Rend. Istit. Mat. Univ. Trieste
Suppl. 1 Vol. XXXII, 221–256 (2001)

On Hyperbolic Type Involutions

Luisa Paoluzzi (*)

Summary. - *We give a bound on the number of hyperbolic knots which are double covered by a fixed (non hyperbolic) manifold in terms of the number of tori and of the invariants of the Seifert fibred pieces of its Jaco-Shalen-Johannson decomposition. We also investigate the problem of finding the non hyperbolic knots with the same double cover of a hyperbolic one and give several examples to illustrate the results.*

1. Introduction

There is a vast literature on the study of knots which share the same 2-fold branched cover. On one hand, one tries to understand to which extent a double branched cover determines the knot, while on the other, one tries to describe how different knots with the same double branched cover are related (see Problem 3.25 of Kirby's list [10]). For certain classes of knots these two problems are completely solved. This is the case, for instance, of 2-bridge knots [7], included the trivial one [31], and of doubles of non strongly invertible prime knots [17], which are determined by their 2-fold branched cover, of Montesinos knots, where the double cover is Seifert fibred and all the quotient knots can be reconstructed from the invariants of the fibration (see [16] and [28]), and of π-hyperbolic knots (see [19] and [14]), for which non equivalent knots with the same double cover of a given one can

(*) Author's address: Laboratoire de Topologie, Université de Bourgogne, 9 avenue Alain Savary – BP 47870, 21078 Dijon cédex, France e-mail: paoluzzi@u-bourgogne.fr
Keywords: hyperbolic knots; cyclic branched covers; orbifolds; Bonahon-Siebenmann decomposition, Seifert fibrations.
AMS Subject Classification: Primary 57M25; Secondary 57M12, 57M50, 57M60.

be detected by looking at its symmetries. Indeed, given a knot K the existence of a non equivalent knot K' with the same double cover as K seems to be strictly related to the symmetries of K on one hand ([32], [33], [19]), and on the presence of Conway spheres ([16], [30], [5]) or more in general of a non trivial Bonahon-Siebenmann decomposition of the orbifold $(\mathbf{S}^3, K_2)$, which is topologically the 3-sphere and has K as singular set of order 2, on the other [17].

In the present paper we consider the above problem for the case when K is a hyperbolic knot and the orbifold $(\mathbf{S}^3, K_2)$ has non trivial Bonahon-Siebenmann decomposition [4]. Among all simple knots, these are the only ones left to consider. We recall that a knot is hyperbolic if its complement admits a complete hyperbolic structure of finite volume. The main result of this paper is a description of how one can recover all the hyperbolic knots with the same double cover as K. We shall see that it can be rather difficult to find the non hyperbolic knots (prime but not simple) with the same double cover as K. Indeed, in Section 3, we shall produce examples of non hyperbolic knots with the same double cover of a hyperbolic one which are clearly not obtained by (generalized) Conway mutation on the hyperbolic one and whose covering involutions do not induce symmetries on any of the hyperbolic quotient knots. We shall also discuss under which conditions knots with this property exist. The examples of Section 3 show that, in general, one cannot avoid to understand the double cover in order to recover all quotient knots. This situation is then different from the π-hyperbolic case, where it is sufficient to consider successive quotients of the knot [19], [14] and from the Montesinos case where the Seifert invariants of the covering manifold can be computed from a rational tangle decomposition of the knot.

To obtain the above result, a characterization, up to equivalence, of "hyperbolic type" involutions on the 2-fold branched cover will be given in Section 2. We shall say that an involution is of *hyperbolic type* if the quotient orbifold is topologically the 3-sphere with singular set a hyperbolic knot. This characterization will allow us to give a bound on the number of hyperbolic knots sharing the same double cover. Such bound depends on the number of elements in the Bonahon-Siebenmann decomposition and on its fibred parts.

Remark that it is already known that the number of knots with a common double cover admitting a non trivial Jaco-Shalen-Johannson decomposition [8], [9] is finite [6]. Unfortunately, it seems difficult to give an estimate for the number of all quotient knots for a generic manifold.

As an application, in Section 4, we shall consider the case when the double cover has Heegaard genus 2. The case when this manifold is hyperbolic was studied in detail in [13] using the main result of [21]. Here we shall use the the characterization of all possible Jaco-Shalen-Johannson decompositions for Heegaard genus 2 manifolds given in [11] to describe the actions of hyperelliptic involutions and the existence of hyperbolic type involutions. This will allow us to understand when two hyperelliptic involutions of a Heegaard genus 2 manifold commute, thus extending, in some negative sense the main results of [21] on the commutativity of hyperelliptic involutions in the case of atoroidal manifolds.

Marco and I discussed the main ideas of this paper while I was visiting him in Sardinia, only few days before his tragic death. Writing the paper down was a humble way to keep alive the memory of a mathematician I deeply valued and, most of all, of a good friend...

"ut te postremo donarem munere mortis" (C. V. Catullus, CI)

The author wishes to thank J. Crisp and D. Lines for guiding her through the intricated fibrations of Seifert spaces, F. Bonahon for suggesting useful references and providing some of his papers and C. Bonatti for valuable discussion.

2. Hyperbolic type involutions

Let M be the 2-fold branched cover of a hyperbolic knot K in the 3-sphere $\mathbf{S}^3$ with covering involution h. M admits a Jaco-Shalen-Johannson decomposition [8], [9] along incompressible tori $T \in \mathcal{T}$. Such decomposition can be chosen to be equivariant with respect to h (see [15]) and projects to the Bonahon-Siebenmann decomposition [4] of the orbifold $(\mathbf{S}^3, K_2)$ where the tori of the decomposition project either to tori or to Conway spheres (for basic definitions about orbifolds see [24]). Recall that a Conway sphere is a sphere meeting a knot in four points and which is incompressible and ∂-incompressible in the complement of the knot. As a consequence of Thurston's hyperbolization theorem [27], [22], [23] (and of Thurston's geometrization theorem for orbifolds in the case of trivial decomposition -see [26], [25] and [2] for a proof), the pieces M_i of the decomposition of M are either Seifert fibred or admit a complete hyperbolic structure of finite volume. Recall that one can associate to each decomposition a characteristic graph whose vertices are the pieces of the decomposition, while the edges connecting two vertices represent the incompressible tori adjacent to both components.

Since K is hyperbolic, its complement in $\mathbf{S}^3$ contains no incompressible non-boundary parallel tori, so the toric 2-orbifolds of the Bonahon-Siebenmann decomposition of $(\mathbf{S}^3, K_2)$ are all Conway spheres. In particular, h fixes setwise each torus of the Jaco-Shalen-Johannson decomposition of M and $Fix(h)$ intersects each torus in exactly four points. An involution whose action on a torus is of this type will be called *standard*. Notice, moreover, that each piece M_i of the decomposition of M projects to an orbifold $\bar{M}_i$ which is topologically a 3-ball with some (perhaps none) 3-balls removed from its interior. In fact, $\bar{M}_i$ is topologically determined by the number of boundary components of M_i, and not by the specific action of h.

Let now u be any involution of M; from now on all involutions will be understood to be orientation-preserving. Up to isotopy, u can be chosen to preserve the given Jaco-Shalen-Johannson decomposition. We shall say that u is *hyperelliptic* if the quotient M/u is $\mathbf{S}^3$ and we shall say that a hyperelliptic involution is *of hyperbolic type* if the singular set $K(u)$ of the quotient orbifold M/u is a hyperbolic knot. The aim of this Section is to classify up to equivalence the hyperbolic type involutions of M. We start with

Proposition 2.1. *Suppose that M has non-trivial Jaco-Shalen-Johannson decomposition. An involution u is of hyperbolic type if and only if the quotient M_i/u is topologically homeomorphic to $\bar{M}_i$ for each i.*

Proof. Suppose that u is of hyperbolic type. As remarked above, u must fix setwise each torus of the Jaco-Shalen-Johannson decompo-

sition of M and act on it as a standard involution. This means that, for each hyperbolic type involution u, the characteristic tree of the decomposition of $(\mathbf{S}^3, K(u)_2)$ is combinatorially the same as that of M and all edges are Conway spheres. Since any tamed 2-sphere in $\mathbf{S}^3$ separates the 3-sphere into two 3-balls, the conclusion follows.
Suppose now that M_i/u is homeomorphic to $\bar{M}_i$ for each i. We have to prove that M/u is topologically $\mathbf{S}^3$ and that $K(u)$ is a hyperbolic knot. The result of identifying the M_i/u's along their boundaries clearly does not depend on the particular glueings, for the boundary components are spheres. This proves that M/u is $\mathbf{S}^3$. Notice that the fixed-point set of u is connected because M, being the 2-fold branched cover of a knot, is a $\mathbb{Z}_2$-homology sphere. This means that $K(u)$ is a knot. To prove hyperbolicity, it is sufficient to prove that $K(u)$ is simple: $K(u)$ cannot be a torus knot for M is not Seifert fibred. This is easily seen to be true since the complement of $K(u)$ does not contain non-peripheral incompressible tori. Some care must be taken when considering Seifert fibred pieces of the decomposition. In principle they can contain incompressible tori even if all their boundary components are spheres. However, these fibrations cannot be contained in $\mathbf{S}^3$.

$\square$

Remark that in Proposition 2.1 we do not need to know that M is the double cover of a hyperbolic knot. We only need to know that M is the double cover of a knot to ensure that the fixed-point set of an involution is connected and that the quotient knots of involutions acting locally as hyperbolic type ones are atoroidal. Indeed, in the case of links, the M_i's which are Seifert fibred can have base orbifold of genus different from 0 (see [16]).

Using Proposition 2.1, we want to reduce our original problem of classifying hyperbolic type involutions on M to the local problems of classifying equivalence classes of involutions on the components M_i and of extending local involutions via glueings of the components. From now on we shall consider the following stronger notion of equivalence among involutions of M. We shall say that two involutions u and v of M are *equivalent* if there exists a homeomorphism ϕ of M which acts as the identity on the characteristic tree of the decomposition of M and conjugates u to v. Notice that this assumption can only increase the number of conjugacy classes when the manifold M admits *global symmetries*, i.e. homeomorphisms which exchange some pieces of the decomposition. However, this is a very special case and not the generic one. Let u be a hyperbolic type involution and let, for each i, u_i be the induced involution on M_i. Suppose that v is another hyperbolic type involution, equivalent to u. Then, clearly, u_i is conjugate to v_i on M_i. The following Proposition gives a converse of the preceding discussion.

PROPOSITION 2.2. *For each i let τ_i be an involution of M_i such hat M_i/τ_i is homeomorphic to $\bar{M}_i$. Then there exists a hyperbolic type involution u of M such that $u_i = \tau_i$ for all i. Moreover, for each v such that v_i is conjugate to τ_i for all i, there exists an involution v', equivalent to v, such that $v'_i = \tau_i$.*

Proof. The existence of a global involution comes from the fact that each τ_i acts on the boundary tori of M_i as a standard involution. It is well known that, given an isotopy class of homeomorphisms of the torus, there exists a representative which commutes with the standard involution (see [17] for similar considerations). Such involution is of hyperbolic type because of Proposition 2.1.
Suppose now that there exist homeomorphisms ϕ_i such that $v_i = \phi_i \tau_i \phi_i^{-1}$ on M_i for all i. For each pair of distinct i, j there is at most one boundary component $T \in \mathcal{T}$ common to M_i and M_j. Denote by T_{ij} (respectively T_{ji}) the copy of T (if any) embedded in M_i (respectively M_j). The manifold M' obtained by glueing the pieces M_i along their common boundary components via the glueing diffeomorphisms $g_{ij} := \phi_j^{-1}\phi_i : T_{ij} \longrightarrow T_{ji}$ is diffeomorphic to M by the diffeomorphism Φ defined as ϕ_i on M_i. The involution $v' = \Phi^{-1} v \Phi$ is equivalent to v by construction and its restriction to M_i is equal to τ_i.

□

Proposition 2.2 says that any involution of M of hyperbolic type can be reconstructed, up to equivalence, by extending, in all possible ways, arbitrarily chosen local representatives of conjugacy classes of all involutions, satisfying the requirements of Proposition 2.1. The possible ways to extend a family of local involutions are determined by all the glueings along the boundary tori of the components M_i, which give a manifold homeomorphic to M. Clearly all glueings which are isotopies between T_{ij} and T_{ji} give manifolds homeomorphic to M, however, a priori, there might be other glueings satisfying this property. This is not a generic case for a manifold: it admits, in fact, a *local symmetry* Φ, i.e. a diffeomorphism acting as the identity on the characteristic tree of the decomposition, but whose restriction to some pieces of the decomposition is not isotopic to the identity. Assume that, for a pair of distinct indices i, j, g_{ij} is not isotopic to the identity on T. Then at least one between ϕ_i and ϕ_j is not isotopic to the identity on T. Suppose ϕ_i is not isotopic to the identity on T. If M_i is hyperbolic, we shall say that v_i, τ_i are *conjugate* only if ϕ_i can be chosen to be isotopic to the identity on at least one boundary component. Remark that in this case ϕ_i is isotopic to the identity on all boundary components of M_i. In the case when M_i is Seifert fibred, one has to use the following facts: up to multiplication with τ_i, one can assume ϕ_i to preserve the orientation of fibres; the base orbifold of M_i is orientable of genus 0 since M_i admits a quotient which is topologically $\bar{M}_i$ and M is the double cover of a knot (and not a link with more than one component) [16]; up to isotopy, the group of diffeomorphisms of M_i splits as the semidirect product of a vertical subgroup acting trivially on the base space (and generated by Dehn twists along fibred annuli) and the group of diffeomorphisms of the base orbifold [12]; up to isotopy, τ_i commutes with the vertical subgroup (see again [12, Proposition 3.5.8.c]) since one can choose as generating set for the group of vertical diffeomorphisms the Dehn twists along the annuli containing the fixed-point set of τ_i; for any diffeomorphism of M_i, preserving the orientation of the fibres and inducing a diffeomorphism of some boundary component which is not isotopic to the identity, there is a vertical diffeomorphism with the same action on the boundary, up to isotopy (indeed, up to isotopy, vertical diffeomorphisms and standard involutions are the only possible diffeomorphisms of a torus which preserve a given fibration). To conclude, let λ be the vertical diffeomorphism of M_i acting as ϕ_i on the boundary. One has $v_i = \phi_i \tau_i \phi_i^{-1} = \phi_i \lambda^{-1} \tau_i \lambda \phi_i^{-1}$ where now $\phi_i \lambda^{-1}$ acts by isotopies on the boundary. We can then assume that the glueings are isotopies.

Remark 2.3. *The glueings g_{ij} commute with v' on T. Indeed, one* *has* $v'g_{ij}v' = \phi_j^{-1}v^2\phi_i = g_{ij}$ *since* v *is well-defined on* T.

We now want to estimate the number of different glueings that we can have along a boundary component.

PROPOSITION 2.4. *Let* u *be a hyperbolic type involution of* M *and let* $T \in \mathcal{T}$. *Let* X_i, $i = 1, 2$, *be the two components obtained by cutting* M *along* T. *Then there are at most four non equivalent hyperbolic type involutions of* M *(included* u*), whose restrictions to* X_i, $i = 1, 2$, *coincide with the restrictions of* u.

Proof. We need the following result whose proof can be found in [29, page 8, Theorem 2].

PROPOSITION 2.5 (TOLLEFSON). *Let* $M = X_1 \cup X_2$ *be a* 3*-manifold such that* $X_1 \cap X_2$ *is a* 2*-torus* T. *Let* u *be an involution of* M *which fixes* T *setwise and* g *be an isotopy of* T *that commutes with the restriction of* u *to* T. *We can define a manifold* $M' = X_1 \cup_g X_2$ *diffeomorphic to* M *and an involution* u_g *acting on it such that its restriction to* X_i, $i = 1, 2$ *is equal to the restriction of* u *to* X_i, $i = 1, 2$.
Suppose that $Fix(u) \cap T$ *is* 0*-dimensional. If* g *fixes one point of* $Fix(u)$ *then* u *and* u_g *are equivalent.*

Let g be a glueing along T which yields M. As we have seen, g is isotopic to the identity and, by Remark 2.3, commutes with u on T. In particular T permutes the four points of $Fix(u)$. By Proposition 2.5, if we want u_g not to be equivalent to u, g must freely permute the four points. Since there are three such permutations, the conclusion follows.

□

The four possible glueings are illustrated in Figure 1.

We are now left to consider the conjugacy classes of involutions with quotient $\bar{M}_i$ on each M_i. In particular we would like to bound the cardinality of $\Sigma_i := \{[\tau_i] \mid M_i/\tau_i \cong \bar{M}_i\}$ where $[\tau_i]$ denotes the conjugacy class of τ_i. We distinguish two cases according to the fact that M_i is hyperbolic or Seifert fibred.

PROPOSITION 2.6. *If* M_i *is hyperbolic,* Σ_i *contains at most four elements.*

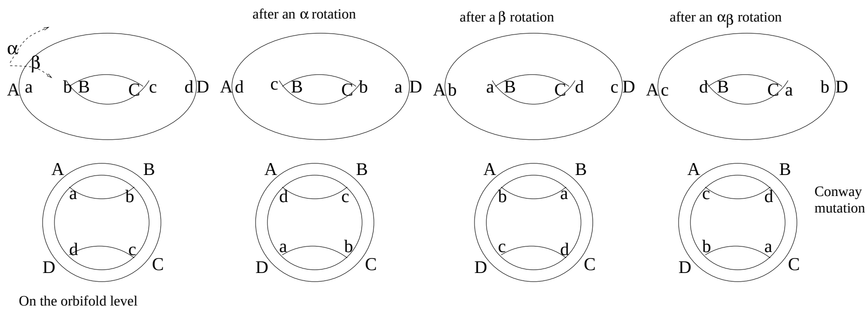


Figure 1.

Proof. Let $[\tau_i]$ be a fixed element of Σ_i. The involution τ_i has nonempty fixed-point set which, in particular, intersects each boundary component of M_i in four points: in particular it induces a standard involution on each boundary component. By Thurston's orbifold geometrization theorem, τ_i is equivalent to an orientation-preserving isometry. Let T be a fixed toric boundary component of M_i. By Mostow's rigidity theorem (see [1] for basic results in hyperbolic geometry) the number of elements of Σ_i is bounded by the number of conjugacy classes of involutions in the orientation-preserving isometry group of M_i which induce a standard involution on T. The group of isometries of M_i which leaves invariant T is a finite subgroup G of $\mathbb{Z}_n \ltimes (\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z})$ where n is 6 or 4. By hypothesis the intersection of G with the group $\mathbb{Z}_n$ contains a standard involution which acts on the normal subgroup of *free rotations* of T by sending each element to its inverse. A subgroup of this kind contains at most four conjugacy classes of involutions (more precisely, it can contain one, two or four classes) which act as standard involutions; a fortiori there are at most four conjugacy classes of involutions of M_i, inducing a standard involution on T. Remark that the elements of $\mathbb{Z}_n$ are the only finite order elements in $PSL(2, \mathbb{Z})$. Notice that if the intersection of G with $\mathbb{Z}_n$ is of order 4 or 6 then the non trivial free rotations of order 2 in G are either three or none. If there are no non trivial free rotations of order 2 or if the intersection of G with $\mathbb{Z}_n$ is of order 6, then there is only one conjugacy class of standard involutions, else there are two conjugacy classes. Recall, however, that we are only interested in the conjugacy classes of involutions inside the maximal subgroup G' of G whose intersection with $\mathbb{Z}_n$ is exactly $\mathbb{Z}_2$.

□

In Figure 2 we represent the action of the four different conjugacy classes on a toric component. Notice that the presence of two (respectively four) conjugacy classes imply the existence of one (respectively three) non trivial free rotations of order 2.

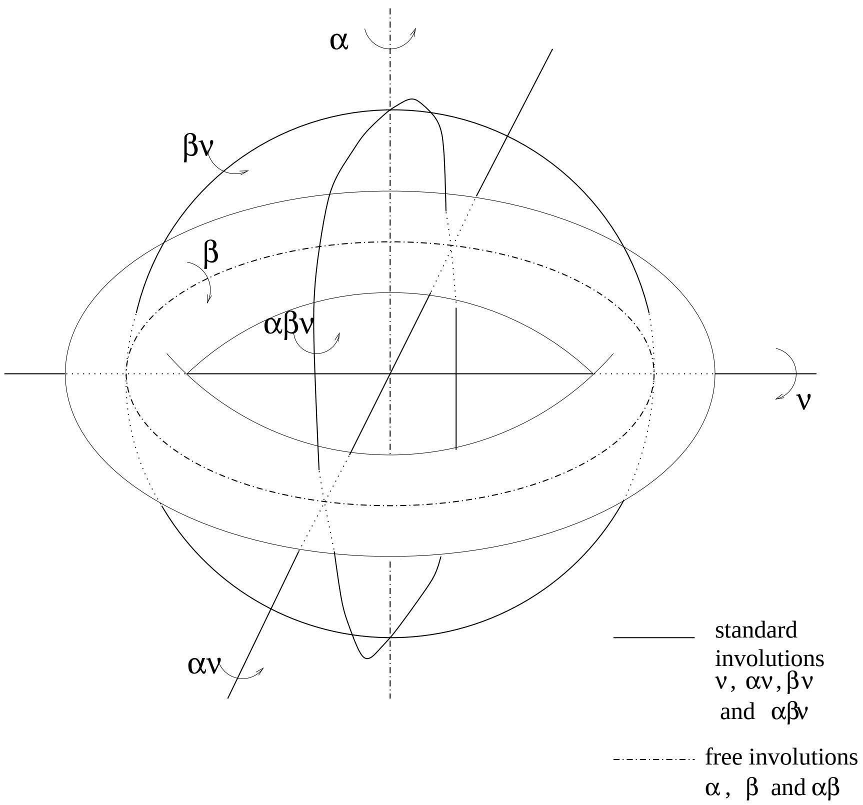
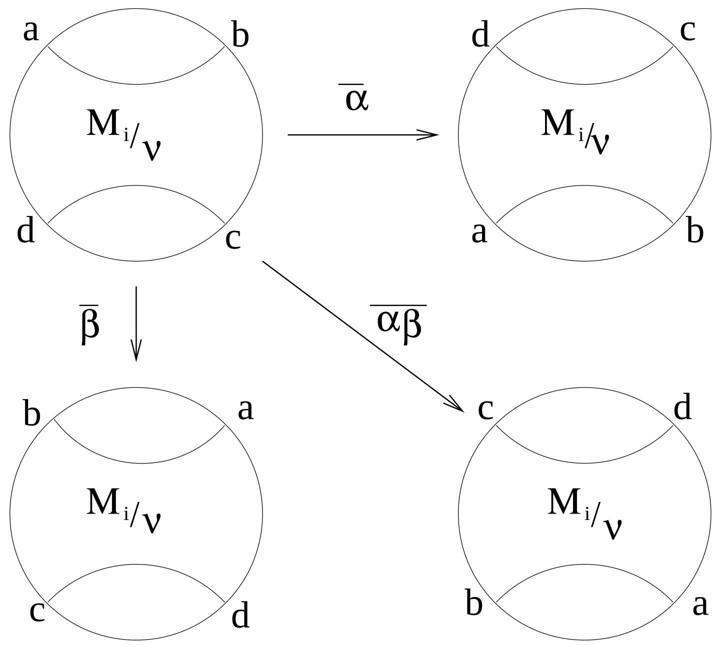


Figure 2.

Remark 2.7. *In general, the number of glueings yielding non equivalent involutions can decrease due to certain symmetries of M_i/u_i and of M/u. For instance, if M_i has a unique boundary component and if there are two (respectively four) conjugacy classes of standard involutions, then the number of possible glueings is only two (respec-*

tively one) because of the symmetries coming from the free rotations of order 2 *which commute with the glueing (compare Figure 3).*



Symmetries induced on M_i/ν by the free involutions α, β and $\alpha\beta$

Figure 3.

We now consider the Seifert fibred case. In this case the number of conjugacy classes can be arbitrarily large. Notice that by [29] each involution of M_i acts fibre-preservingly. The classification of non-equivalent conjugacy classes of involutions such that the quotient is topologically $\bar{M}_i$ coincides with the classification of Montesinos tangles. This classification, on its turn, its analogous to the classification of Montesinos links. More precisely, the quotient M_i/u_i has the structure of a Seifert fibred orbifold [3]; its base space has underlying topological space the 2-disc D. Along the boundary of D there are corner reflectors of angles $\pi/\alpha_1, ..., \pi/\alpha_r$ corresponding to the rational tangles associated to singular fibres and disjoint intervals $I_1, ..., I_s$ corresponding to the boundary components. The base space depends on the order in which we meet corner reflectors and intervals along ∂D, up to cyclic permutation and reversal of order, while M_i does not. Base spaces are in one-to-one correspondence with conjugacy classes of involutions. We thus have:

PROPOSITION 2.8. *If M_i is Seifert fibred, Σ_i contains at most $\nu_i := (m_i - 1)!/2$ elements, where $m_i := r_i + s_i$ is the sum of the number of boundary components s_i and the number of exceptional fibres r_i of M_i.*

Proof. The proof is a straightforward consequence of the above discussion. Here we just want to stress that $m_i \geq 3$ since all boundary components are incompressible. Indeed, if $s_i = 1$, then there must be at least two exceptional fibres for the boundary component not to be compressible while if $s_i = 2$ then there must be at least one exceptional fibre for the two boundary components not to be parallel. □

REMARK 2.9. *In the Seifert case we have only half the number of expected glueings. Indeed, a Seifert piece always admits a natural $\mathbf{S}^1$-action by rotation along fibres, which fixes setwise each boundary component. Let ρ be the rotation of order 2 along the fibres and g the isotopy induced on T. Up to isotopy, the map ρ can be chosen in such a way that g commutes with the restriction of u to T. Since ρ commutes with u on T it follows that u and u_g are equivalent (compare Figure 4).*

REMARK 2.10. *Let M_i be a hyperbolic piece of the decomposition of M and let τ be the restriction of a hyperbolic type involution to $M - M_i$. Let s_i be the number of boundary components of M_i. By the proof of Proposition 2.6, τ can be extended in at most $4^{s_i}|\Sigma_i|$ possible ways to M, since there are four possible glueings along each boundary component. More precisely, if τ_i is the representative of an element of Σ_i there are at most $2^{2s_i - t}$ hyperbolic type involutions u such that $u_i = \tau_i$ and $u_{|(M-M_i)} = \tau$, where t is the number of non trivial involutions of M_i/τ_i which are induced by involutions of M. For instance, if M_i has a unique boundary component, then τ extends in at most four ways to M, independently of the number of elements in Σ_i (compare Remark 2.7).*

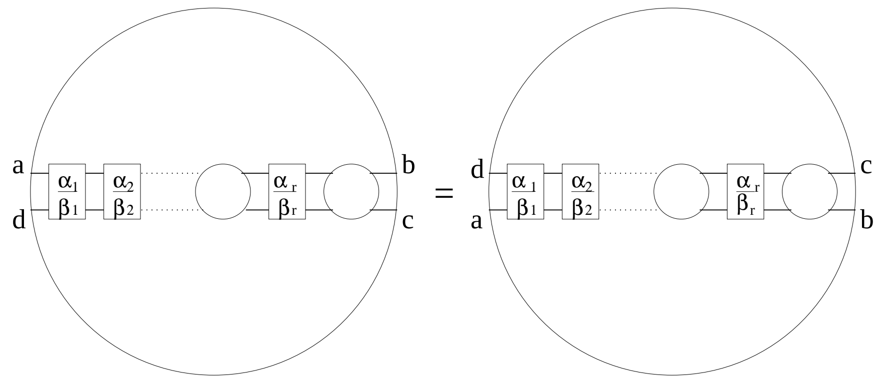


Figure 4.

We can now summarize the above results in the following:

THEOREM 2.11. *Let M be the double branched cover of a hyperbolic knot in the* 3*-sphere and assume that its Jaco-Shalen-Johannson decomposition is non-trivial. Let s be the number of tori and m the number of hyperbolic pieces of the decomposition. Then M is the double branched cover of at most $2^{3m+s-1}\prod_j \nu_j$ hyperbolic knots. Here ν_j is defined as in Proposition 2.8 and j varies over all the $s+1-m$ Seifert fibred pieces of the decomposition.*

The given estimate is obviously not best possible, as discussed in the above Remarks. More precise case by case estimates can be done using the above Propositions. As an example, in Section 4 we shall treat the case when M has Heegaard genus 2 (see also the examples of Section 3).

COROLLARY 2.12. *Let u and v be two distinct hyperbolic type involutions of M. They commute if and only if $u_i \neq v_i$ but $[u_i, v_i] = 1$ for all i. In particular, if there exist two distinct hyperbolic type involutions which commute on M, then all the pieces of the Jaco-Shalen-Johannson decomposition of M are hyperbolic.*

Proof. The two involutions commute if and only if their product has order 2. In particular uv cannot act as a Dehn twist along any of the tori of the decomposition. This means that the product cannot be trivial on -say- M_i and non trivial on an adjacent component M_j. $\square$

3. Hyperelliptic involutions

In this Section we shall try to understand how one can reconstruct the (hyperbolic) knots K' having the same 2-fold branched cover as K, simply by looking at K. The description for hyperbolic K''s is in fact given by the results of Section 2. The first step is to find the Bonahon-Siebenmann decomposition of $(\mathbf{S}^3, K_2)$. Each piece $\bar{M}_i$ of the decomposition is double covered by exactly one piece M_i of the Jaco-Shalen-Johannson decomposition of the double cover M of K. The next step is to find the orbifolds M_i/τ_i where the τ_i's are involutions satisfying the requirements of Proposition 2.1 and are taken exactly one from each conjugacy class. If M_i is Seifert fibred, all the possible quotients are easily obtained once we know the invariants of M_i (see Proposition 2.8) or equivalently a rational tangle decomposition for $K \cap \bar{M}_i$. If $\bar{M}_i$ is a hyperbolic orbifold it is sufficient to consider its $\mathbb{Z}_2$ subgroups of isometries with non-empty fixed-point set. If there are no isometries of this type then $\bar{M}_i$ is the unique possible quotient of M_i; if there is one (respectively there are two) then there are two (respectively four) conjugacy classes of standard involutions on M_i. In this second situation, to recover all quotient orbifolds, it is sufficient to construct the $\mathbb{Z}_2$ (respectively $\mathbb{Z}_2 \oplus \mathbb{Z}_2$) common quotient $\bar{M}_i/\mathbb{Z}_2$ (respectively $\bar{M}_i/\mathbb{Z}_2 \oplus \mathbb{Z}_2$) and all its possible $\mathbb{Z}_2$ (respectively $\mathbb{Z}_2 \oplus \mathbb{Z}_2$) lifts checking whether they are topologically $\bar{M}_i$. The final step is now to reglue the pieces together in all possible ways (compare Figures 1, 3 and 4).

In the case of non hyperbolic knots the description can be rather difficult. Indeed, one cannot give a reasonable characterization of hyperelliptic involutions, since in general it is difficult to decide whether a 3-manifold is $\mathbf{S}^3$. Notice, moreover that in this case, we cannot in general discard the possibility for a glueing to be non isotopic to the identity. However, certain non hyperbolic knots with the same double cover as K can be reconstructed easily.

We need some definitions first. We shall call *symmetry of a knot* K any finite order diffeomorphism of the pair $(\mathbf{S}^3, K)$, preserving the orientation of $\mathbf{S}^3$. In particular a symmetry with non-empty fixed-point set and of order n is called *n-periodic symmetry* if its fixed point set does not intersect K and *strong inversion* otherwise (in this case the symmetry has order 2 and its fixed-point set intersects

K in exactly two points). Notice that a strong inversion induces a standard involution on the complement of the knot. Finally, if the symmetry (and all its non-trivial powers) has no fixed points then it will be called *free*. Remark that these are the only possibilities for symmetries of order 2, while in general there can be symmetries which do not have fixed points but some of whose non-trivial powers are periodic. For other basic definitions in knot theory and for standard notation of knots and links up to ten crossings we refer to [20].

Proposition 3.1. *Let M be the double cover of a hyperbolic knot and let u be an involution of M with the property that M/u is the 3-sphere and that $Fix(u)$ maps to a non-hyperbolic knot $K(u)$. Up to isotopy, u preserves a given Jaco-Shalen-Johannson decomposition for M. Assume that in each component of the decomposition which is fixed by u there is a boundary component which is setwise fixed and that u acts as a free rotation on all invariant tori of the decomposition. Then there exists a hyperbolic knot K, double covered by M, admitting a 2-periodic symmetry or a strong inversion induced by u.*

Proof. It is sufficient to construct a hyperbolic type involution v commuting with u. All we need to do is to specify the action of v on the different pieces of the decomposition (see Proposition 2.2). For each pair of components M_i and M_j which are exchanged by u, choose v_i to be a representative of any element in Σ_i and let $v_j := uv_iu$. Assume now that M_i is a hyperbolic piece fixed by u. Let T be a fixed boundary component. As usual we can assume that u_i is an isometry of M_i acting as an order 2 rotation on T. Any isometry representing an element of Σ_i acts as a standard involution on T and thus commutes with u_i on T and consequently on M_i. This means that on fixed hyperbolic pieces we can choose v_i to be a representative of any element in Σ_i. If M_i is Seifert fibred, then we distinguish two cases. Recall that in our situation, up to isotopy, u_i preserves the fibration of M_i. If u_i induces the identity on the base space, then u_i is a translation along the fibres (the order 2 element of the $\mathbf{S}^1$-action); indeed it must preserve the orientation of the fibres since it acts as a rotation on at least one boundary component. In this case, u_i commutes with the restriction of any hyperbolic type involution. If u_i does not induce the identity then it must be an involution of a surface of genus 0, sending singular fibres to singular fibres of the same type. One can understand the action by embedding the surface on the 2-sphere: the action of u_i is then a rotation with two fixed points at least one of which is outside the surface. The boundary component surrounding such fixed point, corresponds to a fixed boundary torus. The two situations are shown in Figure 5. It is now clear that one can find a hyperbolic type involution whose restriction v_i commutes with u_i. The reflection axis of v_i on the base of M_i is drawn again in Figure 5. Recall that u_i, being of order 2, cannot induce Dehn twists along annuli. This finishes the proof of the Proposition.

□

Proposition 3.1 shows that certain non simple knots having the same double cover of a hyperbolic one K, can be recovered by considering all the hyperbolic knots with the same double cover as K and looking at their symmetries just like it was done in [32], [33] and in [14]. Unfortunately, not all non simple knots can be recovered this way. We have:

Theorem 3.2. *There exist double branched covers of hyperbolic knots admitting hyperelliptic involutions not of hyperbolic type which do not induce symmetries on any of the hyperbolic quotient knots.*

Proof. We shall prove the Theorem by means of two examples in which exactly one of the hypotheses of Proposition 3.1 is not satisfied thus showing that Proposition 3.1 is best possible. The examples will also illustrate some applications of the results of Section 2. Their construction is inspired by [17]. We start with a hyperelliptic involution which does not act as a rotation on a setwise fixed torus of the decomposition.
We start by constructing the double cover M. In fact we shall give the different pieces of its Jaco-Shalen-Johannson decomposition and describe how they are glued together. The first piece M_0 is the complement of the three component daisy chain 6^3_1 shown in Figure 6.

Claim 3.3. *M_0 is hyperbolic.*

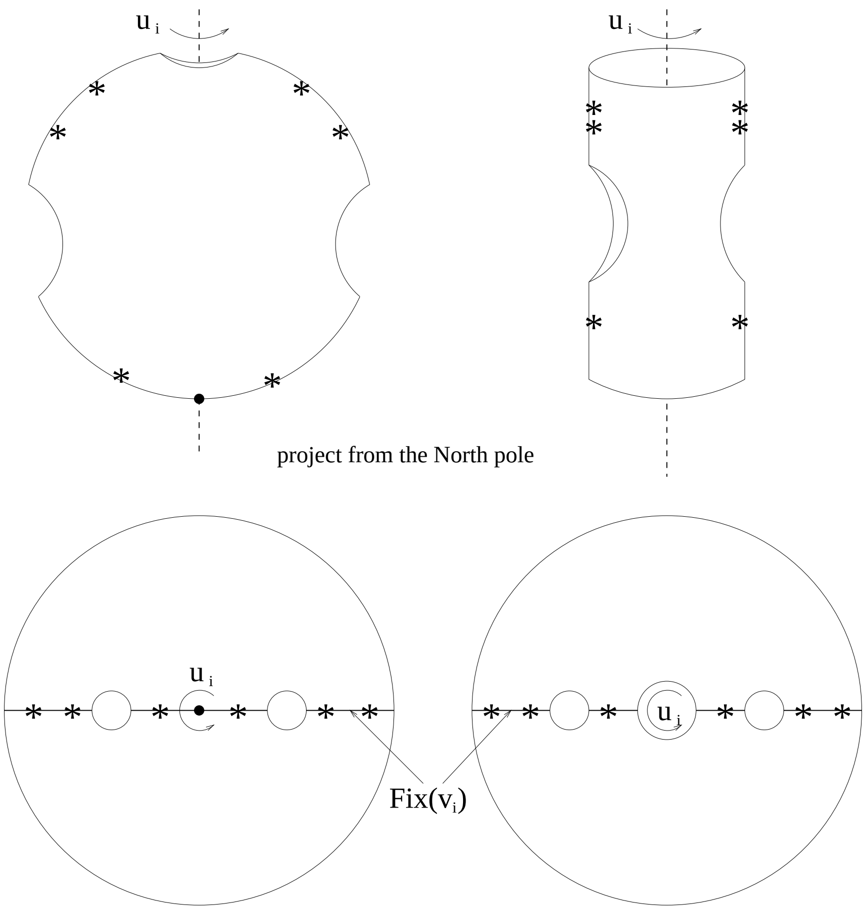


Figure 5.

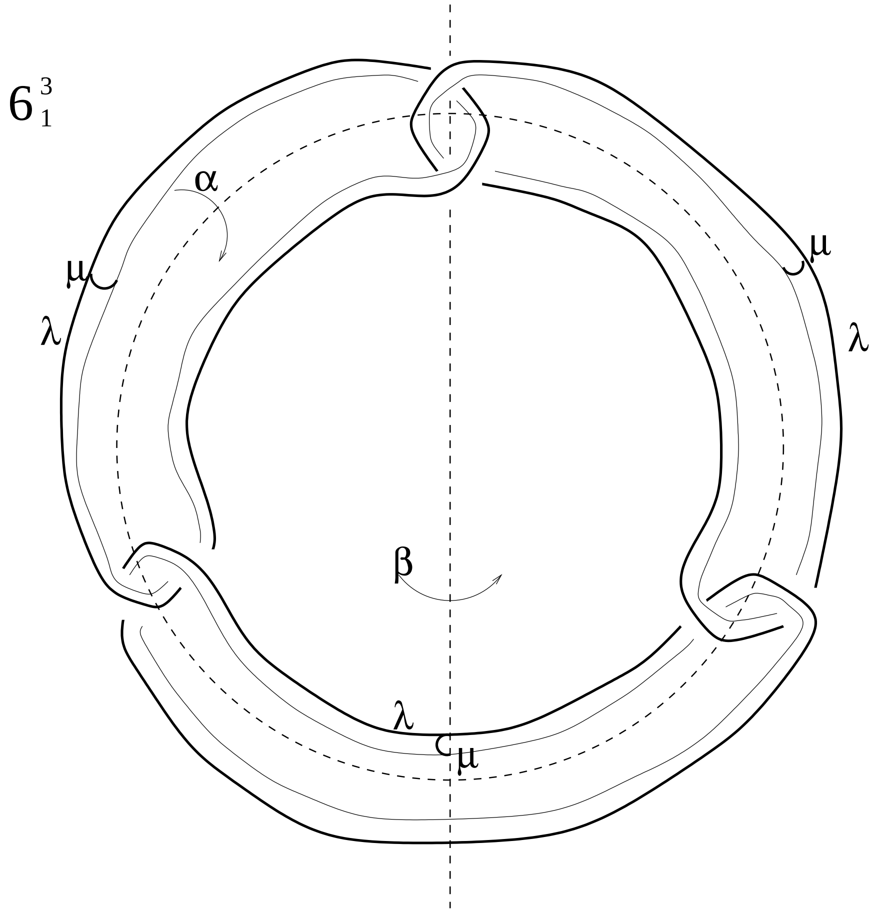


Figure 6.

Proof. To prove this it is sufficient to prove that the three component link 6^3_1 is hyperbolic or, equivalently (because of Thurston's orbifold geometrization theorem), $2\pi/n$ hyperbolic for some n. We choose $n = 3$ and we consider the orbifold whose underlying topological space is $\mathbf{S}^3$ and whose singular set of order 3 is 6^3_1. This orbifold admits a symmetry of order 3 with non-empty fixed point set exchanging the three components of the singular set. The quotient orbifold is topologically $\mathbf{S}^3$ with singular set of order 3 the Whitehead link 5^2_1 which is hyperbolic and $2\pi/3$-hyperbolic. This implies that the orbifold $(\mathbf{S}^3, (6^3_1)_3)$ is a hyperbolic orbifold and the Claim follows.

□

Hyperbolicity of 6^3_1 can also be checked using J. Week's SnapPea, which also computes the symmetry group of the link (it coincides with the group of orientation-preserving isometries of the manifold). In this case, the group of orientation-preserving isometries is dihedral of order 12. Its subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ of index 3 consists of one non trivial isometry α contained in the normal cyclic subgroup of order 6 which fixes setwise each component and acts as a strong inversion on each of them, of an isometry β which acts as a strong inversion on one component and exchanges the other two and of an isometry $\alpha\beta$ which act as a 2-periodic symmetry on one component and exchanges the other two. These three isometries are representatives of the three conjugacy classes of involutions in $\mathbf{D}_6$. Their fixed-point sets are shown in Figure 6 where also a longitude-meridian system for each component is drawn. Notice that the involutions preserve the longitude-meridian systems (up to isotopy of the meridians). On the three boundary components of M_0 glue three copies of M_1, the exterior of the 9_{49} knot: it is hyperbolic (and even π-hyperbolic) and admits, up to conjugation, a unique symmetry of order 2 which is a strong inversion. The group of orientation-preserving isometries of the exterior of 9_{49} is a dihedral group of order 6. The knot together with its strong inversion τ and longitude-meridian system is shown in Figure 7.

To obtain M use the same glueing on the three boundary components: identify the longitude of 9_{49} with the meridian of 6^3_1 and the meridian of 9_{49} with the longitude minus the meridian of 6^3_1. This identification is clearly compatible with the involution β. We thus have sixty-four possibly non-equivalent hyperbolic type involutions of M which restrict necessarily to α on M_0 and to τ on the three copies of M_1. However M admits a homeomorphism of order 3 which extends the isometry of order 3 of M_0 and three homeomorphisms of order 2 which fix exactly one M_1 each and exchange the other two. So there are three non trivial involutions of M_0/α induced by involutions of M thus we have only eight possibly non-equivalent hyperbolic type involutions (see Remark 2.10). In Figure 8 we show the quotients M_0/α, M_0/β and M_1/τ, while in Figure 9 we give the

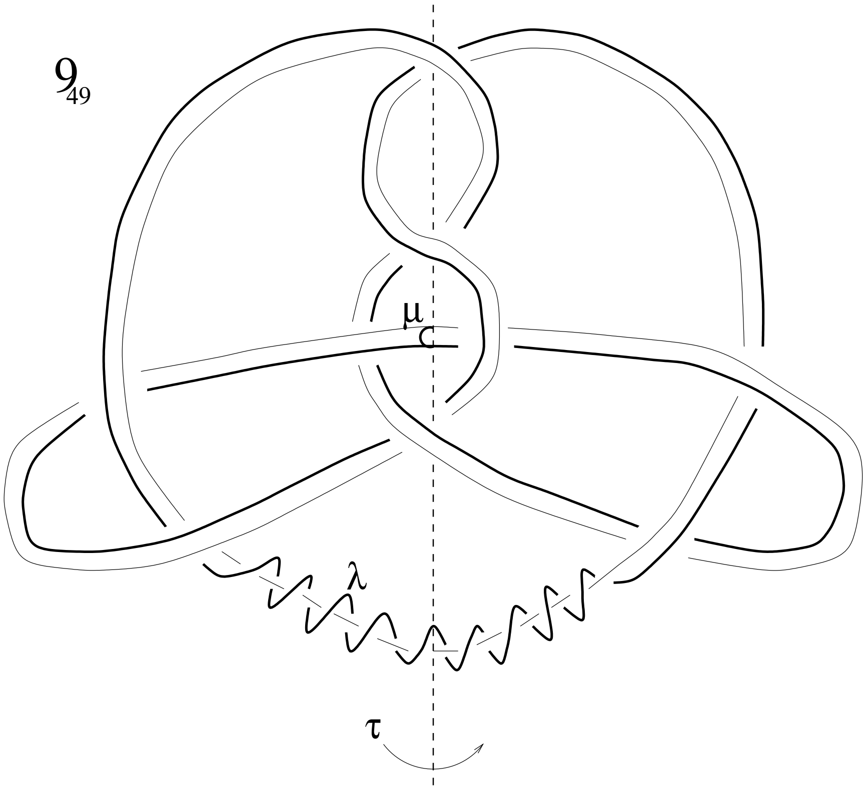


Figure 7.

eight hyperbolic quotient knots.

Remark now that it is easy to construct an involution of M whose restriction on M_0 is β. Necessarily its restriction on the fixed copy of M_1 is τ: in fact there are two such involutions v, w. These involutions are hyperelliptic (we are plugging a solid torus inside a knot complement), not of hyperbolic type and they do not commute with any of the hyperbolic type involutions. Indeed, consider any hyperbolic type involution u and one of the hyperelliptic involutions, say v. The product uv is not an involution since it acts as a Dehn twist along the setwise fixed torus: the restriction of uv is the identity on the M_1 while acts as a rotation on the fixed boundary component of M_0. We thus obtain that u and v do not commute. In Figure 10 we give the two non hyperbolic quotient knots.

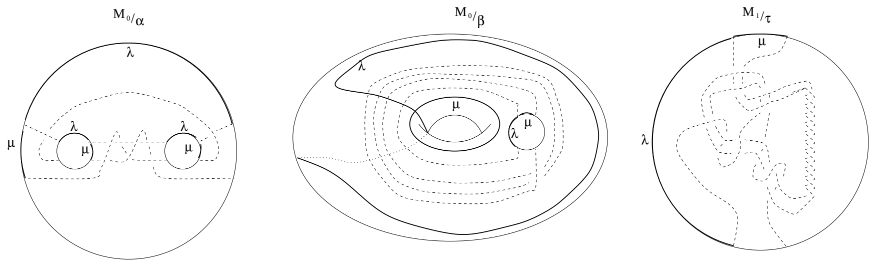


Figure 8.

To conclude, notice that the involution $\alpha\beta$ cannot extend to any global involution of M, for there are no involutions of M_1 which act as rotations of the boundary component. Before passing to the following example remark that we could have replaced 6_1^3 with any hyperbolic link admitting an involution which acts as a strong inversion on each component and an involution which acts as a strong inversion on all but two components which are exchanged. Moreover we need that the quotient of the two exchanged components by the action of the second involution is the trivial knot. Similarly 9_{49} can be replaced by any hyperbolic knot admitting a unique involution which is a strong inversion. Also, we could have chosen a different glueing (we aimed to obtain the simplest possible knots). For a similar example, see Section 4, Case (iv) and Figure 18.

Remark 3.4. *One must be careful not to choose glueings in such a way that the fixed-point sets of involutions are not connected.*

In this second example we shall consider a hyperelliptic involution which does not fix setwise any of the tori of the Jaco-Shalen-Johannson decomposition. As before we describe the different pieces of the decomposition of M'. The first piece M_0' is the complement of the two component link L pictured in Figure 11, where a longitude-meridian system is also shown. Each component of the link is the figure-eight knot 4_1.

As it is clear from the Figure, the link admits two involutions, γ, δ which act as strong inversions on each component, an involution η exchanging the two components and conjugating γ to δ and finally an involution $\gamma\delta$ which acts as a 2-periodic symmetry on each component. The quotient orbifolds M_0'/γ and M_0'/η are represented in

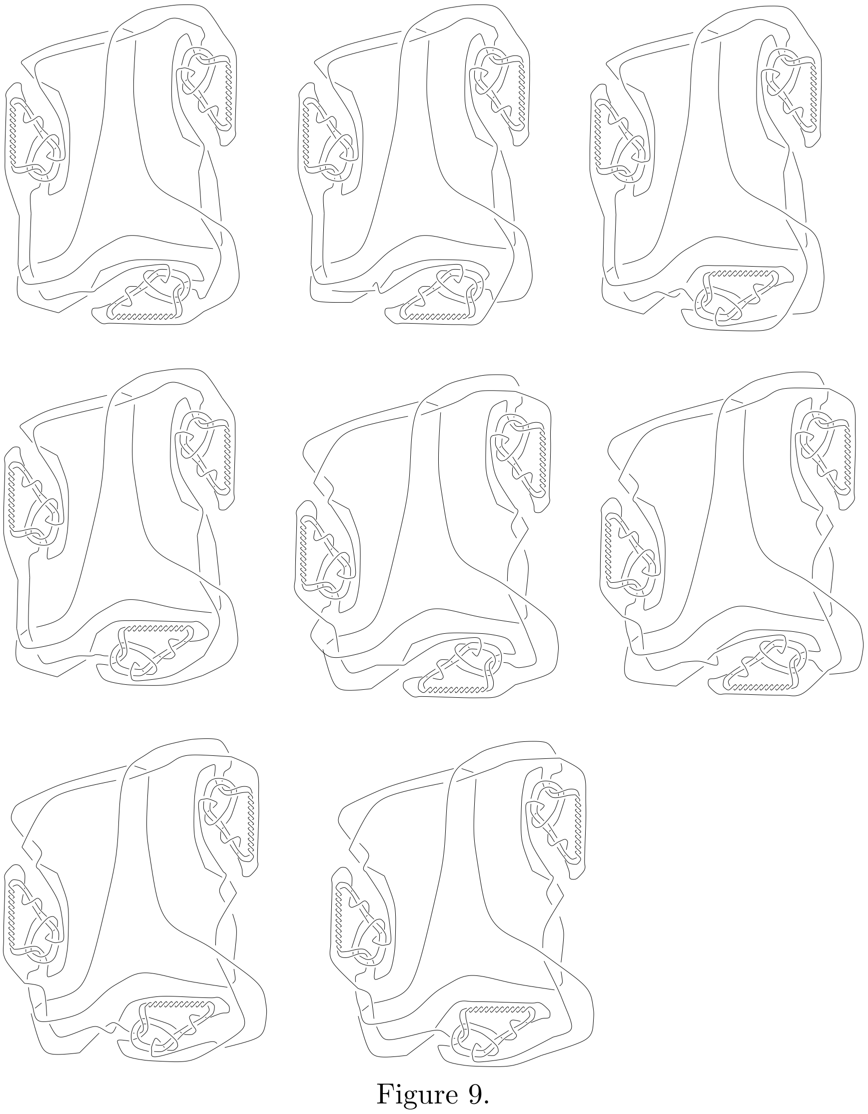


Figure 9.

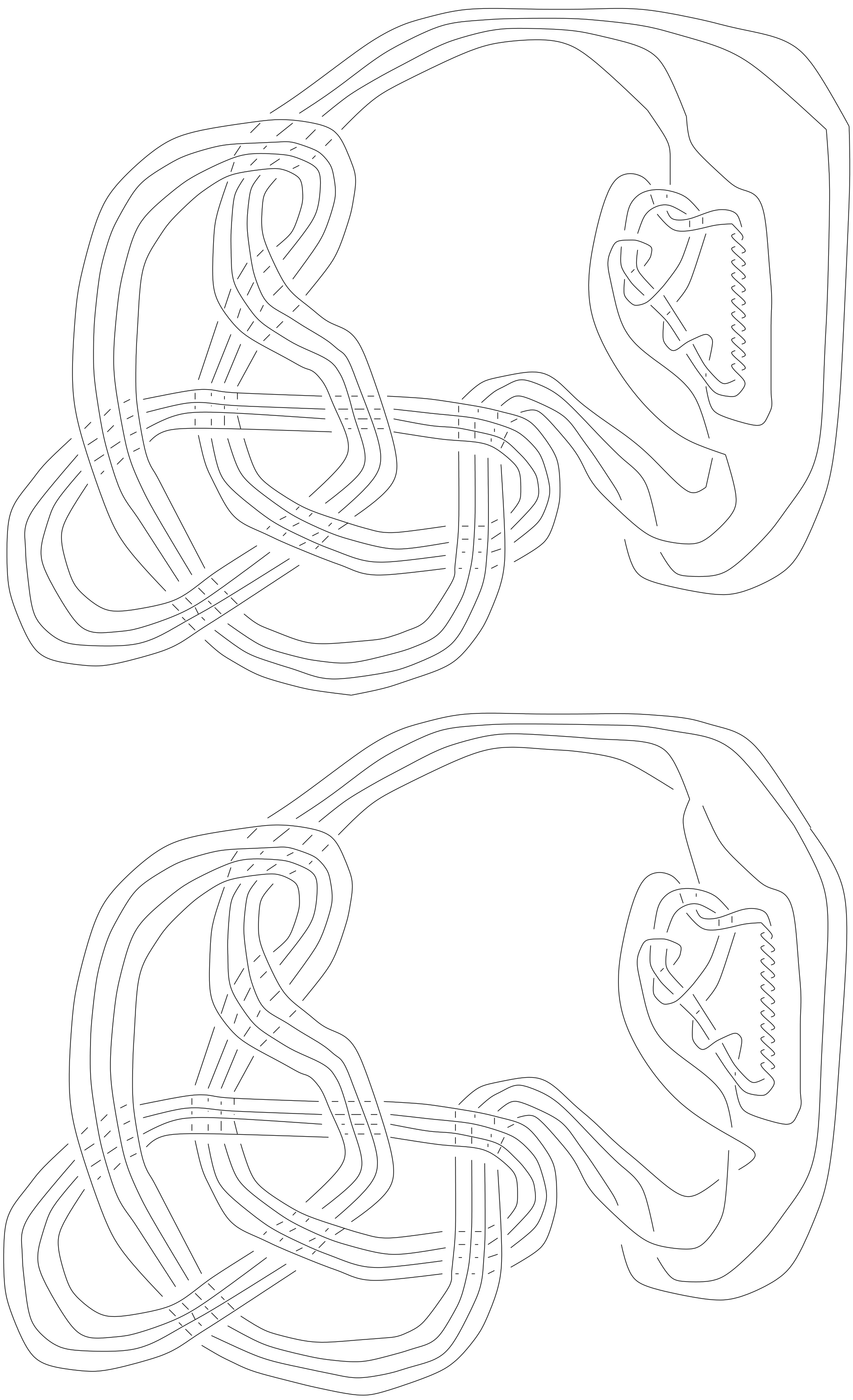


Figure 10.

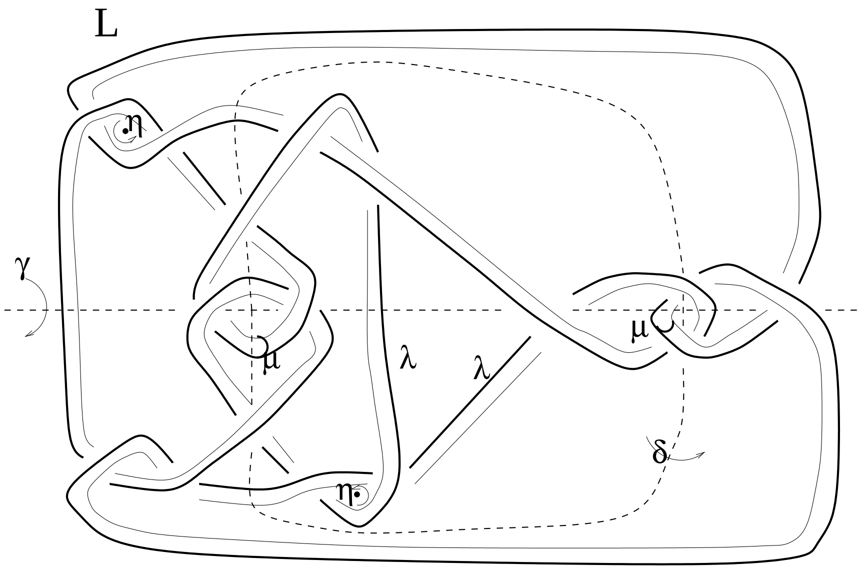


Figure 11.

Figure 12.

CLAIM 3.5. *M_0' is hyperbolic.*

Proof. It is sufficient to show that L is non-split, simple and not a torus link. Clearly L is not split since the linking number of its two components is not 0 and it is not a torus link since its two components are hyperbolic. Consider the link L' which consists of the image of L and of $Fix(\eta)$ in M_0'/η: it is a Montesinos link with three tangles, thus it contains no incompressible tori which are not boundary parallel and no Conway spheres. This means that L is simple and the conclusion follows.

□

The group of orientation preserving isometries of M_0' is dihedral of order 8 and is generated by γ, δ and η, and γ, η and $\gamma\delta$ are representatives of the three conjugacy classes of involutions of $\mathbf{D}_4$. Notice that these involutions preserve the longitude-meridian system. Along the two boundary components of M_0' we shall glue two copies of $M_1' = M_1$, the complement of the knot 9_{49}. We identify the longitude of 9_{49} with the meridian of each boundary component of

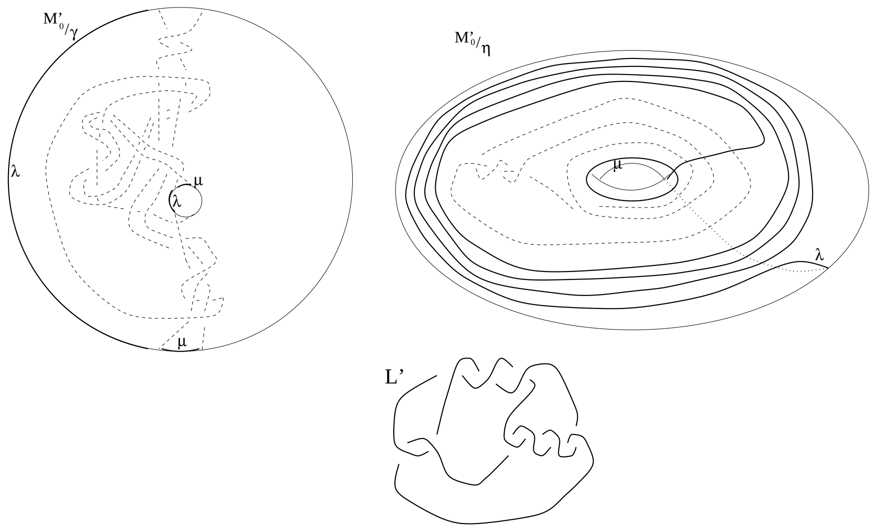


Figure 12.

M_0' and the meridian of 9_{49} with the longitude minus four times the meridian of each boundary component of M_0'. Notice that all involutions of M_0' extend to involutions of M', however η does not induce an involution of M_0'/γ, while δ does, thus there are at most eight non equivalent hyperbolic type involutions on M' and each of them restricts to γ on M_0' and to τ on the two copies of M_1. The possible hyperbolic knots are shown in Figure 13. Call v' the involution of M' whose restriction to M_0' is η. Clearly v' cannot commute with any hyperbolic type involution of M' since the product $\eta\gamma$ has order 4 on M_0'. On the other hand, v' is hyperelliptic as it is easy to see by considering the orbifold M_0'/η (which is topologically a solid torus) and the given glueing. The resulting knot is drown in Figure 14. To conclude, remark that we could have chosen instead of L' any other two component hyperbolic link with two involutions acting as strong inversions on both components and admitting an involution with non-empty fixed-point set which conjugates the two strong inversions and such that the quotient of L' by its action is the trivial knot. In fact, we need to find an involution which does not commute with all strong inversions. Similarly we could have replaced 9_{49} by any strongly invertible simple knot.

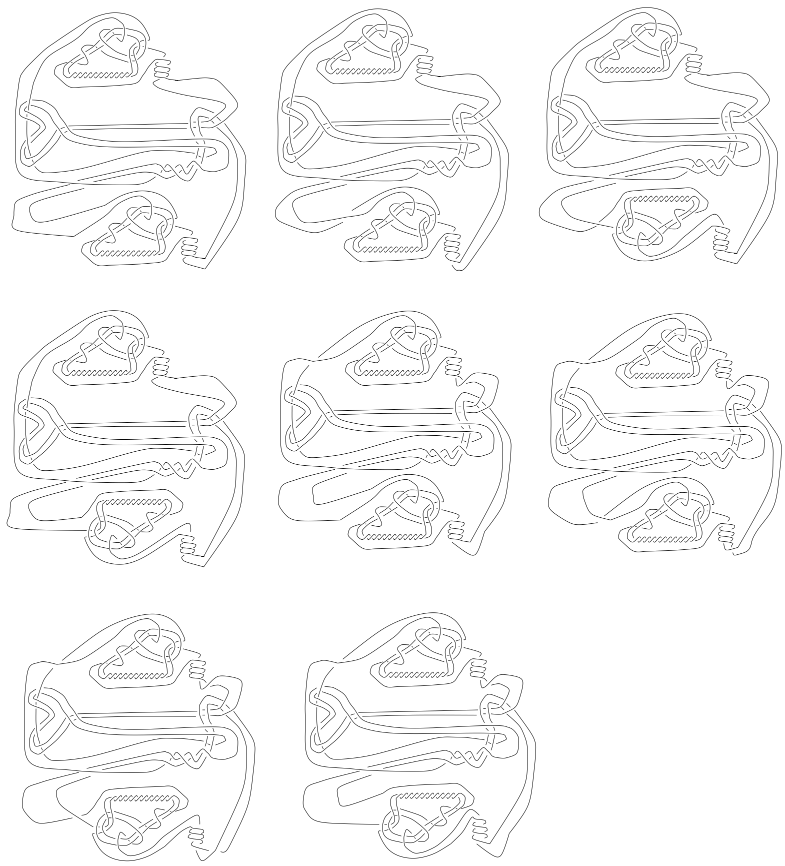


Figure 13.

□

4. Double covers of Heegaard genus 2

In this Section we want to apply the results of Sections 2 and 3 to the case when the manifold M has Heegaard genus 2 and non trivial Jaco-Shalen-Johannson decomposition. A characterization of all possible decompositions for manifolds of Heegaard genus 2 is given in [11]. Here we shall try to understand for each of the five cases listed in [11, page 437, Theorem], whether the manifold M admits hyperbolic type involutions and hyperelliptic involutions, not of hyperbolic type.

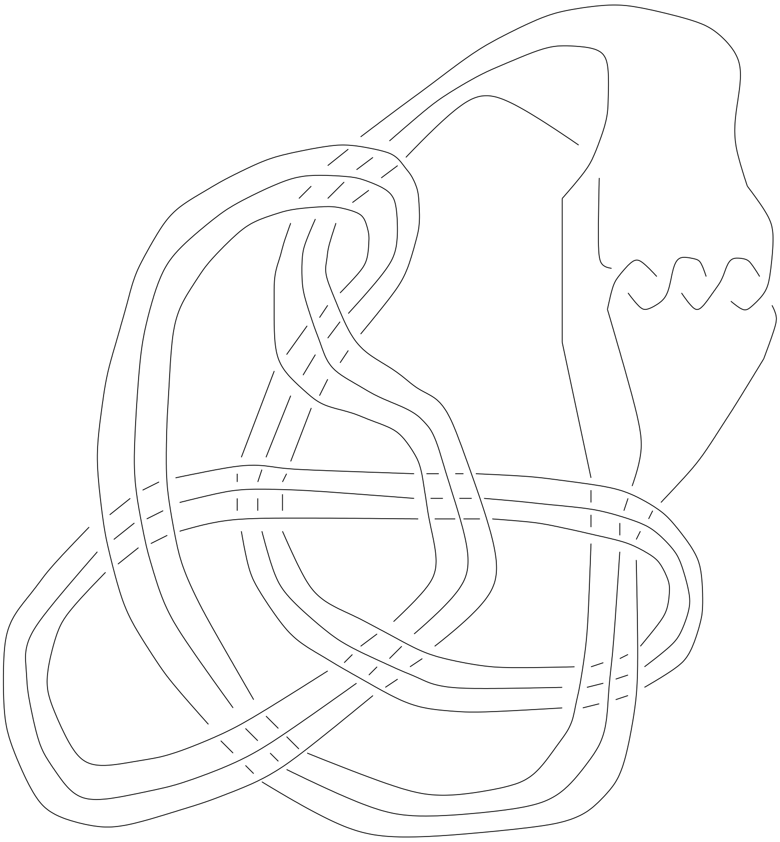


Figure 14.

Case (i): *M is obtained from M_1, a Seifert fibred manifold with base orbifold the disc with two singular points, and M_2, the complement of a 1-bridge knot in a lens space, by identifying the regular fibre of M_1 to the meridian loop of M_2.*

Notice first of all that M_2 admits an involution of standard type. This can be seen by thinking of M_2 as the manifold obtained by glueing together two copies of H, a solid torus with an arc drilled out. H together with the axis of a partial standard involution is shown in Figure 15. Such involution extends to a global one since it commutes up to isotopy with any diffeomorphism of the boundary torus. Unfortunately we are not able to say whether such a

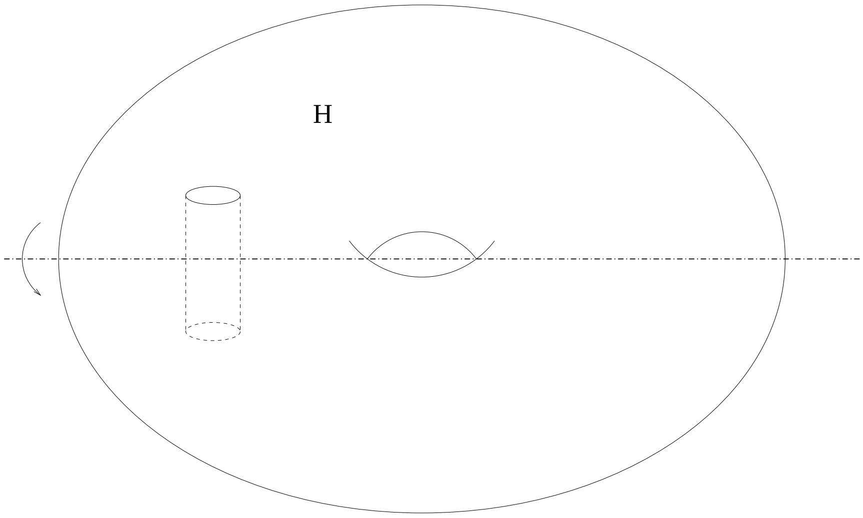


Figure 15.

standard involution is unique. Notice that, in the case when M_2 is Seifert fibred, the base space of M_2 is either a disc with two singular points, the Möbius band with one singular point or the twisted I-bundle over the Klein bottle [11, Lemma 5.2]. In these two latter cases, however, M is necessarily the double cover of a link with more than one component [16]. In the former case, as well as for M_1, there is a unique standard involution, whose quotient is given in Figure 16. Since there is a unique torus in the decomposition of M, by Proposition 3.1 any hyperelliptic involution commutes with some hyperbolic type involution. Hyperelliptic involutions can be of two types: either their fixed point set is contained in M_1 or in M_2. In both cases, the quotient of the piece containing the singular set must be a solid torus, while the other one is a knot complement. In the first case, M_1 is the 2-fold branched cover of a non trivial torus knot $T(p,q)$ where the preimage of a singular fibre (the one of even order, if there is one) is removed. If pq is even then the involution acts as a rotation of the base orbifold. This extends to M if M_2 is the double (non branched) cover of a knot complement in $\mathbf{S}^3$ where the longitude is mapped to the generating element of $\mathbb{Z}_2$ (since the free involution on M_1 acts as a rotation of the longitude of M_2): this is impossible. Suppose now that pq is odd. The involution of

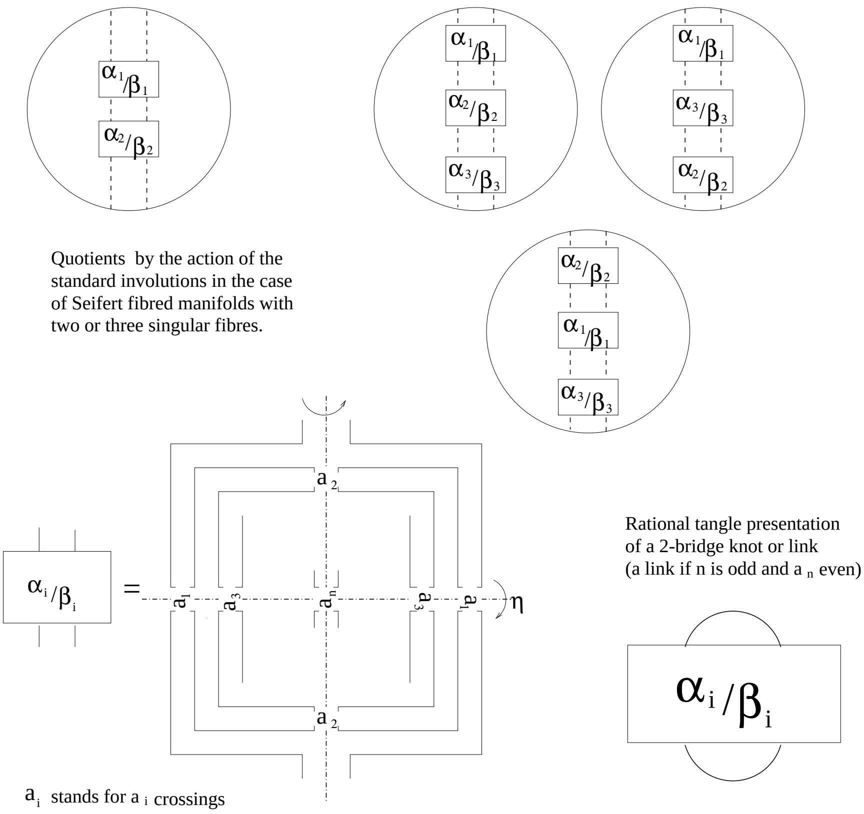


Figure 16.

M_1 consist of a translation along the fibres. In this case there is an hyperelliptic involution of M extending the involution on M_1 if M_2 is the double cover of a knot in $\mathbf{S}^3$ where the preimage of the knot is removed. The involution of M restricts to the covering involution on M_2. In the case when the quotient of M_2 is a solid torus, we require that the image of the meridian is of the form longitude plus k times the meridian of the solid torus. In this case M_1 must be the double branched cover of a non trivial torus knot $T(p, q)$, $(p, q) = (2, 3)$ or pq odd, from which the preimage of $T(p, q)$ has been removed, if we want the global involution to be hyperelliptic.

Case (ii): *M is obtained from M_1, a Seifert fibred manifold with* *base orbifold the Möbius band with at most two singular points, and M_2, the complement of a 2-bridge knot, by identifying the regular fibre of M_1 to the meridian loop of M_2.*

In this case M is the 2-fold branched cover of a link with more than one component [16].

Case (iii): *M is obtained from M_1, a Seifert fibred manifold with base orbifold the disc with two or three singular points, and M_2, the complement of a 2-bridge knot, by identifying the regular fibre of M_1 to the meridian loop of M_2.*

As in (i) we have exactly one standard involution on M_1 if the base orbifold has two singular points and three standard involutions if the base orbifold has three singular points (see Figure 16). The complement of a 2-bridge knot has two conjugacy classes of standard involutions if the knot is hyperbolic and one if it is a torus knot. So, in general, we have four (respectively two) or twelve (respectively six) hyperbolic type involutions according as M_1 has two or three singular fibres and M_2 is hyperbolic (respectively Seifert fibred). As in (i) all hyperelliptic involutions commute with some hyperbolic type involution for there is only one torus in the decomposition. Let us analyse the possible hyperelliptic involutions. Assume that M_2 is hyperbolic. It does not admit free involutions but it does admit a 2-periodic symmetry. This may extend to a hyperelliptic involution if the image of the knot in the quotient by the action of the 2-periodic symmetry is the trivial knot. As before, M_1 must be the double branched cover of a non trivial torus knot $T(p,q)$, where now p and q are arbitrary since M_1 can have two or three singular fibres. Since in this case M_2 does not admit free involutions, there cannot be hyperelliptic involutions with fixed point set inside M_1, since we are assuming that M is the double cover of a knot. Let us now assume that M_2 is fibred. If it admits a 2-periodic symmetry, the analysis is the same as the one given above. On the other hand, if M_1 admits an involution with non-empty fixed-point set and with quotient a solid torus, such involution extends to a global hyperelliptic involution of M only if M_2 is the double branched cover of a non trivial torus knot $T(p,q)$, $(p,q)=(2,3)$ or pq odd, since M_2 has only two singular fibres, where the preimage of the $T(p,q)$ has been removed.

CASE (iv): *M is obtained from M_1 and M_2, Seifert fibred manifolds with base orbifold the disc with two singular points, and M_3, the complement of a 2-bridge link, by identifying the regular fibre of M_i, $i = 1, 2$ to the meridian loop of M_3.*

Any hyperbolic 2-bridge link L admits three involutions easily detectable by looking at the presentation of L as a closed rational tangle (see Figure 16). One of them, τ, acts as a strong inversion on both components of L, the other two, η and $\tau\eta$, have non-empty fixed-point set and act by exchanging the two components; indeed, because of Smith's conjecture [18], they cannot both act as 2-periodic symmetries on the two components of L but they exchange in the same way the fixed points of τ lying on L. In general L can admit other involutions; e.g. the hyperbolic 2-bridge link 8^2_6 admits an involution with non-empty fixed-point set acting as a 2-periodic symmetry on both components while the hyperbolic 2-bridge link 5^2_1 admits an involution acting as strong inversion on one component and as 2-periodic symmetry on the other. If L is a Seifert fibred 2-bridge link, then the base space of M is an annulus with one singular fibre ([11, Lemma 4.4], if there are no singular fibres then M_1 is glued to M_2 by sending regular fibres to regular fibres, so that M admits a global fibration). The possible involutions are: a standard one, a rotation along the singular fibre and a translation along the fibres. In any case M admits hyperbolic type involutions: four if L has no 2-periodic symmetries or is Seifert fibred and eight otherwise. Assume now that we have a hyperelliptic involution v, not of hyperbolic type. We consider different cases according to the action of v_3. Assume that v_3 has non-empty fixed-point set and exchanges the boundary components of M_3. In this case we must have that $M_1 = M_2$ is the complement of a non-trivial torus knot and, moreover, the image of L in the quotient by the action of v_3 is the trivial knot. Notice that v commutes with a hyperbolic type involution if and only if their restrictions commute on M_3 (compare the construction of the second example of Theorem 3.2). If v_3 has non-empty fixed-point set and acts as a 2-periodic symmetry on both components, then the topological space underlying M_3/v_3 must be the intersection of two solid tori embedded in $\mathbf{S}^3$. The only possibility for the glueing to give $\mathbf{S}^3$ is that M_3/v_3 is topologically as pictured in Figure 17 (where longitude-meridian systems are also shown). In this case L needs to

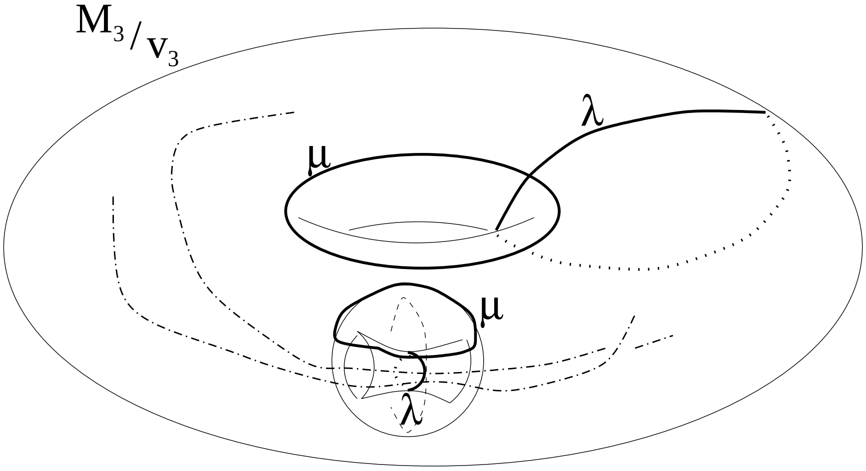


Figure 17.

be hyperbolic and its two components have 0 linking number. M_1 and M_2 must be 2-fold (unbranched) covers of torus knots of type $T(p,q)$, $(p,q)=(2,3)$ or pq odd (compare Case (i)). Notice that, according to Proposition 3.1, v commutes with some hyperbolic type involution. If v_3 has non-empty fixed-point set, acts as a 2-periodic symmetry on the first component and as a strong inversion on the second then it extends to an hyperelliptic involution of M if the image of the first component of the 2-bridge link in the quotient of $\mathbf{S}^3$ by the action of v_3 is the trivial knot. We must also require that M_1 is the 2-fold (unbranched) cover of the complement of a torus knot of type $T(p,q)$, $(p,q)=(2,3)$ or pq odd. In this case v_1 is the covering involution and v_2 is the standard involution of M_2. Notice that the complement of the Whitehead link 5^2_1 satisfies the conditions we have just discussed, so there exists hyperelliptic involutions on manifolds of Heegaard genus 2 which do not commute with any hyperbolic type involution (see Figure 18). If v_3 acts freely fixing setwise the two boundary components, then the topological space underlying M_3/v_3 must be the complement of a knot K' in a solid torus; equivalently, M_3 is the double cover of the complement of a

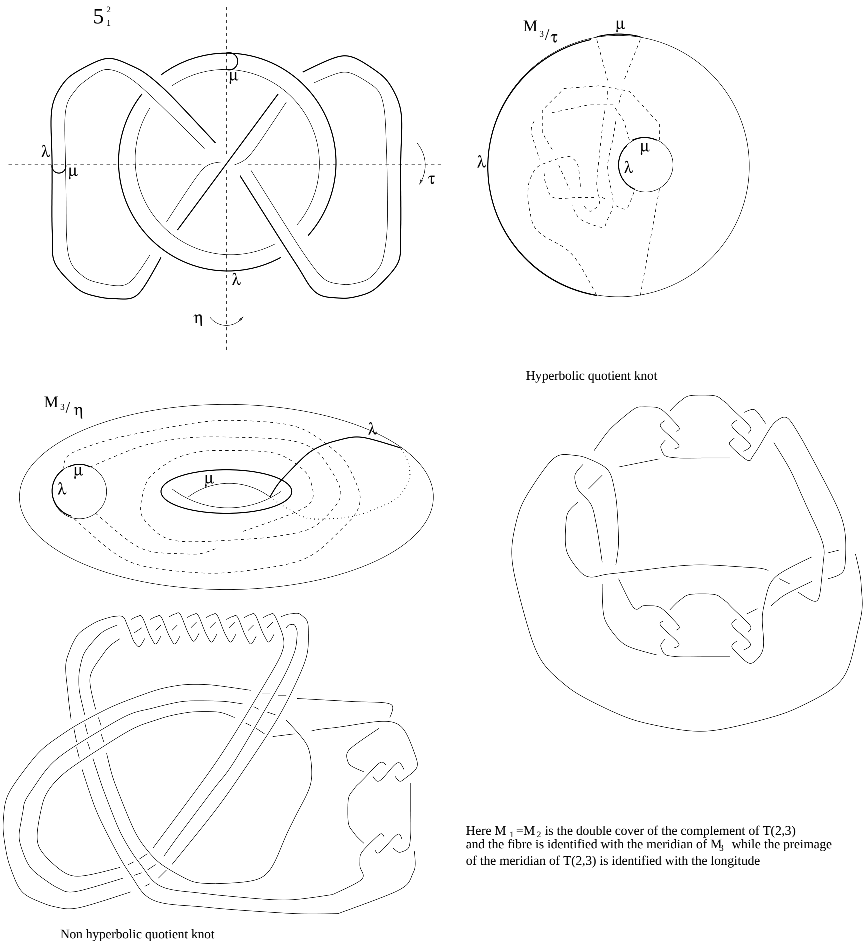

5_1^2
μ
λ
μ
τ
λ
η
M_3/τ
μ
λ
μ
λ
Hyperbolic quotient knot
M_3/η
λ
μ
λ
μ
Here $M_1=M_2$ is the double cover of the complement of T(2,3)
and the fibre is identified with the meridian of M_3 while the preimage
of the meridian of T(2,3) is identified with the longitude
Non hyperbolic quotient knot

Figure 18.

link with two components, one of which is trivial. However, since the regular fibre of -say- $M_1/v_1 = \mathbf{D}^2 \times \mathbf{S}^1$ is glued to the meridian of K', v_3 cannot be hyperelliptic. If v_3 acts freely and exchanges the two boundary components, then the topological space underlying M_3/v_3 must be the complement of a knot K' in the 3-sphere. This is impossible if K' is not trivial for the same reason seen in (i).

CASE (v): *M is obtained from M_1, a Seifert fibred manifold with base orbifold the annulus with at most two singular points, and M_2, the complement of a 2-bridge link, by identifying the regular fibre of M_1 to the meridian loop of M_2.*

In this case M is the double cover of a link with more than one component since M cannot be a $\mathbb{Z}_2$-homology sphere for the characteristic graph associated to its Jaco-Shalen-Johannson decomposition is not a tree. Moreover, such link is not hyperbolic, since the characteristic graph associated to its Bonahon-Siebenmann decomposition is the same as the characteristic graph associated to the Jaco-Shalen-Johannson decomposition of M and must be a tree. To conclude remark that if M_1 has no singular fibres the Jaco-Shalen-Johannson decomposition consists in fact only of M_2 where its two boundary components are glued to one another.

We summarize some properties of the hyperelliptic involutions of manifolds with Heegaard genus 2 and non-trivial Jaco-Shalen-Johannson decomposition in the following (compare [21]):

COROLLARY 4.1. *Let M be a manifold with Heegaard genus* 2 *and non-trivial Jaco-Shalen-Johannson decomposition. Two distinct hyperbolic type involutions of M never commute. Moreover, there exist hyperelliptic involutions not of hyperbolic type which do not commute with any involution of hyperbolic type (an example is given in Figure 18). If v is a hyperelliptic involution not of hyperbolic type, then $K(v)$ is the satellite of one or two torus knots or, possibly, a cable knot.*

References

[1] R. Benedetti and C. Petronio, *Lectures on hyperbolic geometry*, Universitext, Spinger–Verlag, Berlin, 1992.

[2] M. Boileau and J. Porti, *Geometrization of 3-orbifolds of cyclic type*, Prépublication, Laboratoire de Math. E. Picard, 1998.

[3] M. Bonahon and L. C. Siebenmann, *The classification of Seifert fibred 3-orbifolds*, Low dimensional topology (Sussex, 1982) (Cambridge Univ. Press, ed.), London Math. Soc. Lecture Notes, vol. 95, R. A. Fenn, ed., Cambridge, 1985, pp. 19–85.

[4] M. Bonahon and L. C. Siebenmann, *The characteristic toric splitting of irreducible compact 3-orbifolds*, Math. Ann. **278** (1987), 441–479.

[5] J. Conway, *An enumeration of knots and links and some of their algebraic properties*, Computational problems in abstract algebra, Proceedings of the Oxford conference 1967 (New York), Pergamon, 1970, pp. 329–358.

[6] E. Flapan, *The finiteness theorem for symmetries of knots and 3-manifolds with non-trivial characteristic decompositions*, Topology Appl. **24** (1986), 123–131.

[7] C. Hodgson and J. H. Rubinstein, *Involutions and isotopies of lens spaces*, Knot theory and manifolds (Vancouver, 1983) (Springer-Verlag, ed.), Lecture Notes in Math., vol. 1144, D. Rolfsen ed., Berlin, 1985.

[8] W. H. Jaco and P. B. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc. **220** (1979).

[9] K. Johannson, *Homotopy equivalence of 3-manifolds with boundary*, Lecture Notes in Math., vol. 761, Spinger–Verlag, Berlin, 1979.

[10] R. Kirby, *Problems in low-dimensional topology*, Available at http://math.berkeley.edu/~kirby.

[11] T. Kobayashi, *Structures of the Haken manifolds with Heegaard splittings of genus 2*, Osaka J. Math. **21** (1984), 437–455.

[12] D. McCullough, *Virtually geometrically finite mapping class groups of 3-manifolds*, J. Diff. Geom. **33** (1991), 1–65.

[13] M. Mecchia and M. Reni, *Hyperbolic 2-fold branched coverings of 3-bridge knots*, To appear in Kobe J. Math.

[14] M. Mecchia and M. Reni, *Hyperbolic 2-fold branched coverings of links and their quotients*, Preprint.

[15] W. H. Meeks and P. Scott, *Finite group actions on 3-manifolds*, Invent. Math. **86** (1986), 287–346.

[16] J. M. Montesinos, *Variedades de Seifert que son recubridadores ciclicos ramificados a dos hojas*, Bol. Soc. Mat. Mex. **18** (1973), 1–32.

[17] J. M. Montesinos and W. Whitten, *Construction of two-fold branched covering spaces*, Pacific J. Math. **125** (1986), 415–446.
[18] J. Morgan and H. Bass, *The Smith conjecture*, Academic Press, New York, 1984.
[19] M. Reni, *On π-hyperbolic knots with the same 2-fold branched coverings*, Math. Ann. **316** (2000), 681–697.
[20] D. Rolfsen, *Knots and links*, Publish or Perish, Berkeley, 1976.
[21] H. Rubinstein and M. Scharlemann, *Genus two Heegaard splittings of orientable three-manifolds*, Proceedings of the Kirbyfest, Geometry and Topology Monographs, vol. 2, 1999, pp. 489–553.
[22] W. P. Thurston, *Hyperbolic structures on 3-manifolds, II: Surface groups and manifolds which fiber over S^1*, Preprint.
[23] W. P. Thurston, *Hyperbolic structures on 3-manifolds, III: Deformations of 3-manifolds with incompressible boundary*, Preprint.
[24] W. P. Thurston, *The geometry and topology of 3-manifolds*, Princeton Univ. Press, Princeton, 1979.
[25] W. P. Thurston, *3-manifolds with symmetry*, Preprint, 1982.
[26] W. P. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357–381.
[27] W. P. Thurston, *Hyperbolic structures on 3-manifolds, I: Deformations of acylindrical manifolds*, Ann. Math. **124** (1986), 203–246.
[28] J. L. Tollefson, *A 3-manifold admitting a unique periodic PL map*, Michigan Math. J. **21** (1974), 7–12.
[29] J. L. Tollefson, *Involutions of Seifert fiber spaces*, Pacific J. Math. **74** (1978), 519–529.
[30] O. Ya. Viro, *Nonprojecting isotopies and knots with homeomorphic coverings*, J. Soviet Math. **12** (1979), 86–96.
[31] F. Waldhausen, *Über involutionen der 3-sphäre*, Topology **8** (1969), 81–91.
[32] B. Zimmermann, *On hyperbolic knots with the same m-fold and n-fold cyclic branched coverings*, Topology Appl. **79** (1997), 143–157.
[33] B. Zimmermann, *On hyperbolic knots with homeomorphic cyclic branched coverings*, Math. Ann. **311** (1998), 665–673.

Received January 11, 2001.