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Genus Reducing Knots in 3-Manifolds

Yo'av Rieck (*)

To the memory of Marco Reni

Summary. - *A genus reducing knot is a knot that has infinitely many surgeries after which the Heegaard genus of the manifold reduces. We study certain aspects of this question, in particular solving it for totally orientable Seifert Fibered Spaces, where we find examples of manifolds of arbitrarily high genus containing no such knot.*

1. Introduction

The purpose of this paper is to investigate some cases of the following question: what 3-manifolds contain a knot with infinitely many surgeries yielding a manifold of lower genus? We solve this question for totally orientable Seifert Fibered Spaces, and find out that although most of them do, already in this class of manifolds we see examples of manifolds of arbitrarily high genus containing no such knot.

Before all else, let us define the new concepts we shall study:

Definitions 1.1. *1. A* genus reducing knot *in 3-manifold is a knot with infinitely many surgeries yielding manifolds of lower Heegaard genus.*

2. Let M be a manifold and Σ a Heegaard surface for it. A knot γ disjoint from Σ is called a destabilizing knot *for Σ if there*

(*) Author's address: Department of Mathematics, University of Arkansas, AR 72701, email: yoav@mail.uark.edu
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are infinitely many surgeries on γ after which Σ destabilizes to a surface that is not isotopic (in the surgered manifold) to a Heegaard surface for $M \setminus N(\gamma)$.

(As we shall see, it follows from Corollary 6.6 of [13] that a genus reducing knot is a destabilizing knot for some minimal genus Heegaard surface.)

For Seifert Fibered Spaces we prove:

THEOREM (6.1). *A totally orientable Seifert Fibered Space (other than a Seifert Fibered Space over S^2 with at most three exceptional fibers) has a genus reducing knot unless it has a horizontal Heegaard surface and is of the following types:*

1. *Seifert Fibered Space over S^2 with $2k+1,\ k>1$ exceptional fibers of multiplicity two and one other of multiplicity $2n+1$, $n \geq 1$;*
2. *Seifert Fibered Space over any surface with one exceptional fiber.*

(Seifert Fibered Spaces over S^2 with at most three exceptional fibers are treated separately in Section 5.) This implies:

COROLLARY (6.3). *There exist manifolds of arbitrarily high Heegaard genus containing no genus reducing knot.*

The definition of destabilizing knot is somewhat cumbersome, and was constructed in order to allow for the surface Σ to destabilize *in the knot exterior* (and hence in all of the surgered manifolds). In that case, we see a *new* Heegaard surface after surgery. One might ask why bother with such phenomenon, but in fact there are examples (*e.g.* certain Seifert Fibered Spaces and the Casson–Gordon examples) that show its importance: in those example new Heegaard surface appear as a result of a *new* destabilization of a surface that was already stabilized.

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2. Background

Most of our definitions (*e.g.* manifold, Heegaard surface, compression *etc.*) are standard and the reader is referred to [6], [7], [15] for discussion. We work in dimension 3, in the smooth and orientable category.

A *stabilization* of a Heegaard surface is the addition of a trivial handle to the surface. As was discussed in [12], if a knot $\gamma \subset M$ lies on a Heegaard surface a single stabilization suffices to get a Heegaard surface for the knot exterior, $M \setminus N(\gamma)$, see also Figure 1.

It is sometimes convenient to consider the collection of all Heegaard surfaces (up to isotopy) arranged as a graph, the isotopy class of each surface represented as a vertex, the vertices arranged in layers according to the genera of the surfaces, and two surfaces connected by an edge if and only if one is a single stabilization of the other. Uniqueness of stabilization implies that this graph is a forest; the Reidemeister-Singer Theorem says that this forest is a tree. For a given manifold M the tree thus obtained is called the *Heegaard Tree* of M.

Corollary 6.6 of [13] is crucial for our work here (*cf.* Theorem 0.1 of [10] and Corollary 4.2 of [12]). It asserts that, when considering fillings on a manifold with boundary torus, only finitely many such will contain a Heegaard surface onto which the core of the attached solid torus is not isotopic. Since we are interested in the case were there are infinitely many genus reductions after surgery on a single knot, this theorem allows us to consider only the case where the genus reduces and the core of the attached solid torus is isotopic onto the new Heegaard surface.

From this point on we shall only be considering surgeries and Heegaard surfaces where the core of the attached solid torus is isotopic into the Heegaard surface. When a destabilization does occur after surgery (or filling), we call it a *tame* destabilization if the core of the attached solid torus is isotopic onto the destabilized surface.

3. The Process

There is only one way in which a Heegaard surface can have a destabilizing knot. We now describe this process.

While reading the explicit description below, it is worth keeping in mind that it yields quite a few corollaries, in particular we shall see that we in fact destabilize a single surface (although $M \setminus N(\gamma)$ may contain infinitely many surfaces of minimal genus), and the same destabilization happens infinitely often. (Destabilization is a replacement of a once punctured torus by a disk, "the same destabilization" means the same once punctured torus.) Note that a core of a one handle in a compression body corresponds to infinitely many meridian disks, that is to say there are infinitely many disks in the handlebody which intersect the given knot once exactly, and therefore the above fact is not obvious.[1] There may well be other destabilizations that occur infinitely often.

Let γ be a destabilizing knot for $\Sigma \subset M$. In infinitely many different surgeries we obtain Σ', a surface obtain by destabilizing Σ once. Picking a sufficiently large filling, the core will be isotopic into Σ'. The picture is given in Figure 1.

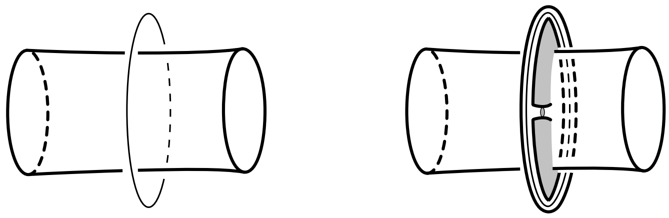

Figure 1: A Tame Destabilization

It is clear from the picture that Σ is indeed a surface for the knot exterior $M \setminus N(\gamma) = M' \setminus N(\gamma')$ and therefore for all surgeries on γ. Furthermore, whenever we have a tame destabilization, it is clear that there are infinitely many filings that yield that destabilization

[1] To clarify this further, given a core of a handle of a compression body H, *i.e.* a knot $\gamma \subset H$ that is isotopic onto ∂H and meets *some* compressing disk for H once, slide this compressing disk over any other compressing disk that is disjoint from the knot, and the resulting disk will be a new compressing disk for H meeting γ once. Their boundaries will not be isotopic in ∂H.

Conversely, given a compressing disk D in a compression body H whose boundary does not separate ∂H, take any curve $\gamma \subset \partial H$ meeting ∂D once, and push it into H. This yield a core of a one handle for which this compressing disk is a meridian.

exactly, namely those fillings were the core of the attached solid torus meets the surface framing (of Σ) once exactly.[2]

The most important observation is the following, in which we refer to the shaded disks in Figure 1:

OBSERVATION 3.1. *If Σ has a destabilizing knot then there exist two disks on opposite sides of Σ meeting minimally at exactly two points.*

For the existence of genus reducing knot we in fact get a necessary and sufficient condition:

OBSERVATION 3.2. *M has a genus reducing knot if and only if there exist $\Sigma \subset M$, a minimal genus Heegaard surface, and two disks on opposite sides of Σ, meeting minimally at two points, so that one of the two separates a solid torus from the handlebody in which it is embedded.*

To prove the the "if" direction, call the side containing the disk that chops off a solid torus the inside and the other side the outside. Note that there are infinitely many fillings after which the outside disk meets the meridian of the attached solid torus once exactly.

Observation 3.1 is all that is necessary to understand our claims about destabilizing knots.

REMARKS 3.3.
1. *Using Observation 3.1, it is easy to construct a Heegaard surface that has no destabilizing knot. Many conditions exist in the literature that imply there do not exist two disks on opposite sides of a Heegaard surface meeting twice. One such condition is Casson's rectangle condition which we shall exploit in Section 6. Another condition can be found, for instance, in Hempel's work ([9]) where he shows that for "many" Heegaard surfaces any two disks on opposite sides fill the surface. Clearly, this is even stronger than we need.*
2. *On the other hand, if Σ is a unique Heegaard surface of minimal genus for some manifold (say of genus g), and γ is a knot, isotopic onto Σ, that is not a core of a 1-handle of one of the complementary handlebodies, Σ will remain a Heegaard surface*

[2]The surface framing is not, in general, an isotopy invariant.

for any surgery of slope meeting the surface framing once exactly. On the other hand, all but finitely many surgeries meeting the surface framing more than once will yield manifolds of genus $g+1$. Therefore these manifolds have a genus reducing knot.

Note that a curve on the boundary of a handlebody is a core if and only if it intersects some compressing disk for that handlebody once exactly.

CLAIMS 3.4. *1. For all but finitely many fillings on a genus reducing knot the genus reduces by one exactly.*

2. A genus reducing knot is a destabilizing knot for some minimal genus surface.[3]

Proof. To prove both claims:

Suppose γ is a genus reducing knot in M. Let M' be a manifold obtained by large surgery on γ for which the genus went down, where by "large" we mean that the core of the attached solid torus is isotopic onto the minimal genus Heegaard surfaces of M', *i.e.* tame. (As mentioned above, by [13] all but finitely many surgeries on a given knot are tame.) Let Σ' be a minimal genus Heegaard surface for M'. The discussion above proves that, firstly, a single stabilization of Σ' yields a Heegaard surface for M. Hence the genus of M' equals that of M minus one. Secondly, Σ' will be a Heegaard surface for infinitely many surgeries on γ, those surgeries which yield a Lickorish twist (see [8]). Since we get the same Heegaard surface destabilizing after infinitely many surgeries, it is clear that a genus reducing knot is a destabilizing knot.

□

Although the phenomena genus reducing knot and destabilizing knot are very similar, it is considerably easier to study the latter as the destabilizing knot may destabilize a Heegaard surface that is already stabilization of some surface. It seems very likely that the Casson–Gordon examples and their generalizations (see [9]) form

[3] Here and throughout this paper we do not use finiteness results about Heegaard surfaces.

destabilizing knots is S^3, although S^3 has no non-stabilized positive genus Heegaard surface by Waldhausen (see [18]). Torus knots, with their lens space filling, provide examples of destabilizing knots in S^3 as well. (To prove that the examples of Casson–Gordon are destabilizing knots, one needs to show that the surfaces they obtain are not Heegaard surfaces for the knot exterior.)

A neat example is provided by small Seifert Fibered Spaces (all examples of Seifert Fibered Spaces are due to Moriah and Schultens's classification of Heegaard surfaces in totally orientable Seifert Fibered Spaces, see [11]).

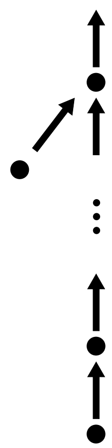

Figure 2: Trees for certain Seifert Fibered Spaces

Consider a Seifert Fibered Space over the disk with two exceptional fibers. As do all Seifert Fibered Spaces with boundary, it fibers over S^1. When the exceptional fibers are chosen carefully, the fiber (in the fiberation over S^1) has a single boundary component. As it branch covers the disk with branch set two points and multiplicity equals that of the exceptional fibers, the genus of this surface can be made arbitrarily large. Call it g.

The surface we get by tubing together two parallel copies of the fiber along the boundary is a Heegaard surface for manifolds obtained by a filling, provided the meridian of the attached solid torus meets the boundary of the fiber once. This surface has genus $2g$ and is non-stabilized. Since the core of the filling is isotopic onto this surface, after a single stabilization we get a Heegaard surface (of genus $2g+1$) for the Seifert Fibered Space over the disk we started with. This surface destabilizes; in fact, a Seifert Fibered Space over D^2 with two exceptional fibers has a unique non-stabilized Heegaard surface of genus 2.

To clarify this, the Heegaard tree of the Seifert Fibered Spaces with a horizontal Heegaard surface are shown in Figure 2. Note the "extra" leaf, which can be of arbitrarily high genus. This leaf does not exist in the Heegaard tree of a Seifert Fibered Space over the disk, where the tree in linear.

Sedgwick observed that these examples are non-Haken manifolds of genus two with non-stabilized Heegaard surfaces of arbitrarily high genus. It is unknown if there are non-Haken manifolds with infinitely many Heegaard surfaces, like the Casson–Gordon examples (which are all Haken).

4. Knots in a Ball

In this section we establish that genus reducing knots are never contained in a ball. It follows from the following lemma, which is of independent interest:

LEMMA 4.1. *Let $\gamma \subset M$ be a genus reducing knot. Then $g(M) = g(M \setminus N(\gamma))$*

Proof. Let $\gamma \subset M$ be a genus reducing knot. Let M' be a large surgery of lower genus. Let Σ' be a Heegaard surface for M' so that $g(\Sigma') < g(\Sigma)$. As discussed above (Section 3), after stabilizing Σ' once we get a Heegaard surface for $M \setminus N(\gamma)$. The lemma follows. □

THEOREM 4.2. *A genus reducing knot is not contained in a ball.*

Proof. Let γ be a knot contained in a ball. By Haken's Lemma ([5]) $g(M \setminus N(\gamma)) = g(M) + g(S^3 \setminus \widehat{\gamma})$, where $\widehat{\gamma}$ is the knot in S^3 that is obtained by capping off the ball in which γ lies with another ball. This number is at least one, so by Lemma 4.1 γ is not a genus reducing knot. □

5. Genus Reduction in Low Genus Manifolds

The sphere: S^3 has no genus reducing knot.

Lens spaces: When a lens space has a genus reduction, pick a large surgery yielding a manifold of lower genus, hence of genus 0 (*i.e.* S^3). By [12] the core of the attached solid torus is isotopic onto a minimal genus Heegaard surface, *i.e.* S^2. So the core is the unknot. Therefore the exterior of such core is a solid torus, and the knot is a core of a solid torus in a Heegaard splitting of the lens space. By Bonahon-Otal (see [1]) there are exactly two such knots, which are sometimes distinct and sometimes isotopic.

Genus 2 manifolds: If a manifold of genus 2 has a genus reduction, pick a large surgery of lower genus (*i.e.* genus one) and isotope the knot onto a Heegaard torus. The exterior of the knot is the union of two solid tori along an annulus, to which we attach a third solid torus. Such manifolds are sometimes reducible (connect sum of two lens spaces), and otherwise they are Seifert Fibered Spaces over S^2 with three exceptional fibers, and then the genus reducing knot is one of the three exceptional fibers. So the only examples are the obvious knots in the obvious manifolds.

The 3-Torus: The 3-dimensional torus T^3 has genus 3. It is homeomorphic to $T^2 \times S^1$. Remove a fiber $\{pt\} \times S^1$. There are infinitely many filling on $(T^2 \setminus \{pt\} \times S^1)$, yielding a manifold of genus two. However, T^3 can be fibered as $T^2 \times S^1$ in infinitely many ways, and in fact any primitive element of $\pi_1(T^3)$ can be realized as $\{pt\} \times S^1$ in some such fiberation. Thus T^3 has infinitely many genus reducing knots.

This example is particularly bad since T^3 has a unique Heegaard surface (up-to isotopy).

The statements above (as well as those in Section 4) can be proved using the Knot Complement Theorem (for genus reducing knots in lens spaces and for Section 4, see [4]) or the Cyclic Surgery Theorem (for genus reducing knots in genus 2 manifolds, see [3]).

We stuck to easier techniques that seem in character with the rest of this paper, and sometimes generalize to higher genus.

6. Seifert Fibered Spaces

We assume the reader is familiar with the work of Moriah-Schultens ([11]) classifying Heegaard surfaces in totally orientable Seifert Fibered Spaces. Since horizontal Heegaard surfaces play an important role in our game we shall describe them here. This generalizes the example of small Seifert Fibered Spaces given towards the end of Section 3.

Let M be a Seifert Fibered Space. Remove a fiber from M. The resulting manifold fibers over S^1, denote the projection by p. The fiber is a punctured surface, assume it has only one puncture (else the construction fails and a horizontal surface does not exist). Decompose S^1 into two intervals I_1 and I_2 meeting at their endpoints and correspondingly decompose M into $p^{-1}(I_1)$ and $p^{-1}(I_2)$. Clearly these are two handlebodies. Now perform a Dehn filling so that the meridian of the attached solid torus intersects the boundary of a fiber exactly once. Break the solid torus along two disks into two balls. Attach one of them to $p^{-1}(I_1)$ and the other to $p^{-1}(I_2)$. Note that we just attached to each handlebody a ball along a disk, hence we still get two handlebodies. Their common boundary is the horizontal Heegaard surface.

The genus of this Heegaard surface is twice the genus of the fiber (in the fiberation over S^1). This fiber branch covers the base orbifold, with multiplicity that equals that of the exceptional fibers, and so its genus goes to infinity with the multiplicities. There are only two cases when this Heegaard surface has minimal genus, and these are the two cases listed in the theorem below. Note that Seifert Fibered Spaces over S^2 with at most three exceptional fibers are slightly different than others; since they were treated in Section 5 (where we saw that they always have genus reducing knots and those are fibers), we do not repeat it here:

THEOREM 6.1. *A totally orientable Seifert Fibered Space (other than a Seifert Fibered Space over S^2 with at most three exceptional fibers)* *has a genus reducing knot unless it has a horizontal Heegaard surface and is of the following types:*

1. *Seifert Fibered Space over* S^2 *with* $2k+1,\ k>1$ *exceptional fibers of multiplicity two and one other of multiplicity* $2n+1$, $n \geq 1$*;*
2. *Seifert Fibered Space over any surface with one exceptional fiber.*

Remark 6.2. *The assumption that the Seifert Fibered Space has a horizontal Heegaard surface is by no means trivial; most Seifert Fibered Spaces that fulfill the description given by item (1) or (2) above do not have such. However, there are infinitely many Seifert Fibered Spaces of each type that do have a horizontal Heegaard surface.*

Proof. **Seifert Fibered Spaces without genus reducing knots:** At this point, most of the work has been done. Recall that Casson's rectangle condition implies any two disks meet at least four times. The spaces described in the theorem have horizontal Heegaard surface as their minimal genus Heegaard surfaces, and Sedgwick has shown ([16]) that horizontal Heegaard surfaces fulfill Casson's rectangle condition whenever the multiplicity of the last fiber attached (in case (1) the fiber of multiplicity $2n+1$, in case (2) the only exceptional fiber) is larger than the least common multiple of the other fibers (or is larger than one in case there are no other exceptional fibers). This holds here.

Observation 3.1 shows that there are two disks on opposite sides meeting twice, contradicting Casson's Rectangle Condition. Therefore the above mentioned Seifert Fibered Spaces have no genus reducing knot.

Seifert Fibered Spaces with genus reducing knots: All other Seifert Fibered Spaces have a minimal genus Heegaard surface that is vertical. We now show that these manifolds contain a genus reducing knot:

Assume first M has two or more exceptional fibers. The genus of such manifold is $2g+e-1$, where g is the genus of the base orbifold, and e is the number of exceptional fibers. On each exceptional fiber there are infinitely many surgeries after which the fiber is no longer exceptional. Those manifolds contain a vertical Heegaard surface of genus $2g + e - 2$, hence the genus of the manifold has been reduced.

If the manifold has one (resp. no) exceptional fibers, and the minimal Heegaard surface is not horizontal then thr manifold has genus $2g + 1$ (in both cases). Then there are infinitely many surgeries on the unique exceptional (resp. regular) fiber yielding a manifold of genus $2g$, with a horizontal Heegaard surface of genus $2g$, as described above. Hence it has a genus reducing knot.

□

Corollary 6.3. *There exist manifolds of arbitrarily high Heegaard genus containing no genus reducing knot.*

Remark 6.4. *Although we do not understand Heegaard surfaces of manifolds other than Seifert Fibered Spaces well enough to classify their genus reducing knots, it is worth emphasizing that Casson's Rectangle Condition, as well as other conditions, prohibit a manifold from having a genus reducing knot. It therefore makes sense to suggest that many manifolds will not have such knot, although the author does not have a precise conjecture to suggest. In that sense, Seifert Fibered Spaces with horizontal Heegaard surfaces look more like a typical hyperbolic manifold than like Seifert Fibered Spaces.*

7. Remarks on Almost Normal Surfaces

In this section we give general remarks, so we do not define many of the terms used, and treat them informally.

Almost normal surfaces have been introduced by Rubinstein to study Heegaard surfaces. Their existence has been established (see [17]), where it is shown that every strongly irreducible Heegaard surface is isotopic to an almost normal one. This holds in any triangulation. By Casson and Gordon's seminal work (see [2]) every non-stabilized Heegaard surface in a non-Haken manifold is strongly irreducible.

We begin by showing why one may expect many Heegaard surfaces to yield an almost normal surface with a tube (rather than an octagon). We then show that the existence of a genus reducing knot implies the existence of almost normal surfaces in *certain* triangulations. While we do use special triangulations, they are quite natural and have been studied by Jaco and Rubinstein extensively. Since their work shows that such triangulations are very useful, it seems helpful to know some of their properties.

The triangulations we work with are obtained by attaching a triangulated solid torus to a triangulation of the knot exterior. In order to make such triangulation useful, Jaco and Rubinstein added various assumption (in essence, to preserve "efficiency"). However, we do not assume efficiency, and the only assumption needed is that the triangulation has a unique vertex, and was obtained by a Dehn filling procedure, as was described by Jaco and Rubinstein.

By triangulating the knot exterior and then attaching a triangulated solid torus one might not get a minimal triangulation; however, it is clearly a very good way of thinking of triangulations.

Heegaard surfaces with tube Let M be a manifold containing a destabilizing knot γ. Assume further that γ is small, *i.e.* the only closed incompressible surface in $M \setminus N(\gamma)$ is $\partial N(\gamma)$. We shall consider triangulations for M, and for other surgeries on γ, obtained by attaching a triangulated solid torus to a triangulation of $M \setminus N(\gamma)$. Consider a large surgery on γ that contains a new Heegaard surface, say Σ, so that the genus of Σ is one less than the genus of $M \setminus N(\gamma)$. Let $\Sigma^* \subset M \setminus N(\gamma)$ be the (perhaps disconnected) twice punctured surface obtained by isotoping γ onto Σ and then drilling it out, in the language of [13], the almost Heegaard surface. In that paper it was shown that Σ^* is essential.

Therefore Σ^* can be made normal (in $M \setminus N(\gamma)$). After filling, $\partial\Sigma^*$ bounds an annulus in the solid torus. Whenever this annulus can be made normal, we attach that normal annulus to get a closed normal surface However, in most fillings one gets a Heegaard surface only after stabilizing Σ. As stabilization is adding a trivial little tube, these surfaces will have tubes in them, not octagons.

It is of course possible that these surfaces can be isotoped to have octagons. However note that Jaco and Sedgwick have shown that the Heegaard torus in *some* lens spaces cannot have an octagon for so-called “layered” triangulations and so it seems unreasonable to expect an almost normal surface with an octagon to appear under general circumstances.

Getting almost normal surfaces We get from the above argument that if M has a genus reducing knot then in some large filling one can find an almost normal surface of genus $g-1$, where g is the genus of M. This surface will exist in the manifold M, although it is of lower genus than the Heegaard genus of M.

Since these almost normal surfaces are not Heegaard surfaces, one can resolve them to normal surfaces. Thus we see almost normal surfaces not corresponding to Heegaard surfaces, and normal surfaces not corresponding to essential ones.

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