By the standard Sobolev imbedding over B and by homogeneity, there is a $C > 0$ such that for every $\alpha \in \mathbf{Z}^{N-1}, j \in \mathbf{Z}$,

$$\int_{B_{\alpha,j}} x_N^{-p_*}|u_k|^p \leq$$
$$C\left(\|u_k\|^2_{\mathcal{D}^{1,2}(B_{\alpha,j})} + \|u_k\|^2_{L^2(B_{\alpha,j},x_N^{-2})}\right)\|u_k\|^{p-2}_{L^p(B_{\alpha,j},x_N^{-p_*})}. \tag{18}$$

Adding the inequalities (18) over all α, j, we arrive at

$$\int_Y x_N^{-p_*}|u_k|^p \leq$$
$$C\left(\|u_k\|^2_{\mathcal{D}^{1,2}(Y)} + \|u_k\|^2_{L^2(Y,x_N^{-2})}\right)\sup_{\eta\in d}\left(\int_B x_N^{-p_*}|u_k(\eta\cdot)|^p\right)^{1-2/p}. \tag{19}$$

Using the Hardy inequality that estimates the $L^2(Y, x_N^{-2})$-norm by the $\mathcal{D}^{1,2}(Y)$-norm, and choosing an appropriate "near-supremum" sequence $\eta_k \in d$, we get from (19)

$$\int_Y x_N^{-p_*}|u_k|^p \leq 2C\left(\int_B x_N^{-p_*}|u_k(\eta_k\cdot)|^p\right)^{1-2/p}. \tag{20}$$

It remains to note that since $u_k(\eta_k\cdot) \rightharpoonup 0$ in $\mathcal{D}^{1,2}(Y)$, from compactness of Sobolev imbedding on B one concludes that $u_k(\eta_k\cdot) \to 0$ in $L^p(B, x_N^{-p_*})$, so that the assertion of the lemma follows. ☐

We now give a sufficient geometric condition for a set to be asymptotically contractive. We recall that for a sequence of sets Ω_k, $\liminf \Omega_k := \bigcup_n \bigcap_{k\geq n} \Omega_k$.

LEMMA 2.7. *Let $\Omega \subset Y$ be an open set, such that $\partial(Y \setminus \overline{\Omega}) = \partial\Omega$. If for every $\eta_k \in d^*$, there exist a $t > 0$ and an $\alpha \in \mathbf{R}^{N-1}$ such that*

$$\liminf \eta_k\Omega \subset \eta\Omega, \tag{21}$$

where $\eta x = (t\bar{x}+\alpha, tx_N)$, the set Ω is asymptotically contractive with respect to D.

Proof. There is a similar statement in [6] with a similar proof, so we give only a brief sketch. Let $u_k \in \mathcal{D}^{1,2}(\Omega)$ be a bounded sequence. It is easy to see from (21) that if $w =$ w-lim $g_{\eta_k^{-1}} u_k$, then $w(x) = 0$ for almost every x in the complement of $\liminf \eta_k\Omega$, and so in the complement of $\eta\Omega$. Consider a regularization of w in the sense of potential theory. Then $w = 0$ quasi-everywhere in the set $\overline{\mathrm{int}(Y \setminus \eta\Omega)}$, and therefore, by assumption of the lemma, in $Y \setminus \eta\Omega$. Then $w \in \mathcal{D}^{1,2}(\eta\Omega)$ by the Hedberg trace theorem. More precisely, since the Hedberg trace theorem is formulated for Sobolev spaces and not for $\mathcal{D}^{1,2}(Y)$, its conclusion applies directly only to $\chi_k(x_N)w$, where $\chi_k(y) = 0$ for $y < 1/k$ or $y > 2k$, $\chi_k(y) = ky - 1$ for $1/k < y < 2/k$ and $\chi(y) = 1$ for $y \in [2/k, k]$. However, it is easy to see that $\chi_k(y)w$ approximates w in the $\mathcal{D}^{1,2}$-norm. □

As examples of asymptotically contractive domains we can give

1) $\Omega = \{(\bar{x}, x_N) : \varphi(\bar{x}) < x_N < \psi(\bar{x})\}$ with $0 \le \varphi \le \lim_{|\bar{x}|\to\infty} \varphi(\bar{x})$ and $\psi \ge \lim_{|\bar{x}|\to\infty} \psi(\bar{x})$;
2) $\Omega = \{(\bar{x}, x_N) : |\bar{x}| < \lambda x_N\}$, $\lambda > 0$;
3) $\Omega = \bigcup_{\eta\in d^*} \eta\omega$, with an open $\omega \subset Y$;
4) Any asymptotically null set as defined below;
5) A union of asymptotically contractive set and a set whose closure is compact in Y.

The functions φ, ψ above are assumed to be continuous.

Remark 2.8. *We will say that an open set $\Omega \subset Y$ is asymptotically null (with respect to the dislocation set D) if for every sequence $g_k \in D$, $g_k \rightharpoonup 0$ and every bounded sequence $u_k \in \mathcal{D}^{1,2}(\Omega)$, w-lim $g_k u_k = 0$. It is easy to see that for a $p \in (2, 2^*)$ the imbedding into $L^p(\Omega, x_N^{-p_*})$ is compact if and only if Ω is asymptotically null (cf. [6]). In particular, this is true if for every sequence $\eta_k \in d^*$ defined by unbounded (α_k, j_k),*

$$|\liminf \eta_k\Omega| = 0. \tag{22}$$

Indeed, (22) easily implies that for any bounded sequence u_k, the dislocated weak limits $w^{(n)}$, $n > 1$, are zero a.e., so that $u_k \overset{D}{\rightharpoonup} w^{(1)}$ and therefore $u_k \to w^{(1)}$ in L^p, $p \in (2, 2^*)$. A set $\varphi(\bar{x}) < x_N < \psi(\bar{x})$

with $\inf\varphi > 0$, $\sup\psi < \infty$ *and* $\lim_{\bar{x}\to\pm\infty}(\varphi(x)-\psi(x)) = 0$ *is asymptotically null, and so is the set* $|\bar{x}| < \psi(x_N)$ *with* $\lim_{y\to 0}\psi(y)/y = \lim_{y\to\infty}\psi(y)/y = 0$. *The functions* φ, ψ *are assumed to be continuous.*

3. Proof of Theorem 1.2.

Proof. Let u_k be a minimizing sequence in (3). We apply to it Theorem 2.3. By asymptotic contractivity of Ω we can rename $g_\eta w^{(n)}$ as $w^{(n)}$ with an appropriate choice of η for every n, so that $w^{(n)} \in \mathcal{D}^{1,2}(\Omega)$. Then

$$\sum \|w^{(n)}\|^2_{\mathcal{D}^{1,2}}(\Omega) \le c(\Omega, p, p_*). \tag{23}$$

At the same time it is easy to see (cf. e.g. [7]) that

$$\sum \|w^{(n)}\|^p_{L^p(\Omega, x_N^{-p_*})} \le = \lim \|u_k\|^p_{L^p(\Omega, x_N^{-p_*})} = 1. \tag{24}$$

¿From (23) and the definition of $c(\Omega, p, p_*)$ follows that

$$\sum \|w^{(n)}\|^2_{\mathcal{D}^{1,2}}(\Omega) \le c(\Omega, p, p_*) \sum t_n^{2/p}, \tag{25}$$

where $t_n = \|w^{(n)}\|^p_{L^p(\Omega, x_N^{-p_*})}$. Note now that (24) can be written now as $\sum t_n = 1$, so that with $p > 2$, $\sum t_n^{2/p} = 1$ only if all but one of t_n, say for $n = n_0$, equal zero. We conclude that $w^{(n_0)}$ is the minimizer. Then so is $|w^{(n_0)}|$ and the strict positivity of $|w^{(n_0)}|$ follows from the maximum principle. □

4. Appendix: proof of inequality (4) for N=2.

Proof. Note that for $N = 2$, $p_* = 2$ independently of p. Let B and $B_{\alpha,j}$ be defined as in (16),(17) respectively, i.e. $B_{i,j} = \{(2^j(i, i+1), (2^j, 2^{j+1})\}$, $i, j \in \mathbf{Z}$. Note that $\bigcup_{i,j\in\mathbf{Z}} B_{i,j}$ is Y up to a set of measure zero. Therefore from the Sobolev inequality that holds on B,

$$\|u\|^p_{L^p(B, x_2^{-2})} \le C(p, B)\left(\|u\|^2_{\mathcal{D}^{1,2}(B)} + \|u\|^2_{L^p(B, x_2^{-2})}\right)^{p/2}, \tag{26}$$

we deduce the inequality with the same constant for every $B_{i,j}$. Adding the inequalities over $i, j \in \mathbf{Z}$, and noticing that the function $s \mapsto s^{p/2}$ with $p/2 \geq 1$ is superadditive, we get

$$\|u\|^p_{L^p(Y,x_2^{-2})} \leq C \left(\|u\|^2_{\mathcal{D}^{1,2}(Y)} + \|u\|^2_{L^p(Y,x_2^{-2})} \right)^{p/2}, \tag{27}$$

Due to the Hardy inequality, the $L^p(Y, x_2^{-2})$-norm is dominated by the $\mathcal{D}^{1,2}$-norm, and we arrive at (4). □

It can be observed that in the case $N = 2$ we can view Y as a Riemannian manifold with the metric $x_2^2(dx_1^2 + dx_2^2)$, in which the square of $\mathcal{D}^{1,2}$-norm is the quadratic form of the Laplace-Beltrami operator and $x_2^{-2} dx_1 dx_2$ is the invariant measure. In this setting the set d is a subset of the group of isometries on Y.

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