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A Code for m-Bipartite Edge-Coloured Graphs

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Summary. - *An* $(n+1)$*-coloured graph* (Γ,γ) *is said to be* m-bipartite *if* m *is the maximum integer so that every* m*-residue of* (Γ,γ) *(i.e. every connected subgraph whose edges are coloured by only* m *colours) is bipartite; obviously, every* $(n+1)$*-coloured graph, with* $n \geq 2$*, results to be* m*-bipartite for some* m*, with* $2 \leq m \leq n+1$*. In this paper, a numerical* code *of length* $(2n-m+1)\times q$ *is assigned to each* m*-bipartite* $(n+1)$*-coloured graph of order* $2q$*. Then, it is proved that* any two such graphs have the same code if and only if they are colour-isomorphic, *i.e. if a graph isomorphism exists, which transforms the graphs one into the other, up to permutation of the edge-colouring. More precisely, if* H *is a given group of permutations on the colour set, we face the problem of algorithmically recognizing* H-isomorphic *coloured graphs by means of a suitable definition of* H-code.

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1. Introduction and preliminary notations

The basic notions of graph theory used in this paper follow [10]. In particular, we recall that in a *multigraph* loops are forbidden, but multiple edges are allowed; moreover, by a *(proper) edge-colouring* on a multigraph Γ we mean a map γ from the edge-set $E(\Gamma)$ to a set $\mathcal{C}$ (called the *colour-set*), which associates different colours to any pair of adjacent edges.

Definition 1.1. *An* $(n+1)$-coloured graph *is a pair* (Γ, γ)*, where:*

(i) $\Gamma = (V(\Gamma), E(\Gamma))$ *is a multigraph whose vertices have either degree* $n+1$ (internal vertices) *or degree* n (boundary vertices)*;*

(ii) $\gamma : E(\Gamma) \to \Delta_n = \{c \in \mathbb{Z} \ / \ 0 \leq c \leq n\}$ *is a proper edge-colouring on* Γ*, such that the subgraph* $\Gamma_{\hat{n}} = (V(\Gamma), \gamma^{-1}(\Delta_{n-1}))$ *is a regular multigraph of degree* n*.*

Note that the order $\#V(\Gamma)$ of Γ is always an even positive integer.

If Γ has no boundary-vertex, we will say that (Γ, γ) has empty boundary (denoted by $\partial\Gamma = \emptyset$); otherwise, (Γ, γ) is said to have *non empty boundary*. Note that, if $\partial\Gamma \neq \emptyset$, then the number of boundary vertices of Γ is an even positive integer, too.

From now on, each $(n+1)$-coloured graph (Γ, γ) will be assumed to be connected.

Two vertices v, w of (Γ, γ) will be called *c-adjacent with respect to the colouring* γ (for $0 \leq c \leq n$), iff they are the endpoints of a c-coloured edge of (Γ, γ) (i.e., an edge $e \in E(\Gamma)$ with $\gamma(e) = c$); if no confusion arises, v and w are often said to be *c-adjacent*, without explicit mention of the colouring γ.

Definition 1.2. *For every* $\mathcal{F} \subseteq \Delta_n$*, an* $\mathcal{F} - residue$ *of* (Γ, γ) *is a connected component* Ξ *of the subgraph* $\Gamma_{\mathcal{F}} = (V(\Gamma), \gamma^{-1}(\mathcal{F}))$*, with the induced edge-colouring; if the cardinality* $\#\mathcal{F}$ *of* $\mathcal{F}$ *is* m *(with* $0 \leq m \leq n+1$*), then* Ξ *will be called an* $m - residue$ *of* (Γ, γ)*. Of course, the 0-residues are the vertices of* Γ*, the 1-residues* $\Gamma_{\{c\}}$*,* $c \in \Delta_n$*, are the c-colored edges and (in case* $c = n$*) the boundary vertices of* Γ*, while the 2-residues* $\Gamma_{\{c,d\}}$*,* $c, d \in \Delta_n$*,* $c \neq d$*, are bicoloured cycles and/or (in case* $n \in \{c, d\}$*) bicoloured paths joining two boundary-vertices of* Γ*.*

Definition 1.3. *An* $(n+1)$*-coloured graph* (Γ,γ) *is said to be* m-bipartite *(for* $0 \le m \le n+1$*) if* m *is the maximum integer such that every* m*-residue of* (Γ,γ) *is bipartite.*

Note that *(n+1)-bipartite simply means bipartite.* Furthermore, *every 2-coloured graph is 2-bipartite* (i.e., bipartite) and so, *every (n+1)-coloured graph, with* $n \ge 2$*, is m-bipartite for some m, with* $2 \le m \le n+1$.

Hence, for every m-bipartite (n+1)-coloured graph (Γ,γ), the following integer $\bar{m}(\Gamma)$ is well defined, with $2 \le \bar{m} \le n+1$:

$$\bar{m}(\Gamma) = \begin{cases} n & \text{if } (\Gamma,\gamma) \text{ is bipartite with non empty boundary} \\ m & \text{otherwise} \end{cases}$$

Let now H be any subgroup of the group $\mathcal{S}_{n+1}$ of all permutations $\sigma : \Delta_n \to \Delta_n$.

Definition 1.4. *Two* $(n+1)$*-coloured graphs* (Γ,γ) *and* (Γ',γ') *will be called* H-isomorphic *if there exists a permutation* $\sigma \in H$ *and a graph isomorphism* $\phi : \Gamma \to \Gamma'$ *such that*

$$\gamma' \circ \phi = \sigma \circ \gamma.$$

If $H = \{Id\}$*, then* (Γ,γ) *and* (Γ',γ') *will be called* strictly-isomorphic*. If* $H = \mathcal{S}_{n+1}$*, then* (Γ,γ) *and* (Γ',γ') *will be called* colour-isomorphic*, or simply* isomorphic*.*

Note that isomorphic graphs may be not H-isomorphic, for a fixed subgroup H of $\mathcal{S}_{n+1}$, while H-isomorphism (for any H) trivially implies isomorphism.

In this paper we face the problem of algorithmically recognizing isomorphic (or, more precisely, H-isomorphic, for a given H) m-bipartite $(n+1)$-coloured graphs, by means of the introduction of a numerical "code" $c(\Gamma)$, whose length depends on the integer m; the algorithm computing $c(\Gamma)$ [1] has been implemented in the language C, and a copy of the program is available upon request.

[1]Obviously, the integer m - and hence the length of the code - is directly computed by the algorithm itself.

The most interesting cases - from our point of view - are those for $m = n$ and $m = n + 1$. In fact, a representation theory for PL-manifolds of arbitrary dimension n exists, which makes use of particular n-bipartite or $(n + 1)$-bipartite $(n + 1)$-coloured graphs, according to the orientability of the manifold (see [3], [7], [1], [9], [4, Chapter 13] and their bibliography). Thus, since the code allows to directly verify whether two given $(n+1)$-coloured graphs are isomorphic (and hence - obviously - represent the same PL-manifold), this is a tool which makes possible the creation of "sufficiently essential" catalogues of graphs representing manifolds. In particular, the investigation about the 3-dimensional orientable (resp. non-orientable) case has been already started in [5] (resp. in [2]), by means of specific "codes", of which the present one is a generalization. The 4-dimensional case will be the matter of a forthcoming paper.

2. The code

Let (Γ, γ) be an $(n + 1)$-coloured graph of order $2q$. By a *vertex-labelling* of Γ we mean a bijective map $l : V(\Gamma) \to I_{2q}$, where I_{2q} is any subset of the integer set $\mathbb{Z}$, with $0 \notin I_{2q}$. We shall assume $I_{2q} = \{i \in \mathbb{Z} \ / \ 1 \leq i \leq 2q\}$, unless otherwise stated. For each $i \in I_{2q}$, we shall call v_i the vertex of Γ labelled i by l.

Definition 2.1. *Given an $(n+1)$-coloured graph (Γ, γ) and a vertex-labelling l of it, we define*

$$\mathcal{A} = \mathcal{A}(\Gamma, \gamma, l) = (a^i_c)$$

to be the $[2q \times (n + 1)]$-matrix, with entries in $I_{2q} \cup \{0\}$, where for $i \in I_{2q}$ and for $0 \leq c \leq n$,

$$a^i_c = \begin{cases} 0 & \textit{if } c = n \textit{ and } v_i \textit{ is a boundary-vertex of } \Gamma \\ k \in I_{2q} & \textit{if } v_i \textit{ and } v_k \textit{ are } c\textit{-adjacent in } \Gamma \textit{ with respect to} \\ & \textit{the colouring } \gamma \end{cases}$$

As a straightforward consequence of the definitions, eachone of the first n columns of $\mathcal{A}$ results to be a permutation of I_{2q}, having exactly q orbits, each of size 2. Moreover, the last column of $\mathcal{A}$ has an even number of 0-entries, corresponding to the labels of $\bar{I} = \{i \in I_{2q} \ / v_i$ is a boundary-vertex$\}$, while the remaining entries constitute a permutation of $I_{2q} - \bar{I}$, having exactly $q - \bar{q}$ orbits, with $2\bar{q} = \#\bar{I}$. These properties may be summarised in the following way, for every $i, j, k \in I_{2q}$ and $c \in \Delta_n$:

1) $(a_c^i = k) \iff (a_c^k = i)$;

2) $(a_c^i = 0) \iff (c = n \text{ and } i \in \bar{I})$;

3) $(a_c^i = a_c^j \neq 0) \implies (i = j)$.

Of course, $\mathcal{A}(\Gamma, \gamma, l) = \mathcal{A}(\Gamma', \gamma', l')$ iff there exist a permutation $\sigma \in \mathcal{S}_{n+1}$ and a graph isomorphism $\phi : \Gamma \to \Gamma'$ such that $\gamma' \circ \phi = \sigma \circ \gamma$ and $l = l' \circ \phi$.

It is not difficult to check that, in case Γ, Γ' having non empty boundary, the above permutation $\sigma \in \mathcal{S}_{n+1}$ always satisfies the condition $\sigma(n) = n$; this leads to the following definition.

Definition 2.2. *Let* (Γ, γ) *be an* $(n+1)$*-coloured graph. Then, the set of* admissible colour permutations *for* (Γ, γ) *is defined to be*

$$\bar{H}(\Gamma) = \begin{cases} \mathcal{S}_{n+1} & \textit{if } \ \partial\Gamma = \emptyset \\ \{\sigma \in \mathcal{S}_{n+1} \ / \ \sigma(n) = n\} & \textit{if } \ \partial\Gamma \neq \emptyset \end{cases}$$

Suppose now (Γ, γ) to be m-bipartite, for $2 \leq m \leq n+1$. We want to introduce an algorithmic procedure for labelling the vertices of Γ, which only depends on the choice of a starting vertex $r \in V(\Gamma)$ (called the *root*) and of an admissible colour permutation $\pi = (\pi(0), \pi(1), \ldots, \pi(n)) \in \bar{H}(\Gamma)$.

First of all, we need the following preliminary construction. Let Ξ be a regular and bipartite $\bar{m}$-coloured graph of order $2q(\Xi)$; let further s be any positive integer, $x \in V(\Xi)$ be any vertex of Ξ and $\sigma = (\sigma(c_0), \sigma(c_1), \ldots, \sigma(c_{\bar{m}-1}))$ be any permutation of the colour set $\mathcal{C} = \{c_0, , c_1, \ldots, c_{\bar{m}-1}\}$. We define

$$\tilde{N} = \tilde{N}_{x,\sigma,s} : V(\Xi) \to \{\pm i \in \mathbb{Z} - \{0\} \ / \ s \leq i \leq s + q(\Xi) - 1\}$$

as follows:

1. $\tilde{N}(x) = -s$;

2. $\tilde{N}(x') = +s$, where x' is the vertex of Ξ $\sigma(c_0)$-adjacent to x.

3. For $i = 1, 2, \dots, q(\Xi) - 1$: let v be the last element of the ordered sequence
$$(\tilde{N}^{-1}(+s), \tilde{N}^{-1}(+s+1), \dots, \tilde{N}^{-1}(+s+i-1))$$
such that the set of its $\sigma(c)$-adjacent vertices, with $1 \leq c \leq \bar{m}-1$, is not a subset of $\tilde{N}^{-1}(\{-j \in \mathbb{Z} \ / \ s \leq j \leq s+i-1\})$; further, if y_r denotes the vertex of Ξ $\sigma(c_r)$-adjacent to v, with $r = 1, 2, \dots, \bar{m}-1$, let $\bar{y}$ be the first element of the $(\bar{m}-1)-$ple $(y_1, y_2, \dots, y_{\bar{m}-1})$ not belonging to $\tilde{N}^{-1}(\{-j \in \mathbb{Z} \ / \ s \leq j \leq s+i-1\})$. Then:
   - $\tilde{N}(\bar{y}) = -(s+i)$;
   - $\tilde{N}(\bar{y}') = +s+i$, where $\bar{y}'$ is the vertex of Ξ $\sigma(c_0)$-adjacent to $\bar{y}$.

By the construction itself, $\tilde{N}$ is a bijection; moreover:

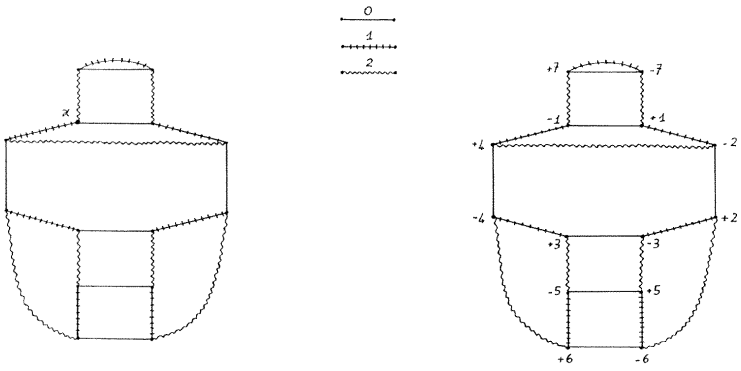
a) the two bipartition classes of $V(\Xi)$ are exactly $\tilde{N}^{-1}(\{-i \ / \ s \leq i \leq s+q(\Xi)-1\})$ and $\tilde{N}^{-1}(\{+i \ / \ s \leq i \leq s+q(\Xi)-1\})$;

b) for every $i \in \mathbb{Z}$, $s \leq i \leq s+q(\Xi)-1$, the vertices $\tilde{N}^{-1}(-i)$ and $\tilde{N}^{-1}(+i)$ are $\sigma(c_0)$-adjacent.

For example, if Ξ is the regular and bipartite 3-coloured graph (Γ_1,γ_1) depicted in Figure 1(a), and if the integer $s = 1$, the canonical permutation $\sigma = Id = (0, 1, 2)$ and the vertex $x \in V(\Gamma_1)$ pointed out in Figure 1(a) are chosen, then the vertex-labelling $\tilde{N} = \tilde{N}_{x,\sigma,1}$ is visualized in Figure 1(b).

Coming back to our m-bipartite graph (Γ, γ), for each root $r \in V(\Gamma)$ and for each admissible colour permutation $\pi \in \bar{H}(\Gamma)$, we define
$$N = N_{r,\pi} : V(\Gamma) \to \{\pm i \in \mathbb{Z} - \{0\} \ / \ 1 \leq i \leq q\}$$
as follows:

1. Let Ξ_1 be the (regular and bipartite) $\{\pi(0), \pi(1), \dots, \pi(\bar{m}-1)\}$-residue of (Γ, γ), which contains r; then, set $N_{r,\pi}|_{V(\Xi_1)} = \tilde{N}_{r,\pi,1}$.



(Γ_1, γ_1)

Figures 1(a) - 1(b)

2. Further, for $i = 1, 2, \dots t-1$, (t being the number of $\{\pi(0), \pi(1), \dots, \pi(\bar{m}-1)\}$-residues of (Γ, γ)):

   - let u be the last element of the ordered sequence

   $$(N^{-1}(-1), N^{-1}(+1), N^{-1}(-2), \dots$$
   $$\dots, N^{-1}(-\sum_{j=1}^{i} q(\Xi_j)), N^{-1}(+\sum_{j=1}^{i} q(\Xi_j)))$$

   such that the set of its $\pi(c)$-adjacent vertices, with $\bar{m} \leq c \leq n$, is not a subset of $N^{-1}(\{\pm r \in \mathbb{Z} - \{0\} \ / \ 1 \leq r \leq \sum_{j=1}^{i} q(\Xi_j)\})$;
   - if x is the first vertex of the $(n-\bar{m}+1)$-ple $(y_{\bar{m}}, \dots, y_n)$, with y_c $\pi(c)$-adjacent to u ($\bar{m} \leq c \leq n$), which does not belong to $N^{-1}(\{\pm r \in \mathbb{Z} - \{0\} \ / \ 1 \leq r \leq \sum_{j=1}^{i} q(\Xi_j)\})$ and Ξ_{i+1} is the (regular and bipartite) $\{\pi(0), \dots, \pi(\bar{m}-1)\}$-residue of (Γ, γ) which contains x, then set

   $$N_{r,\pi|_{V(\Xi_{i+1})}} = \tilde{N}_{x',\pi,1+\sum_{j=1}^{i} q(\Xi_j)}$$

   where

   $$x' = \begin{cases} x & \text{if } N(u) > 0 \\ \text{the vertex } \pi(0)\text{-adjacent to } x & \text{if } N(u) < 0 \end{cases}$$

Note that, if (Γ, γ) is regular and bipartite (i.e. if $\bar{m} = n+1$), then the bijection N is completely defined by the rule of point (1.), i.e. by setting $N_{r,\pi} = \tilde{N}_{r,\pi,1}$. [2]

On the contrary, if $\bar{m} \leq n$, the rule (2.) says how to choose - at every step - the subsequent $\{\pi(0), \pi(1), \dots, \pi(\bar{m}-1)\}$-residue of (Γ, γ): for every colour $\pi(c)$, $\bar{m} \leq c \leq n$, and for every element u of the queue of visited vertices, if the $\pi(c)$-adjacent vertex x of u has not been visited, $N_{r,\pi}$ labels the vertices of the $\bar{m}$-residue containing x by means of the function $\tilde{N}_{x',\pi,s}$ where

$$x' = \begin{cases} x & \text{if } u \text{ is (-)-labelled} \\ \text{the vertex } \pi(0)\text{-adjacent to } x & \text{if } u \text{ is (+)-labelled} \end{cases}$$

and s is the first not used positive integer.

The properties of the algorithm defining $N_{r,\pi}$ are collected into the following:

Proposition 2.3. *Let (Γ, γ) be an order $2q$ m-bipartite $(n+1)$-coloured graph, with $\bar{m}(\Gamma) = \bar{m}$. Then, for every chosen root $r \in V(\Gamma)$ and permutation $\pi = (\pi(0), \pi(1), \dots, \pi(n)) \in \bar{H}(\Gamma)$, the function*

$$N = N_{r,\pi} : V(\Gamma) \to \{j \in \mathbb{Z} - \{0\} \ / \ -q \leq j \leq +q\}$$

is a vertex-labelling of Γ (with $I_{2q} = \{j \in \mathbb{Z} - \{0\} \ / \ -q \leq j \leq +q\}$) such that:

a) for every $\{\pi(0), \pi(1), \dots, \pi(m-1)\}$-residue Ξ of (Γ, γ), one of the two bipartition classes of $V(\Xi)$ is a subset of $(N_{r,\pi})^{-1}(\{-i \in \mathbb{Z} \ / \ 1 \leq i \leq q\})$, and the other one is a subset of $(N_{r,\pi})^{-1}(\{+i \in \mathbb{Z} \ / \ 1 \leq i \leq q\})$;

b) for every $i \in \mathbb{Z}$, $1 \leq i \leq q$, $\quad (N_{r,\pi})^{-1}(-i)$ and $(N_{r,\pi})^{-1}(+i)$ are $\pi(0)$-adjacent vertices of (Γ, γ).

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[2] It is easy to check that this "simplified" algorithm coincides with the *rooted numbering algorithm*, described in [4, Chapter 13] and in [5]. Moreover, note that similar problems have been faced in [8].

Proof. Both the bijectivity of N and property b) are direct consequences of the construction itself, and of the homonimous properties of function $\tilde{N}$, applied to every $\{\pi(0), \dots, \pi(\bar{m}-1)\}$-residue of (Γ, γ).

As far as property a) is concerned, note that obviously, for each $\{\pi(0), \pi(1), \dots, \pi(\bar{m}-1)\}$-residue Ξ of (Γ, γ), the two bipartition classes of $V(\Xi)$ are subsets of $N^{-1}(\{-i \ / \ 1 \leq i \leq q\})$ (i.e. the set of $(-)$-labelled vertices) and of $N^{-1}(\{+i \ / \ 1 \leq i \leq q\})$ (i.e. the set of $(+)$-labelled vertices) respectively; this proves statement a) for every $(n+1)$-coloured graph (Γ, γ) satisfying $m = \bar{m}$.

On the other hand, if $\bar{m} = n$ and $m = n+1$ hold, we have only to check that the "jump" between two $\{\pi(0), \dots, \pi(\bar{m}-1) = \pi(n-1)\}$-residues always respects the bipartition property of the whole graph: in fact, the new partial root (which is always $(-)$-labelled) is chosen either as the $\pi(n)$-adjacent of a $(+)$-labelled vertex, or as the $\pi(0)$-adjacent of the $\pi(n)$-adjacent of a $(-)$-labelled vertex. $\square$

The choice of the vertex-labelling $N_{r,\pi}$, together with the choice of the "permuted" edge-colouring $\gamma' = \pi \circ \gamma$, enables to represent (Γ, γ) by means of a matrix $A_{r,\pi}(\Gamma) = A(\Gamma, \pi \circ \gamma, N_{r,\pi})$, where many entries may be recovered from the other ones.

PROPOSITION 2.4. *Let* (Γ, γ) *be an order* $2q$ m*-bipartite* $(n+1)$*-coloured graph, with* $\bar{m}(\Gamma) = \bar{m}$*. Then, for every chosen root* $r \in V(\Gamma)$ *and admissible colour permutation* $\pi = (\pi(0), \pi(1), \dots, \pi(n)) \in \bar{H}(\Gamma)$*, the matrix*

$$A_{r,\pi}(\Gamma) = A(\Gamma, \pi \circ \gamma, N_{r,\pi}) = (a_c^i)$$

is completely determined by its elements of type a_c^i*, for* $i \in \{j \in \mathbb{Z} \ / \ -q \leq j \leq -1\}$ *and* $c \in \{1, \dots, n\}$*, and (if* $m \neq n+1$*) by its elements of type* a_c^i*, for* $i \in \{j \in \mathbb{Z} \ / \ 1 \leq j \leq +q\}$ *and* $c \in \{m, \dots, n\}$*.*

Proof. In order to prove the statement, it is necessary to show that the $(2n - m + 1) \times q$ above listed elements allow to reconstruct the whole $((2q) \times (n+1))$-matrix $A_{r,\pi}(\Gamma) = A(\Gamma, \pi \circ \gamma, N_{r,\pi})$.

First, we note that the properties of $N = N_{r,\pi}$ induce the following properties of $A_{r,\pi}(\Gamma) = (a_c^i)$:

(a) for every $i \in \{j \in \mathbb{Z} - \{0\} \ / \ -q \leq j \leq +q\}$, then $a_0^i = -i$;

(b) for every $c \in \{0, 1, \ldots, \bar{m} - 1\}$ and for every $i \in \{j \in \mathbb{Z} \ / \ 1 \leq j \leq +q\}$, then

$$\begin{aligned} & a_c^{-i} \in \{j \in \mathbb{Z} \ / \ 1 \leq j \leq +q\}, \\ & a_c^{+i} \in \{j \in \mathbb{Z} \ / \ -q \leq j \leq -1\}, \\ & \text{and} \quad (a_c^{-i} = +k) \iff (a_c^{+k} = -i); \end{aligned}$$

As a consequence of property (a), the first column of $A_{r,\pi}(\Gamma)$ may be recovered (since it is always of a standard type); moreover, as a consequence of property (b), the "second half" of the $(c+1)$-th column of $A_{r,\pi}(\Gamma)$, for $c \in \{1, \ldots, \bar{m} - 1\}$, may also be recovered (since it may be reconstructed by means of the "first half" of the same column).

Hence, the statement results to be proved for every $(n+1)$-coloured graph (Γ, γ) satisfying $m = \bar{m}$.

In order to complete the proof, we have now to consider the case of a bipartite $(n+1)$-coloured graph with non empty boundary, i.e. the case $m = n + 1$ and $\bar{m} = n$. By Proposition 2.3 (property a)), the vertex-labelling $N_{r,\pi}$ is such that the $\pi(n)$-adjacent of a $(+)$-labelled (resp. $(-)$-labelled) vertex, if any, is surely a $(-)$-labelled (resp. $(+)$-labelled) vertex; thus, the "second half" of the $(n+1)$-th column of $A_{r,\pi}(\Gamma)$ may be reconstructed from the "first half" of the same column, by means of the following rule (for every $i \in \{j \in \mathbb{Z} \ / \ 1 \leq j \leq +q\}$):

$$a_n^{+i} = \begin{cases} 0 & \text{if } +i \notin \{a_n^{-j} \ / \ 1 \leq j \leq q\} \\ -k & \text{if } a_n^{-k} = +i \end{cases}$$

□

REMARK 2.5. *The 0-elements of the matrix $A_{r,\pi}(\Gamma)$, if any, always belong to the $(n+1)$-th column. This is a consequence of the "admissibility" of the colour permutation $\pi \in \bar{H}(\Gamma)$: in fact, if (Γ, γ) has non empty boundary, the permutation $\pi = (\pi(0), \pi(1), \ldots, \pi(n))$ of Δ_n is assumed to have $\pi(n) = n$.*

DEFINITION 2.6. *Let (Γ,γ) be an order 2q m-bipartite (n+1)-coloured graph. Then, for every chosen pair $(r, \pi) \in V(\Gamma) \times \bar{H}(\Gamma)$, the (r,π)-*code $c_{r,\pi}(\Gamma)$ *of (Γ, γ) is the $((2n - m + 1) \times q)$-tuple*

$$(c_{1,1}, c_{1,2}, \ldots, c_{1,q}; c_{2,1}, c_{2,2}, \ldots, c_{2,q}; \ldots$$

$$\dots ; c_{2n-m+1,1}, c_{2n-m+1,2}, \dots, c_{2n-m+1,q}),$$

which contains exactly the essential elements of the matrix $A_{r,\pi}(\Gamma)$, *in the following order: for every* $j \in \{1, 2, \dots, q\}$, *set*

$$c_{i,j} = \begin{cases} a_i^{-j} & \text{if } i \in \{1, 2, \dots, n\} \\ a_{m+i-n-1}^{+j} & \text{if } i \in \{n+1, n+2, \dots, 2n-m+1\} \end{cases}$$

DEFINITION 2.7. *Let* (Γ,γ) *be an order 2q m-bipartite (n+1)-coloured graph, and let* H *be any subgroup of the group* $\bar{H}(\Gamma)$. *For each pair* $(r,\pi) \in V(\Gamma) \times H$, *juxstapposition of the elements of the* (r,π)*-code* $c_{r,\pi}(\Gamma)$ *yields a length* $((2n-m+1) \times q)$ *"word"* $w_{r,\pi}(\Gamma)$ *in the alphabet* $I_{2q} \cup \{0\} = \{j \in \mathbb{Z} \ / \ -q \leq j \leq +q\}$. *Then, if the alphabet is ordered according to*

$$-1 < -2 < \dots < -q < 0 < +1 < +2 < \dots < +q,$$

the H-code $c_H(\Gamma)$ *of* (Γ,γ) *is the lexicographic maximum among the "words"* $w_{r,\pi}(\Gamma)$, *for every pair* $(r,\pi) \in V(\Gamma) \times H$.

In particular, if $H = \bar{H}(\Gamma)$, *then the* $\bar{H}(\Gamma)$*-code is simply said to be the* code *of* (Γ,γ), *and is denoted by* $c(\Gamma)$.

REMARK 2.8. *In case* $\bar{m} = n+1$ *(resp. in case* $\bar{m} = n$ *and* $m = n+1$*), i.e. in case* (Γ,γ) *being a regular bipartite* $(n+1)$*-coloured graph (resp. i.e. in case* (Γ,γ) *being a bipartite* $(n+1)$*-coloured graph with non empty boundary), then the code* $c(\Gamma)$ *reduces to a length* nq *word in the alphabet* $\{+i/1 \leq i \leq q\}$ *(resp. in* $\{+i/1 \leq i \leq q\} \cup \{0\}$*). By deleting all (positive) signs, a numerical code is obtained, which exactly coincides (resp. which is a reasonable extension) with the one already defined in [4, Chapter 13] and in [5].*

According to Definition 2.7, the computation of the code of an order $2q(n+1)$-coloured graph (Γ,γ) with empty (resp. non empty) boundary would imply to determine $2q \times (n+1)!$ (resp. $2q \times n!$) (r,π)-codes $c_{r,\pi}(\Gamma)$ of (Γ,γ); really, this job is not entirely necessary, since the choice of the pair (r,π) may be restricted to those with particular properties:

PROPOSITION 2.9. *Let* (Γ,γ) *be an (n+1)-coloured graph, and let* H *be any subgroup of the group* $\bar{H}(\Gamma)$. *If the* H*-code* $c_H(\Gamma)$ *is obtained* *from the elements of the $(\bar{r},\bar{\pi})$-code $c_{\bar{r},\bar{\pi}}(\Gamma)$, then the $\{\bar{\pi}(0),\bar{\pi}(1)\}$-residue of Γ containing the root $\bar{r}\in V(\Gamma)$ attains a maximum length among all $\{c,d\}$-residues of (Γ,γ), with $\{c,d\}=\{\pi(0),\pi(1)\}$ for some $\pi\in H$.*

In particular, if (Γ,γ) is an (n+1)-coloured graph, with empty (resp. non empty) boundary, and the code $c(\Gamma)$ is obtained from the elements of the $(\bar{r},\bar{\pi})$-code $c_{\bar{r},\bar{\pi}}(\Gamma)$, then the $\{\bar{\pi}(0),\bar{\pi}(1)\}$-residue of (Γ,γ) containing the root $\bar{r}\in V(\Gamma)$ attains a maximum length among all the 2-residues of (Γ,γ) (resp. of $\Gamma_{\hat{n}}$).

Proof. It is easy to check that the first part of the algorithm defining $N_{r,\pi}$ visits and labels all vertices of the $\{\pi(0),\pi(1)\}$-residue containing the root r, starting from r itself, following alternatively $\pi(0)$- and $\pi(1)$-adjacencies, until the $\pi(1)$-adjacent vertex of r is reached.

Thus, the thesis directly follows from the fact that, for every pair (r,π), the first element of the (r,π)-code $c_{r,\pi}(\Gamma)$ is $c_{1,1}=a_1^{-1}\in A_{r,\pi}(\Gamma)$, i.e. the positive integer $+l$ such that the (regular) $\{\pi(0),\pi(1)\}$-residue of (Γ,γ) containing r has length $2l$. □

PROPOSITION 2.10. *Let (Γ,γ), (Γ',γ') be two (n+1)-coloured graphs, and let H be any subgroup of the group $\bar{H}(\Gamma)$. Then, (Γ,γ), (Γ',γ') are H-isomorphic if and only if $c_H(\Gamma)=c_H(\Gamma')$.*

In particular, (Γ,γ), (Γ',γ') are isomorphic if and only if $c(\Gamma)=c(\Gamma')$.

Proof. If (Γ,γ), (Γ',γ') are H-isomorphic graphs (for any fixed subgroup H of $\bar{H}(\Gamma)$), they have obviously the same order - $2q$, say - and the same bipartition and regularity properties: thus, $\bar{H}(\Gamma)=\bar{H}(\Gamma')$, $\bar{m}(\Gamma)=\bar{m}(\Gamma')=\bar{m}$, and the length of any (r,π)-code of (Γ,γ) equals the length of any (r',π')-code of (Γ',γ'), with $r\in V(\Gamma)$, $r'\in V(\Gamma')$, $\pi,\pi'\in\bar{H}(\Gamma)=\bar{H}(\Gamma')$. Moreover, the H-isomorphism of (Γ,γ) and (Γ',γ') means the existence of a permutation $\sigma\in H$ and a graph isomorphism $\phi:\Gamma\to\Gamma'$ such that

$$\gamma'\circ\phi=\sigma\circ\gamma.$$

It is now easy to check that this implies, for every vertex $r\in V(\Gamma)$ and for every colour permutation $\pi=(\pi(0),\dots,\pi(n))\in H\subset\bar{H}(\Gamma)$, $A_{r,\pi}(\Gamma)=A_{\phi(r),\sigma\circ\pi}(\Gamma')$. The equality $c_H(\Gamma)=c_H(\Gamma')$ directly follows.

On the other hand, let us assume $c_H(\Gamma) = c_H(\Gamma')$, for any fixed subgroup H of $\bar{H}(\Gamma)$.

The common order of Γ and Γ' is $2q$, q being the maximum integer such that $+q$ belongs to the word $c_H(\Gamma) = c_H(\Gamma')$; further, the integer m may be easily computed, by recalling that the length of $c_H(\Gamma) = c_H(\Gamma')$ is $((2n - m + 1) \times q)$. Moreover, both $\partial\Gamma$ and $\partial\Gamma'$ are empty if and only if $c_H(\Gamma) = c_H(\Gamma')$ contains no 0 element; so, the equality $\bar{H}(\Gamma) = \bar{H}(\Gamma')$ obviously holds. Then, $c_H(\Gamma) = c_H(\Gamma')$ implies the existence of a vertex $r \in V(\Gamma)$, a vertex $r' \in V(\Gamma')$ and two colour permutations $\pi, \pi' \in H \subset \bar{H}(\Gamma) = \bar{H}(\Gamma')$, such that $A_{r,\pi}(\Gamma) = A_{r',\pi'}(\Gamma')$. It is now easy to check the existence of a graph isomorphism $\phi : \Gamma \to \Gamma'$, compatible with the permutation $\pi' \circ \pi^{-1} \in H$, uniquely determined by $\phi(r) = r'$. This concludes our proof. $\square$

In case of sufficiently "small" graphs (i.e., for $q \leq 26$), it is convenient - for sake of notational simplicity - to write down an alphanumerical version of the code $c(\Gamma)$, by substituting the elements of $\{-i \in \mathbb{Z}/1 \leq i \leq q\}$ (resp. $\{+i \in \mathbb{Z}/1 \leq i \leq q\}$) with the first q small (resp. capital) letters, in orderly way.

Thus, for example, the following codes

$$c(\Gamma_2) = CABCABBcbbcaCaA$$

$$c(\Gamma_3) = CABBcbCA0CaAb0a$$

$$c(\Gamma_4) = CABDDCBA00000D0B$$

$$c(\Gamma_5) = CABDDCBA00D0$$

$$c(\Gamma_6) = CABDDCBAACdcaDbB$$

$$c(\Gamma_7) = CABDDCBAACDB$$

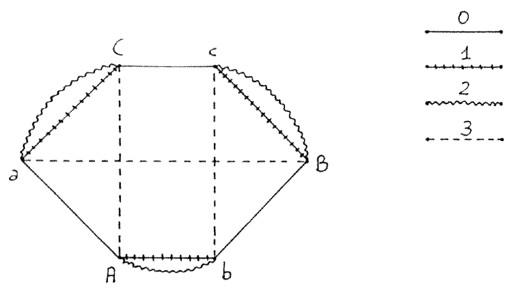
$$c(\Gamma_8) = CABDGEFHDCBAHFEGEGDH0C0F$$

$$c(\Gamma_9) = DABCFEHGHGECDBFADAGFHBCE$$

$$c(\Gamma_{10}) = CABDFEGDCBAFEGE000AGFeD0Bagf$$

$$c(\Gamma_{11}) = FABCDEHGHEDCBAFGFgBCDAbGfcdeHahE$$

identify the edge-coloured graphs (Γ_j, γ_j) $(j = 2, 3, \ldots, 11)$ depicted in Fig. j, and every isomorphic graph. In every graph, the pair

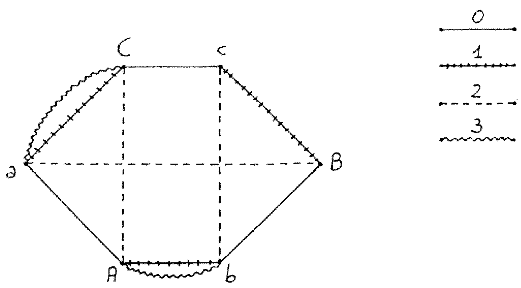


(Γ_2, γ_2)

Figure 2

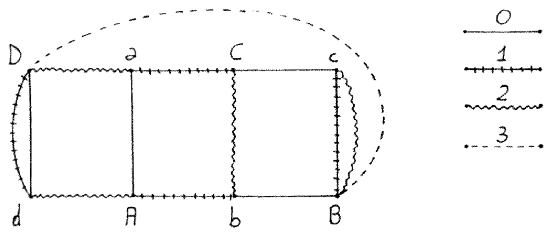
(r, π) attaining the code is identified by means of the labelling of the vertices and of the legend of the colour set.

In every graph, the pair (r, π) attaining the code is identified by means of the labelling of the vertices and of the legend of the colour set.



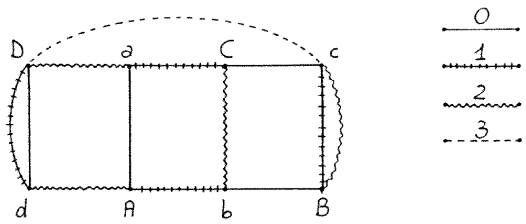
(Γ_3, γ_3)

Figure 3



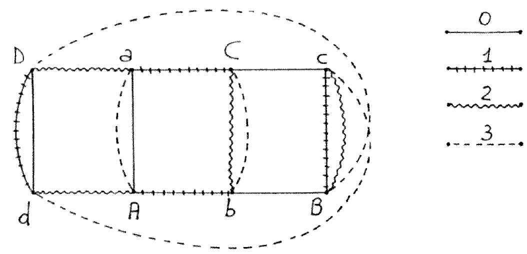
(Γ_4, γ_4)

Figure 4



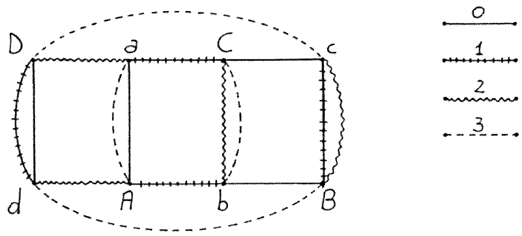
(Γ_5, γ_5)

Figure 5



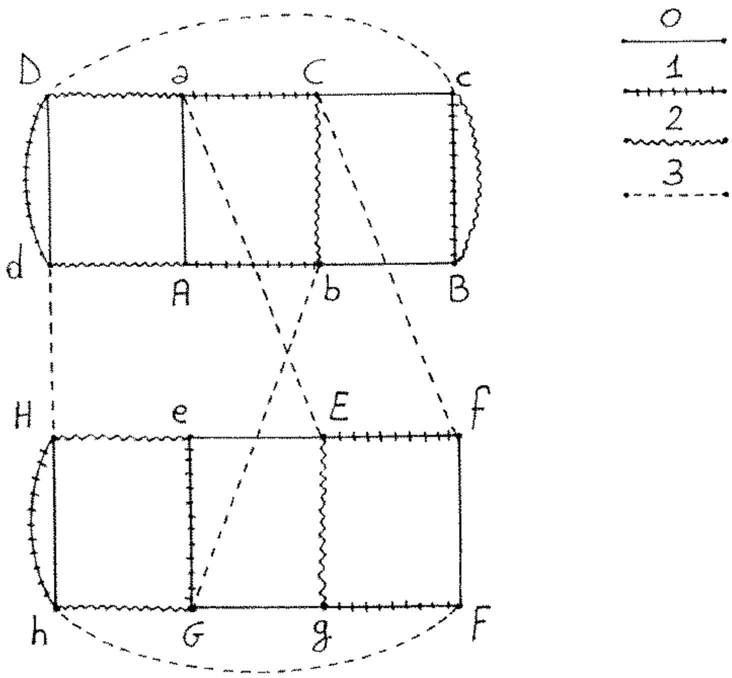
(Γ_6, γ_6)

Figure 6



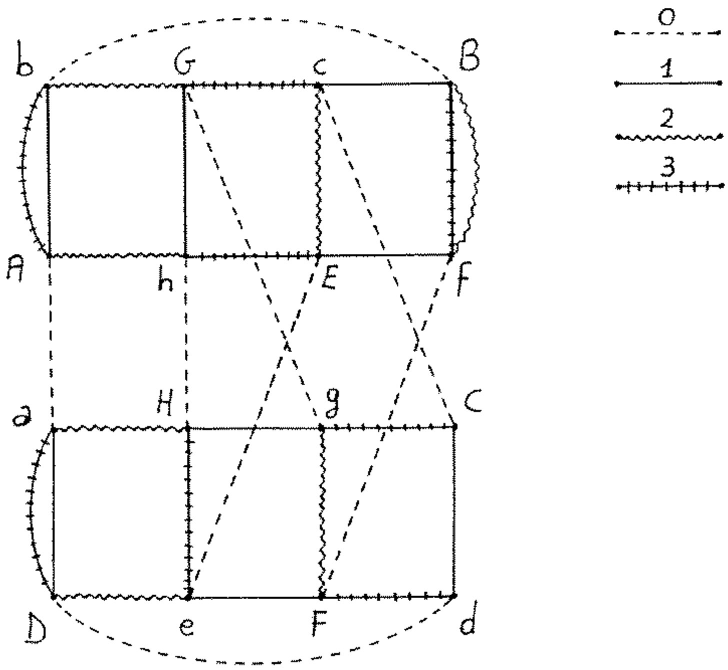
(Γ_7, γ_7)

Figure 7



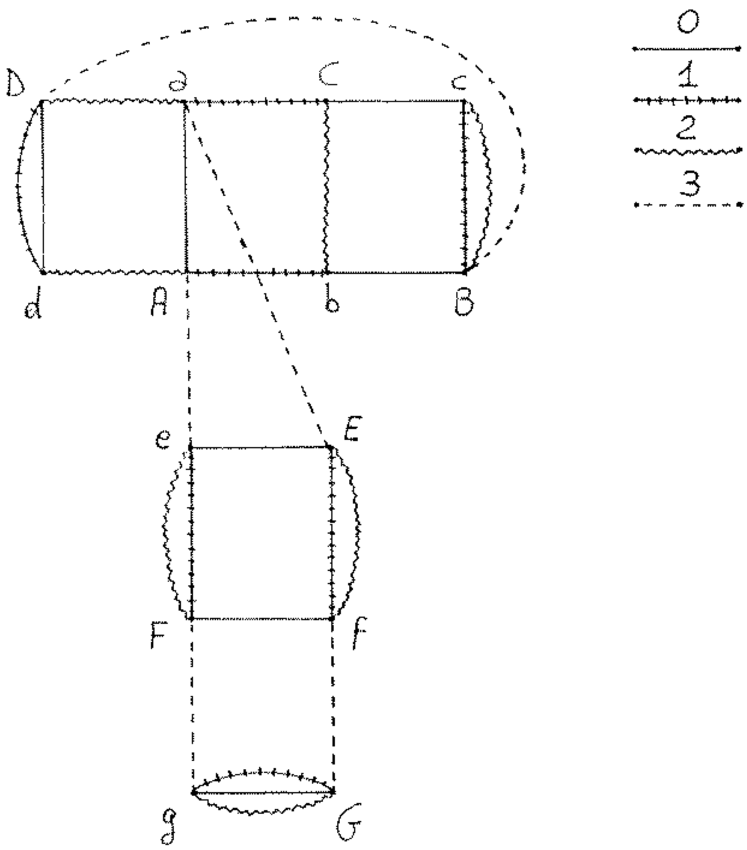
(Γ_8, γ_8)

Figure 8



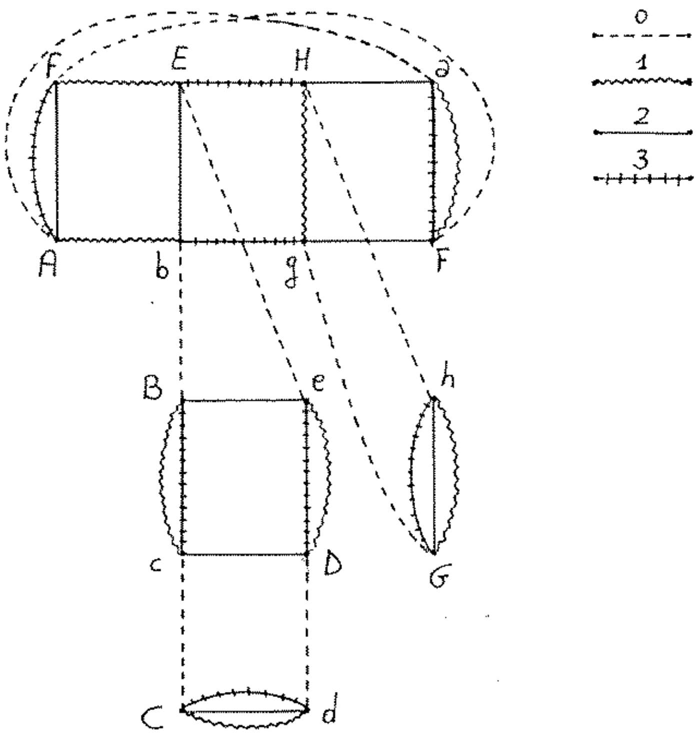
(Γ_9, γ_9)

Figure 9



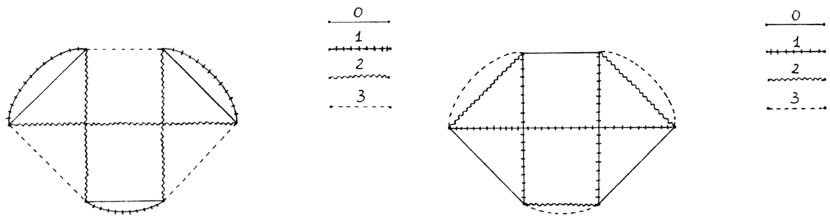
$(\Gamma_{10}, \gamma_{10})$

Figure 10



$(\Gamma_{11}, \gamma_{11})$

Figure 11



(Λ_1, λ_1) (Λ_2, λ_2)

Figure 12

Finally, note that both graphs (Λ_1, λ_1) and (Λ_2, λ_2) depicted in Fig. 12 are (colour)-isomorphic to the graph (Γ_2, γ_2) depicted in Fig. 2, but neither (Λ_1, λ_1) nor (Λ_2, λ_2) is strictly isomorphic to (Γ_2, γ_2), as the following codes say:

$$c(\Gamma_2) = c(\Lambda_1) = c(\Lambda_2) = CABCABBcbbcaCaA$$

$$c_{\{id\}}(\Gamma_2) = CABCABBcbbcaCaA$$

$$c_{\{id\}}(\Lambda_1) = ABCCABBbaAbcacCB$$

$$c_{\{id\}}(\Lambda_2) = CABbaAbaAcCBcCB$$

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