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A Note on Harmonic Calculus in m-convex Algebras

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Summary. - *We prove a version of the maximum modulus principle, for harmonic vector valued functions, in complete locally m-convex Q-$*$-algebras. This is used to generalize some extended versions of von Neumann's inequality.*

Introduction

Let f be a complex function holomorphic on the open unit disk D and $\mathcal{H}$ be a complex Hilbert space. In [4] Ky Fan has proved that if $f(D) \subset D$, then the inequality $\|f(T)\| < 1$ holds for every proper contraction T on $\mathcal{H}$. It is known that Ky Fan's theorem is an equivalence formulation to the important inequality of von Neumann given also in [4]. Generalizations of this result, in hermitian algebras, are obtained in [2]. Tao Zhiguang ([7]) has generalized Ky Fan's theorem to analytic operator functions. Using a maximum principle, we have extended, in [3], Ky Fan's theorem and von Neumann's inequality to harmonic functions, in hermitian Banach algebras. In this paper, we prove that tha above mentioned results remain valid in the general context of locally m-convex algebras.

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1. Preliminaries

Let (E,τ) be a locally convex space the topology of which is given by a family $\{|.|_\lambda : \lambda \in \Lambda\}$ of seminorms. If E is endowed with an algebra structure such that $|xy|_\lambda \leq |x|_\lambda |y|_\lambda$, for every $x, y \in E$ and $\lambda \in \Lambda$, we say that $(E, (|.|_\lambda)_{\lambda\in\Lambda})$ is a locally multiplicatively convex (*l. m. c. a.* in short). It is known that a complex *l. m. c. a.* $(E, (|.|_\lambda)_{\lambda\in\Lambda})$ is the projective limit of the normed algebras $(E_\lambda, \|.\|_\lambda)$, where $E_\lambda = E/N_\lambda$ with $N_\lambda = \{x \in E : |x|_\lambda = 0\}$ and $\|\overline{x}\|_\lambda = |x|_\lambda$. An element x of E is written $x = (x_\lambda)_\lambda = (\pi_\lambda(x))_\lambda$, where $\pi_\lambda : E \longrightarrow E_\lambda$ is the canonical surjection. The algebra $(E, (|.|_\lambda)_{\lambda\in\Lambda})$ is also the projective limit of the Banach algebras $\widehat{E_\lambda}$, the completions of E_λ's. The norm in $\widehat{E_\lambda}$ will also be denoted by $\|.\|_\lambda$. If E is endowed with an algebra involution $x \longmapsto x^*$ such that $|x|_\lambda = |x^*|_\lambda$, for any $x \in E, \lambda \in \Lambda$, then $(E, (|.|_\lambda)_{\lambda\in\Lambda})$ is called a locally multiplicatively convex $*$-algebra (*l. m. c.* $*$-*a.* in short). In this case, each $\widehat{E_\lambda}$, $\lambda \in \Lambda$, becomes an involutive Banach algebra. A *l. m. c. a.* $(E, (|.|_\lambda)_{\lambda\in\Lambda})$ is called a Q-algebra if the group $G(E)$ of its invertible elements is open. Denote by $Rea = \frac{1}{2}(a + a^*)$ the real part of an element a in E, by U the set of all unitary elements of E (i.e., all a such that $a^*a = aa^* = e$) and by U_e the identity component of U. Recall that the algebra E is said to be hermitian if hermitian elements (i.e., all a such that $a^* = a$) have real spectrum. If E is a *l. m. c.* $*$-*a.* which is a Q-algebra, define $|a| = \rho(a^*a)^{\frac{1}{2}}$ for a in E.

Let Ω be an open subset of C and $f : \Omega \longrightarrow E$ a $C^{(2)}$ function of two real variables x and y. Recall that f is said harmonic if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ on Ω. The set of all harmonic E-valued functions on Ω is denoted $h(\Omega, E)$. If f is holomorphic on Ω, then f is harmonic; so we have $H(\Omega, E) \subset h(\Omega, E)$. For the scalar functions on Ω, we simply put $H(\Omega) = H(\Omega, C)$ and $h(\Omega) = h(\Omega, C)$. In the sequel, e will denote the unit and for $z \in C$ we simply write z instead of ze. Also the open unit and the closed disk in C will be denoted by D and $\overline{D}$ respectively. The spectrum and the spectral radius of an element $a \in E$ will be denoted by Spa and $\rho(a)$ respectively.

2. Harmonic calculus and a form of the maximum principle

The functional calculus for harmonic E-valued functions ([3]) can be extended to locally m-convex algebras as follows.

DEFINITION 2.1. *Let $(E, (|.|_\lambda)_{\lambda\in\Lambda})$ be a unitary and* complete l. m. c. $*$-a., *Ω an open subset of C, $z_0 \in \Omega$ such that $\overline{D(z_0, r)} \subset \Omega$, $(r > 0)$, $a \in E$ with $Spa \subset D(z_0, r)$ and $f \in h(\Omega, E)$. Then*

$$f(a) = \frac{1}{2\pi}\int_{|z-z_0|=r} f(z) Re\left[(z + a - 2z_0)(z - a)^{-1}\right]\frac{|dz|}{r}. \qquad (1)$$

The fundamental properties of this functional calculus are contained in the following result. The proof, being straightforward, is omitted.

PROPOSITION 2.2.

1) *The mapping $f \longmapsto f(a) = (f(a_\lambda))_{\lambda\in\Lambda}$ is an involutive homomorphism from $h(\Omega, E)$ into E that extends the algebra homomorphism $f \longmapsto f(a)$ from $H(\Omega, E)$ into E given by the holomorphic functional calculus.*

2) *If K is a compact neighbourhood contained in Ω and containing Spa, then the mapping $f \longmapsto f(a)$ is continuous with respect to the uniform convergence on K.*

3) *If $x \longmapsto x^*$ is a hermitian involution and a is normal, then $f(Spa) = Spf(a)$ for every $f \in h(\Omega)$.*

Let $a \in E$ such that $\rho(a) < 1$ and $|a| < 1$. The characteristic function Φ_a is defined by

$$\Phi_a(z) = (e - aa^*)^{-\frac{1}{2}}(z + a)(e + za^*)^{-1}(e - a^*a)^{\frac{1}{2}}$$

for $z \in C$ satisfying $|z|\rho(a) < 1$. It is holomorphic in a neighbourhood of $\overline{D}$ and takes the unit circle into U_e.

The following result will be needed later on.

Proposition 2.3. *Let* $(E, (|.|_\lambda)_{\lambda\in\Lambda})$ *be a complex, unitary and complete* l. m. c. $*$-a. *which is a* Q*-algebra,* Ω *be an open subset of* C *and* $f \in H(\Omega, E)$*. Then the function* $z \longmapsto \rho\left(\Phi_a(z)\right)$ *is subharmonic in* Ω.

Proof. By a result of Vesentini ([6]), the function $z \longmapsto \rho_\lambda\left(\pi_\lambda \circ f(z)\right)$ is subharmonic, for every λ. On the other hand, the spectral radius ρ is *u. s. c.* for E is a Q-algebra. Then the function $z \longmapsto \rho\left(f(z)\right)$ is *u. s. c.* in Ω. It follows that the function $z \longmapsto \rho\left(f(z)\right) = \sup\limits_{\lambda\in\Lambda} \rho_\lambda\left(\pi_\lambda \circ f(z)\right)$ is subharmonic in Ω. □

As a first application of the harmonic calculus, we give a proof of J.W.M. Ford's square root lemma. Our approach enlighten more the fact that the square root is hermitian.

Proposition 2.4. *Let* $(E, (|.|_\lambda)_{\lambda\in\Lambda})$ *be a complex, unitary and complete* l. m. c. $*$-a. *and* h *be a hermitian element of* E *such that* $Sph \subset \{z \in C : Rez > 0\}$*. Then there exists a hermitian element* $k \in E$ *such that* $k^2 = h$.

Proof. Put $\Omega = \{z \in C : z \notin R^-\}$. There is a holomorphic function f in Ω such that $f^2(z) = z$ and $f(1) = 1$. Let Γ be a closed curve such that Sph is contained is its interior $int\Gamma$ and $int\Gamma \cup \Gamma$ is contained in Ω. Put $k = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma f(z)(z-h)^{-1}dz$. It is clear that $k = (k_\lambda)_{\lambda\in\Lambda}$, where $k_\lambda = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma f(z)(z-h_\lambda)^{-1}dz$ for $Sph_\lambda \subset Sph$. Moreover, by 1) of Proposition 2.2, we have

$$k^2 = \left({k^2}_\lambda\right)_{\lambda\in\Lambda} = \left(\frac{1}{2\pi i}\int_\Gamma z(z-h_\lambda)^{-1}dz\right)_{\lambda\in\Lambda} = (h_\lambda)_{\lambda\in\Lambda} = h.$$

It remains to show that k_λ is hermitian for every $\lambda \in \Lambda$. Let r and r' such that $0 < r < r'$, $Sph \subset D(r', r)$ and $\overline{D(r', r)} \subset \Omega$. We check that

$$k_\lambda = \frac{1}{2\pi}\int_{|z-r'|=r} f(z) Re\left[(z + h_\lambda - 2r')(z-h_\lambda)^{-1}\right]\frac{|dz|}{r}.$$

Since $x \longmapsto x^*$ is continuous, we have

$$k^*{}_\lambda = \frac{1}{2\pi}\int_{|z-r'|=r} \overline{f(z)} Re\left[(z + h_\lambda - 2r')(z-h_\lambda)^{-1}\right]\frac{|dz|}{r}.$$

But $f(\overline{z}) = \overline{f(z)}$, for every $z \in \Omega$, hence $k^*{}_\lambda = k_\lambda$. This completes the proof. $\square$

We now prove the following version of the maximum principle for harmonic functional calculus in the context of m-convex algebras. It is the main result of this note.

PROPOSITION 2.5. *Let* $(E, (|.|_\lambda)_{\lambda\in\Lambda})$ *be a complex, unitary and complete* l. m. c. $*$-a. *which is a* Q*-algebra,* Ω *be an open subset of* C *containing* $\overline{D}$*,* $f \in h(\Omega, E)$*,* $E_0 = \{a \in E : \ max\,(|a|, \rho(a)) \leq 1\}$ *and* $E_1 = \{a \in E_0 : f$ *commutes with both* a *and* $a^*\}$.

1) If $Sp\Phi_a(z) \subset \overline{D}$*, for all* $a \in E_1$ *and* $|z| = 1$*, then* $f(E_1) \subset \overline{C_0} f(U_e)$*, where* $\overline{C_0} f(U_e)$ *is the closure of the convex hull of* $f(U_e)$

2) If E *is hermitian, then* $|f(a)| \leq \sup\{|f(u)| : u \in U_e\}$ *for all* a *in* E_1.

Proof. 1) Assume, without loss of generality, that $|a| < 1$ and $\rho(a) < 1$ for $a \in E_0$. Since Φ_a is holomorphic on a neighbourhood of $\overline{D}$, it follows from Proposition 2.3 that $\rho\left(\Phi_a(z)\right)$ is subharmonic function of z in a neighbourhood of $\overline{D}$. Then, by the maximum modulus principle, we have $Sp\Phi_a(\xi) \subset \overline{D}$, for all $|\xi| \leq 1$. Let $z_0 \in D$ and $r_1 > 0$ such that $\overline{D(z_0, r_1)} \subset D$ and put $C_1 = \{z \in C : |z - z_0| = r_1\}$. Again, by the maximum modulus principle, there exists $0 < r_2 < 1$ such that $Sp\Phi_a(\xi) \subset D(0, r_2)$ for every $|\xi - z_0| \leq r_1$. Put $C_2 = \{w \in C : |w| = r_2\}$. For $|\xi - z_0| < r_1$ we have, by (1),

$$f\left(\Phi_a(\xi)\right) = \frac{1}{2\pi}\int_{C_2} f(w) Re\left[(w + \Phi_a(\xi))(w - \Phi_a(\xi))^{-1}\right]\frac{|dw|}{r_2}$$

Since for any fixed $w \in C_2$, the function g defined by

$$g(\xi) = Re\left[(w + \Phi_a(\xi))(w - \Phi_a(\xi))^{-1}\right]$$

is harmonic on $D(z_0, r_1)$ and continuous on C_1, it follows that

$$g(\xi) = \frac{1}{2\pi}\int_{C_1} g(z) Re\left[(z + \xi - 2z_0)(z - \xi)^{-1}\right]\frac{|dz|}{r_1}.$$

Thus, for every $\xi \in D(z_0, r_1)$, we have $f(\Phi_a(\xi)) =$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{C_2}\int_{C_1} f(w)g(z)Re\left[(z+\xi-2z_0)(z-\xi)^{-1}\right]\frac{|dz|}{r_1}\frac{|dw|}{r_2}$$

Let $\phi \in E'$ the topological dual of $(E, (|.|_\lambda)_{\lambda\in\Lambda})$. The reader can prove that the function $\xi \longmapsto \phi(f(\Phi_a(\xi)))$ is harmonic on D. Then the function $F(\xi) = Re(\phi(f(\Phi_a(\xi))))$ defined for $|\xi| < 1$ is harmonic for $|\xi| < 1$. Furthermore F is continuous in $\overline{D}$. The maximum principle tells us that $F(0) \leq \sup\{F(\xi) : |\xi| = 1\}$ and consequently

$$Re\phi(f(a)) \leq \sup\left\{Re\left(\phi\left(f\left(\Phi_a(\xi)\right)\right)\right) : |\xi| = 1\right\}.$$

Since f commutes with a and a^*, it follows that f commutes with $\Phi_a(\xi)$ and $\Phi_a(\xi)^*$, for every $|\xi| = 1$ and hence $Re\phi(f(a)) \leq \sup Re\phi(f(U_e))$. Therefore, by a separation theorem ([1, p. 417]), $f(a) \in \overline{C_0}\, f(U_e)$. This completes the proof of 1).

2) Since E is hermitian, we have $\rho(\Phi_a(z)) = 1$ for every $|z| = 1$ and therefore $Sp\Phi_a(z) \subset \overline{D}$ for all $a \in E_2$ and $|z| = 1$. On the other hand, by [5], $|.|$ is a continuous submultiplicative seminorm on E such that $\rho(x) \leq |x|$ for every $x \in E$. Hence 2) in a consequence of 1).

□

As a consequence, we obtain the following result

Corollary 2.6. *Let $(E, (|.|_\lambda)_{\lambda\in\Lambda})$ be a complex, unitary and hermitian complete* l. m. c. $*$-a. *which is a Q-algebra.*

1) If f is a complex harmonic function on some neighbourhood of $\overline{D}$ such that $f(\overline{D}) \subset \overline{D}$ and $a \in E$ with $|a| \leq 1$, then $|f(a)| \leq 1$.

2) If $f \in h(D)$, $0 < r < 1$ and $M(r) = \max\limits_{|z|=r} |f(z)|$, then $M(r) = \max\limits_{|a|=r} |f(a)|$.

3) If $f \in h(D)$ such that $f(D) \subset D$ and $a \in E$ with $|a| \leq 1$, then $|f(a)| < 1$

Proof.

1) By 2) of Proposition 2.5, it suffices to prove that

$$\sup\{|f(u)| : u \in U_e\} \leq 1.$$

Since $\rho(u) = |u| = 1$, for every $u \in U_e$, we have $Spu \subset \overline{D}$. Hence $f(u)$ is defined. Furthermore $f(u)$ is normal and hence $|f(u)| = \rho(f(u))$. On the other hand, we have $Sp\, f(\pi_i(u)) = f(Sp(\pi_i(u)))$ by the spectral mapping theorem given by 3) of Proposition 2.2. Thus $\rho_i(f(\pi_i(u))) \leq 1\ \forall i$. Now since $\rho(f(u)) = \sup \rho_i(f(\pi_i(u)))$, we obtain $f(u) \leq 1$.

2) Author's proof of [3, Theorem 3.2] applies to this case as well.

3) It is a direct consequence of 2).

□

In [7], Tao Zhiguang has generalized (1) to analytic operator functions. More precisely, let $\mathcal{H}$ be a complex Hilbert space, $L(\mathcal{H})$ the complex Banach algebra of all bounded linear operators on $\mathcal{H}$ and $\Omega = \{z : |z| < 1 + 2\delta\}$ where $\delta > 0$. Tao Zhiguang showed that if $f \in N_{\mathcal{H}}(\Omega)$ with $\|f(z)\| \leq 1$ for $z \in \overline{D}$ and $T \in L(\mathcal{H})$ is a contraction such that T and f are commuting (i.e. $Tf(z) = f(z)T$ for every $z \in \Omega$, then $\|f(T)\| \leq 1$. Here $f(T)$ denotes the operator defined by Riez-Dunford integral ([1, p. 568]) $f(T) = \frac{1}{2\pi i}\int_\Gamma f(z)(z-T)^{-1}dz$, where Γ is any contour that surrounds the spectrum of T in Ω; and $N_{\mathcal{H}}(\Omega)$ the set of all analytic functions on Ω into $L(\mathcal{H})$ such that $f(z)f(w) = f(w)f(z)$ and $f(z)f(z)^* = f(z)^*f(z)\ \forall\, z, w \in \Omega$. If we replace the condition $f \in N_{\mathcal{H}}(\Omega)$ by an harmonic operator function such that $f(z)T^* = T^*f(z)$ for $z \in \overline{D}$, the result of Tao Zhiguang remains valid as the following result shows.

Proposition 2.7. *Let $T \in L(\mathcal{H})$ be a contraction (i.e., $\|T\| \leq 1$) and $f \in h(\Omega, L(\mathcal{H}))$ such that $\|f(z)\| \leq 1$ for every $z \in \overline{D}$ and commutes with both T and T^*. Then $\|f(T)\| \leq 1$.*

Proof. We first prove that $\|f(\eta T)\| \leq 1$, if $\eta \in [0, 1[$. By Proposition 2.5, it suffices to show that the number

$$\theta = \sup\{f(S) : S \in U(I) \text{ and } f \text{ commutes with } S\}$$

satisties $\theta \leq 1$. Let $S \in U(I)$ and $f \in h(\Omega, L(\mathcal{H}))$ such that $Sf(z) = f(z)S$ for all $z \in \Omega$. Let E be the spectral resolution of S. Then, one has $f(S) = \widetilde{f}(S) = \int_{\sigma(T)} f(z)dE_z$ and we deduce that $\|f(S)\| \leq \sup\{\|f(z)\| : z \in \sigma(S)\} \leq 1$. Thus $\|f(\eta T)\| \leq 1$. Now when η increases to 1, $f(\eta z)$ converges to $f(z)$ uniformly on the neighbourhood $\{z : |z| < 1 + \delta\}$ of $\sigma(T)$. By 2) of Proposition 2.2, $f(\eta T)$ converges to $f(T)$ in the norm topology and so $\|f(T)\| \leq 1$. □

References

[1] N. Dunford and J.T. Schwartz, *Linear operators*, vol. I, Interscience, New York, 1953, vol. II, 1963.

[2] A. El Kinani, *Holomorphic functions operating in hermitian Banach algebras*, Proc. Amer. Math. Soc. **111** (1991), 931–939.

[3] A. El Kinani, *Harmonic functions operating in hermitian Banach algebras*, Publicacions Matemátiques **41** (1997), 403–409.

[4] K. Fan, *Analytic functions of a proper contraction*, Math. Z. **160** (1978), 275–290.

[5] M. Fragoulopoulou, *Symmetric Topological ∗-Algebras. Applications*, 3, no. 9, Shriftenreihe der Mathematischen Institus und der Graduiertenkollegs der Universitat Munster, 1993.

[6] Vesentini, *On the subharmonicity of the spectral radius*, Boll. Un. Mat. Ital. **1** (1968), 427–429.

[7] T. Zhiguang, *Analytic operator functions*, J. Math. Anal. Appl. **103** (1984), 293–320.

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