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Verlinde-type Formulae and Twistor Transform

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Summary. - *We study certain aspects of the topology of six moduli spaces of orthogonal vector bundles over Riemann surfaces, with genus between 2 and 7, in order to find generalizations of the well-known Verlinde formula and the Newstead conjectures.*

1. Introduction

In this note we investigate some aspects of the topology (cohomology) of a moduli space $\mathcal{M}_g$ of orthogonal vector bundles over a hyperelliptic Riemann surface Σ of genus g, with $2 \leq g \leq 7$. This space is a Kähler manifold and is endowed with a positive line bundle L. Motivated by the study of the Verlinde formulae for moduli spaces carried out in [11, 12, 13, 1, 15, 4, 6], we set out to compute the following invariants: the dimension of the space of holomorphic sections $H^0(\mathcal{M}_g, \mathcal{O}(L^k))$ (the *Verlinde-type formulae*), and the intersection numbers of a sub-ring of $H^*(\mathcal{M}_g)$ generated by two classes l and v (see below). In fact, the space $\mathcal{M}_g$ is a complex submanifold of a twistor space $\mathcal{F}_g$ of a (quaternion-Kähler) real Grassmannian $\mathcal{G}_g$. We find that all the relevant classes in the Riemann-Roch formula for $H^0(\mathcal{M}_g, \mathcal{O}(L^k))$ are given in terms of the classes l and v which arise from the twistor fibration $\mathcal{F}_g \longrightarrow \mathcal{G}_g$.

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In §2, we give the definition of the moduli space $\mathcal{M}_g$ and quote a theorem of S. Ramanan which identifies it as an intersection of quadratic varieties in a complex Grassmannian [9]. In §3, we study the cohomology of the Grassmannian $\mathcal{G}_g$ by using a K-theoretical decomposition of its tangent bundle and some vanishing theorems. We compute intersection numbers in $\mathcal{G}_g$ which, via twistor transform, will give the required intersection numbers in $\mathcal{M}_g$. In §4, we recall some properties of the flag manifold $\mathcal{F}_g$ as a twistor space of $\mathcal{G}_g$, and compute the total Chern and Pontrjagin classes of $\mathcal{M}_g$. In §5, we compute the intersection numbers for the sub-ring generated by l and v in $H^*(\mathcal{M}_g)$. Moreover, we prove the vanishing of all the Pontrjagin numbers and of the top-two Chern classes of $\mathcal{M}_g$ which, in fact, constitute Newstead-type vanishings (cf. [8, Conectures (a),(b)]). We conclude by computing the Verlinde-type formula $h^0(\mathcal{M}_g, \mathcal{O}(L^k))$. This work is intended as an extension of the results proved in [4, 11].

2. The moduli space $\mathcal{M}_g$

Let Σ be a hyper-elliptic Riemann surface of genus g with involution $\mathrm{i}\colon \Sigma \longrightarrow \Sigma$ and Weierstrass points $\{\omega_1, \ldots, \omega_{2g+2}\}$. Consider the special Clifford group $SC(2g-2) = \mathbb{C}^* \times_{\mathbb{Z}_2} Spin(2g-2)$, which fits in the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & SC(2g-2) & \longrightarrow & SO(2g-2) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & Spin(2g-2) & \longrightarrow & SO(2g-2) & \longrightarrow & 1 \end{array}$$

Let $\mathcal{M}_g$ denote the moduli space of semistable, holomorphic, rank $2g-2$, vector bundles E over Σ with the following properties:

- E is an (orthogonal) vector bundle with structure group $SO(2g-2)$ and with a lift of structure group to $SC(2g-2)$.
- E is i-invariant, i.e. there is a lift of i to E (denoted by the same symbol) such that $E \cong \mathrm{i}^*E$. Thus, we have the restrictions of

i to the fibers over the Weierstrass points $\mathrm{i}\colon E_{\omega_j} \longrightarrow E_{\omega_j}$, for all $j = 1, \dots, 2g+2$. Since $\mathrm{i}^2 = 1$, the eigenvalues of i on these fibers are ± 1, and we denote the eigenspace by $E^{\pm}_{\omega_j}$.

- E is such that $\dim((E \otimes \Lambda)^{-}_{\omega_j}) = 1$ for all $j = 1, \dots, 2g+2$, where Λ is an i-invariant line bundle over Σ of degree $2g-1$.

Examples.

1. Case $g = 2$. Since $SO(2) \cong U(1)$, $\mathcal{M}_2$ is the Jacobian $J(\Sigma)$ of Σ_2 (cf. [9]).

2. Case $g = 3$. The special Clifford group is

$$SC(4) = \{(A, B) \in Gl(2) \times Gl(2) \,|\, \det(A) \cdot \det(B) = 1\}$$

and the homomorphism $SC(4) \longrightarrow SO(4)$ is given by $(A, B) \longrightarrow A \otimes B$. Thus a $SC(4)$-bundle is essentially a pair of $Gl(2)$-bundles M, N with $\det(M) \otimes \det(N) = 1$ a trivial bundle. Since the Clifford group $C(4)$ does not distinguish between M and N, we have that $\mathcal{M}_2$ is the moduli space of (stable) vector bundles of rank 2 and fixed odd determinant (cf. [9]).

In [9, Theorem 3], Ramanan proved that the moduli space $\mathcal{M}_g$ is isomorphic to the variety of 2-dimensional subspaces of $\mathbb{C}^{2g+2}$ which are isotropic with respect to the two quadratic forms

$$\sum_{i=1}^{2g+2} y_i^2, \qquad \sum_{i=1}^{2g+2} \omega_i y_i^2. \tag{1}$$

Therefore, we have a holomorphic embedding of $\mathcal{M}_g$ into the complex partial flag manifold

$$\mathcal{F}_g = \frac{SO(2g+2)}{U(2) \times SO(2g-2)}$$

which parameterizes the 2-dimensional subspaces of $\mathbb{C}^{2g+2}$ which are isotropic with respect to the fist quadratic form.

The flag manifold $\mathcal{F}_g$ is the twistor space of the real Grassmannian

$$\mathcal{G}_g = \frac{SO(2g+2)}{SO(4)\times SO(2g-2)},$$

since the fiber $\mathbb{CP}^1 = SO(4)/U(2)$ parameterizes orthogonal almost complex structures on the real oriented 4-dimensional subspaces of $\mathbb{R}^{2g+2}$, compatible with the orientation [10, 7]. We shall study the topology of $\mathcal{M}_g$ via this embedding into $\mathcal{F}_g$ and the twistor fibration $\mathcal{F}_g \longrightarrow \mathcal{G}_g$.

3. Cohomology of the real Grassmannian $\mathcal{G}_g$

Let $\mathcal{G}_g$ be the real Grassmannian ($g \geq 2$)

$$\mathcal{G}_g = \frac{SO(2g+2)}{SO(4)\times SO(2g-2)}$$

parameterizing real oriented 4-dimensional subspaces of $\mathbb{R}^{2g+2}$. The isotropy group is contained in $Sp(2g-2)Sp(1)$ making $\mathcal{G}_g$ into a quaternion-Kähler manifold [14, 10]. Let W be the tautological $SO(4)$-bundle over $\mathcal{G}_g$ and $W^\perp$ its orthogonal complement in the trivial bundle with fiber $\mathbb{R}^{2g+2}$. The homogeneous bundle $W^\perp$ corresponds to the fundamental representation of $SO(2g-2)$, so the tangent bundle of $\mathcal{G}_g$ factors as follows

$$T\mathcal{G}_g = W \otimes W^\perp.$$

Since $SO(4) \cong Sp(1)Sp(1) \cong SU(2)SU(2)$,

$$W_c = U \otimes_c V$$

where U, V are two copies of the fundamental representation of $SU(2)$, and the subscript $_c$ denotes complexification. Thus,

$$(T\mathcal{G}_g)_c = U \otimes (V \otimes W_c^\perp)$$

where U may be thought of as a (locally defined) quaternionic line bundle.

We shall consider the sub-ring of $H^*(\mathcal{G}_g)$ generated by the *quaternionic* classes (cf. [10])

$$u = -c_2(U) \in H^4(\mathcal{G}_g), \qquad v = -c_2(V) \in H^4(\mathcal{G}_g).$$

Although u and v are not integral classes, their multiples $4u$, $4v$ are integral since the vector bundles S^2U, S^2V are globally defined over $\mathcal{G}_g$. The Poincaré polynomials of $\mathcal{G}_g$ for $2 \leq g \leq 7$ are

$$\begin{aligned}
P_t(\mathcal{G}_2) &= 1 + t^2 + 2t^4 + t^6 + t^8, \\
P_t(\mathcal{G}_3) &= 1 + 3t^4 + 4t^8 + 3t^{12} + t^{16}, \\
P_t(\mathcal{G}_4) &= 1 + 2t^4 + t^6 + 3t^8 + t^{10} + 4t^{12} + t^{14} + 3t^{16} + t^{18} + 2t^{20} \\
&\quad + t^{24}, \\
P_t(\mathcal{G}_5) &= 1 + 2t^4 + 4t^8 + 5t^{12} + 6t^{16} + 5t^{20} + 4t^{24} + 2t^{28} + t^{32}, \\
P_t(\mathcal{G}_6) &= 1 + 2t^4 + 3t^8 + t^{10} + 4t^{12} + t^{14} + 5t^{16} + t^{18} + 6t^{20} + t^{22} \\
&\quad + 5t^{24} + t^{26} + 4t^{28} + t^{30} + 3t^{32} + 2t^{36} + t^{40}, \\
P_t(\mathcal{G}_7) &= 1 + 2t^4 + 3t^8 + 5t^{12} + 6t^{16} + 7t^{20} + 8t^{24} + 7t^{28} + 6t^{32} \\
&\quad + 5t^{36} + 3t^{40} + 2t^{44} + t^{48},
\end{aligned}$$

confirming that there is only one more class apart from u and v appearing in dimension $2g - 2$.

Let l and $\hat{l}$ be formal roots such that $4u = l^2$ and $4v = \hat{l}^2$. Thus,

$$\mathrm{ch}(U) = e^{\frac{l}{2}} + e^{-\frac{l}{2}} = 2 + u + \frac{1}{12}u^2 + \frac{1}{360}u^3 + \frac{1}{20160}u^4 + \dots,$$

$$\mathrm{ch}(V) = e^{\frac{\hat{l}}{2}} + e^{-\frac{\hat{l}}{2}} = 2 + v + \frac{1}{12}v^2 + \frac{1}{360}v^3 + \frac{1}{20160}v^4 + \dots$$

The identity of vector bundles on $\mathcal{G}_g$

$$W \oplus W^{\perp} = 2g + 2,$$

gives in K-theory

$$W^{\perp} = 2g + 2 - W,$$

so that

$$T\mathcal{G}_g = (2g + 2)W - W^{\otimes 2}.$$

Therefore, the total Chern and Pontrjagin classes of $T\mathcal{G}_g$ can be expressed in terms of u and v as follows

$$c((T\mathcal{G}_g)_c) = \frac{(1-2(u+v)+(u-v)^2)^{2g+2}}{(1-4u)^2(1-4v)^2(1-8(u+v)+16(u-v)^2)},$$

$$p(T\mathcal{G}_g) = \frac{(1+2(u+v)+(u-v)^2)^{2g+2}}{(1+4u)^2(1+4v)^2(1+8(u+v)+16(u-v)^2)}.$$

Furthermore,

$$\widehat{A}(\mathcal{G}_g) = \left(\frac{\sqrt{u}+\sqrt{v}}{\sinh(\sqrt{u}+\sqrt{v})}\frac{\sqrt{u}-\sqrt{v}}{\sinh(\sqrt{u}-\sqrt{v})}\right)^{2g+2} \times$$

$$\times \frac{\sinh(2(\sqrt{u}+\sqrt{v}))}{2(\sqrt{u}+\sqrt{v})}\ \frac{\sinh(2(\sqrt{u}-\sqrt{v}))}{2(\sqrt{u}-\sqrt{v})}\left(\frac{\sinh(2\sqrt{u})}{2\sqrt{u}}\ \frac{\sinh(2\sqrt{v})}{2\sqrt{v}}\right)^2.$$

We know that $\mathcal{G}_g$ is a spin manifold [7], and therefore there is a Dirac operator D acting on sections of the spin bundle Δ. Let $E = V \otimes W_c^{\perp}$. Thus, Δ decomposes as $\Delta_+ \oplus \Delta_-$, where

$$\Delta_+ = S^{2g-2}U \oplus S^{2g-4}U \otimes \textstyle\bigwedge_0^2 E \oplus \ldots \oplus \bigwedge_0^{2g-2} E,$$

$$\Delta_- = S^{2g-3}U \otimes E \oplus S^{2g-5}U \otimes \textstyle\bigwedge_0^3 E \oplus \ldots \oplus U \otimes \bigwedge_0^{2g-3} E,$$

over $\mathcal{G}_g$. If F is a vector bundle over $\mathcal{G}_g$ equipped with a connection, one can extend the Dirac operator D to an elliptic operator with coefficients in F

$$D_F\colon \Gamma(\Delta_+ \otimes F) \longrightarrow \Gamma(\Delta_- \otimes F),$$

whose index is by definition $\dim(\ker D_F) - \dim(\mathrm{coker}\ D_F)$. In [10, 7], Salamon proved the following

$$\mathrm{ind}(D_{S^{2g-2+2k}U}) = \begin{cases} 0 & \text{if } k = 1, \ldots, \left[\frac{2g-1}{2}\right], \frac{2g-1}{2}, \\ 1 & \text{if } k = 0. \end{cases} \tag{2}$$

By the Atiyah-Singer index theorem,

$$\mathrm{ind}(D_{S^{2g-2+2k}U}) = \langle \mathrm{ch}(S^{2g-2+2k}U)\widehat{A}(\mathcal{G}_g), [\mathcal{G}_g]\rangle,$$

where

$$\operatorname{ch}(S^p U) = \frac{\sinh((p+1)\sqrt{u})}{\sinh(\sqrt{u})}.$$

The identities in (2) give enough linear equations to compute the intersection numbers

$$\langle u^i v^j, [\mathcal{G}_g]\rangle$$

where $i + j = 2g - 2$, given that

$$\langle u^i v^j, [\mathcal{G}_g]\rangle = \langle u^j v^i, [\mathcal{G}_g]\rangle$$

due to the symmetry between the bundles U and V. We define the *quaternionic volume* of $\mathcal{G}_g$ to be

$$\operatorname{vol}(\mathcal{G}_g) = \langle (4u)^{2g-2}, [\mathcal{G}_g]\rangle = \langle (4v)^{2g-2}, [\mathcal{G}_g]\rangle,$$

which was computed in [3].

PROPOSITION 3.1. *Evaluation on the fundamental class $[\mathcal{G}_g]$ yields*

$$\operatorname{vol}(\mathcal{G}_g) = \frac{2}{g}\binom{4g-3}{2g-1};$$

for $g = 2$,

$$\langle 4^2 uv, [\mathcal{G}_2]\rangle = -\frac{3\operatorname{vol}(\mathcal{G}_2)}{5};$$

for $g = 3$,

$$\langle 4^4 u^3 v, [\mathcal{G}_3]\rangle = -\frac{\operatorname{vol}(\mathcal{G}_3)}{3}, \qquad \langle 4^4 u^2 v^2, [\mathcal{G}_3]\rangle = \frac{5\operatorname{vol}(\mathcal{G}_3)}{21};$$

for $g = 4$,

$$\langle 4^6 u^5 v, [\mathcal{G}_4]\rangle = -\frac{3\operatorname{vol}(\mathcal{G}_4)}{13}, \qquad \langle 4^6 u^4 v^2, [\mathcal{G}_4]\rangle = \frac{15\operatorname{vol}(\mathcal{G}_4)}{143},$$

$$\langle 4^6 u^3 v^3, [\mathcal{G}_4]\rangle = -\frac{35\operatorname{vol}(\mathcal{G}_4)}{429};$$

for $g = 5$,

$$\langle 4^8 u^7 v, [\mathcal{G}_5]\rangle = -\frac{3\operatorname{vol}(\mathcal{G}_5)}{17}, \qquad \langle 4^8 u^6 v^2, [\mathcal{G}_5]\rangle = \frac{1\operatorname{vol}(\mathcal{G}_5)}{17},$$

$$\langle 4^8u^5v^3, [\mathcal{G}_5]\rangle = -\frac{7\,\mathrm{vol}(\mathcal{G}_5)}{221}, \qquad \langle 4^8u^4v^4, [\mathcal{G}_5]\rangle = \frac{63\,\mathrm{vol}(\mathcal{G}_5)}{2431};$$

for $g = 6$,

$$\langle 4^8u^9v, [\mathcal{G}_6]\rangle = -\frac{1\,\mathrm{vol}(\mathcal{G}_6)}{7}, \qquad \langle 4^8u^8v^2, [\mathcal{G}_6]\rangle = \frac{5\,\mathrm{vol}(\mathcal{G}_6)}{133},$$

$$\langle 4^8u^7v^3, [\mathcal{G}_6]\rangle = -\frac{5\,\mathrm{vol}(\mathcal{G}_6)}{323}, \qquad \langle 4^8u^6v^4, [\mathcal{G}_6]\rangle = \frac{3\,\mathrm{vol}(\mathcal{G}_6)}{323},$$

$$\langle 4^8u^5v^5, [\mathcal{G}_6]\rangle = -\frac{33\,\mathrm{vol}(\mathcal{G}_6)}{4199};$$

for $g = 8$,

$$\langle 4^8u^{11}v, [\mathcal{G}_7]\rangle = -\frac{3\,\mathrm{vol}(\mathcal{G}_7)}{25}, \qquad \langle 4^8u^{10}v^2, [\mathcal{G}_7]\rangle = \frac{3\,\mathrm{vol}(\mathcal{G}_7)}{115},$$

$$\langle 4^8u^6v^3, [\mathcal{G}_7]\rangle = -\frac{1\,\mathrm{vol}(\mathcal{G}_7)}{115}, \qquad \langle 4^8u^8v^4, [\mathcal{G}_7]\rangle = \frac{9\,\mathrm{vol}(\mathcal{G}_7)}{2185},$$

$$\langle 4^8u^7v^5, [\mathcal{G}_7]\rangle = -\frac{99\,\mathrm{vol}(\mathcal{G}_7)}{37145}, \qquad \langle 4^8u^6v^6, [\mathcal{G}_7]\rangle = \frac{429\,\mathrm{vol}(\mathcal{G}_7)}{185725}.$$

Note that we have only missed the intersection numbers involving the extra cohomology class appearing in dimension $2g-2$.

4. The space $\mathcal{F}_g$ and the cohomology of $\mathcal{M}_g$

The complex manifold $\mathcal{F}_g$ has complex dimension $4g-3$, and parameterizes complex 2-dimensional subspaces Π of $\mathbb{C}^{2g+2}$ which are isotropic with respect to the standard $SO(2g+2)$-invariant bilinear form. It is a contact Kähler-Einstein manifold [7], which projects onto $\mathcal{G}_g$, $\pi\colon \mathcal{F}_g \longrightarrow \mathcal{G}_g$, by sending Π to the 4-dimensional subspace of $\mathbb{R}^{2g+2}$ whose complexification is $\Pi \oplus \overline{\Pi}$. Each fiber is isomorphic to a rational curve $SO(4)/U(2) \cong \mathbb{CP}^1$ in $\mathcal{F}_g$.

$\mathrm{Pic}(\mathcal{F}_g)$ is generated by a line bundle $L \longrightarrow \mathcal{F}_g$ such that (cf. [7])

1. $L\,|_{\pi^{-1}(x)} = \mathcal{O}(2)$ on $\pi^{-1}(x) \cong \mathbb{CP}^1$.
2. L^{2g-1} is isomorphic to the anti-canonical bundle $K_{\mathcal{F}}^{-1}$ of $\mathcal{F}_g$.
3. If Q denotes the dual of the tautological $U(2)$ bundle over $\mathcal{F}_g$, $L = \det(Q)$.

The holomorphic tangent bundle of $\mathcal{F}_g$ satisfies

$$T^{1,0}\mathcal{F}_g = Q \otimes W_c^{\perp} \oplus \bigwedge^2 Q = Q \otimes W_c^{\perp} \oplus L,$$

where

$$Q \oplus Q^* \oplus W_c^{\perp} = 2g + 2.$$

There is a local C^∞ isomorphism

$$\pi^* U = L^{1/2} \oplus L^{-1/2}.$$

Let $l = c_1(L) \in H^2(\mathcal{F}_g, \mathbb{Z})$, so that by the Leray-Hirsch Theorem

$$\left(\frac{l}{2}\right)^2 + \pi^* c_2(U) = 0,$$

i.e. $l^2 = 4u$ (omitting π^*).

As we mentioned in §2, the spaces $\mathcal{M}_g$ can be identified as the zero set of a non-degenerate holomorphic section $s \in H^0(\mathcal{F}_g, \mathcal{O}(S^2 Q))$, which corresponds to a quadratic form on $\mathbb{C}^{2g+2}$. Note that $Q = L^{1/2} \otimes \pi^* V$ over $\mathcal{F}_g$, so that $S^2 Q = L \otimes \pi^* S^2 V$, where $S^2 V$ is trivial over each fiber $\pi^{-1}(x) \cong \mathbb{CP}^1$. The complex dimension of $\mathcal{M}_g$ is $4g - 6$.

Since the normal bundle of $\mathcal{M}_g$ in $\mathcal{F}_g$ is isomorphic to $S^2 Q$, the holomorphic tangent bundle of $\mathcal{M}_g$ decomposes K-theoretically as follows

$$\begin{aligned} T^{1,0}\mathcal{M}_g &= Q \otimes W_c^{\perp} - \psi^2 Q \\ &= (2g+2) V \otimes L^{1/2} - 2\psi^2 V \otimes L - 2L - \psi^2 V - 2, \end{aligned}$$

where we have dropped π^* from the notation and ψ^2 denotes the second Adams operator on vector bundles [2]. From this we deduce

$$c(\mathcal{M}_g) = \frac{((1 + l/2 + \hat{l}/2)(1 + l/2 - \hat{l}/2))^{2g+2}}{(1 + l + \hat{l})^2 (1 + l - \hat{l})^2 (1 + l)^2 (1 - \hat{l}^2)}, \tag{3}$$

where $\hat{l}$ is defined formally to be $2\sqrt{v}$, and we denote by u and v the pull-backs to $\mathcal{M}_g$ of the quaternionic classes on $\mathcal{G}_g$. Thus,

$$p(\mathcal{M}_g) = \frac{(1 + 2(u+v) + (u-v)^2)^{2g+2}}{(1+4u)^2 (1+4v)^2 (1 + 8(u+v) + 16(u-v)^2)^2}, \tag{4}$$

and,

$$\widehat{A}(\mathcal{M}_g) = \left(\frac{\sqrt{u}+\sqrt{v}}{\sinh(\sqrt{u}+\sqrt{v})} \frac{\sqrt{u}-\sqrt{v}}{\sinh(\sqrt{u}-\sqrt{v})} \right)^{2g+2} \times$$

$$\times \left(\frac{\sinh(2(\sqrt{u}+\sqrt{v}))}{2(\sqrt{u}+\sqrt{v})} \ \frac{\sinh(2(\sqrt{u}-\sqrt{v}))}{2(\sqrt{u}-\sqrt{v})} \ \frac{\sinh(2\sqrt{u})}{2\sqrt{u}} \ \frac{\sinh(2\sqrt{v})}{2\sqrt{v}} \right)^2 . \tag{5}$$

5. Intersection numbers on $\mathcal{M}_g$ and Verlinde-type formulae

THEOREM 5.1. *The intersection numbers $\langle u^i v^j, [\mathcal{M}_g] \rangle$, where $i+j = 4g-6$, are skew-symmetric in u and v. Evaluating on the fundamental class $[\mathcal{M}_g]$ yields: for $g = 2$,*

$$\big\langle u, [\mathcal{M}_2] \big\rangle = 8 = -\big\langle v, [\mathcal{M}_2] \big\rangle;$$

for $g = 3$,

$$\big\langle u^3, [\mathcal{M}_3] \big\rangle = \frac{7}{2}, \qquad \big\langle u^2 v, [\mathcal{M}_3] \big\rangle = -\frac{3}{2};$$

for $g = 4$,

$$\big\langle u^5, [\mathcal{M}_4] \big\rangle = \frac{99}{8}, \qquad \big\langle u^4 v, [\mathcal{M}_4] \big\rangle = -\frac{27}{8}, \qquad \big\langle u^3 v^2, [\mathcal{M}_4] \big\rangle = \frac{15}{8};$$

for $g = 5$,

$$\big\langle u^7, [\mathcal{M}_5] \big\rangle = \frac{715}{512}, \qquad \big\langle u^6 v, [\mathcal{M}_5] \big\rangle = -\frac{143}{512},$$

$$\big\langle u^5 v^2, [\mathcal{M}_5] \big\rangle = \frac{55}{512}, \qquad \big\langle u^4 v^3, [\mathcal{M}_5] \big\rangle = -\frac{35}{512};$$

for $g = 6$,

$$\big\langle u^9, [\mathcal{M}_6] \big\rangle = \frac{4199}{4096}, \qquad \big\langle u^8 v, [\mathcal{M}_6] \big\rangle = -\frac{663}{4096}, \qquad \big\langle u^7 v^2, [\mathcal{M}_6] \big\rangle = \frac{195}{4096},$$

$$\big\langle u^6 v^3, [\mathcal{M}_6] \big\rangle = -\frac{91}{4096}, \qquad \big\langle u^5 v^4, [\mathcal{M}_6] \big\rangle = -\frac{63}{4096};$$

for $g = 7$,

$$\big\langle u^{11}, [\mathcal{M}_7] \big\rangle = \frac{52003}{65536}, \qquad \big\langle u^{10} v, [\mathcal{M}_7] \big\rangle = -\frac{6783}{65536},$$

$$\langle u^9v^2, [\mathcal{M}_7]\rangle = \frac{1615}{65536}, \qquad \langle u^8v^3, [\mathcal{M}_7]\rangle = -\frac{595}{65536},$$
$$\langle u^7v^4, [\mathcal{M}_7]\rangle = \frac{315}{65536}, \qquad \langle u^6v^5, [\mathcal{M}_7]\rangle = -\frac{231}{65536}.$$

Proof. As a submanifold of $\mathcal{F}_g$, $\mathcal{M}_g$ is Poincaré dual to the Euler class $c_3(S^2Q)$, which is easily computed from the identity $S^2Q = L \otimes \pi^* S^2V$ and is equal to $4l(u-v)$. Hence, for example,

$$\langle u^3, [\mathcal{M}_3]\rangle = \langle 4lu^3(u-v), [\mathcal{F}_3]\rangle = 8\langle u^4 - u^3v, [\mathcal{G}_3]\rangle = \frac{7}{2}$$

where the second equality follows from twistor transform. Similarly for all the other pairings. □

The expressions (4) and (5) are symmetric in u and v. Thus, we have the following.

COROLLARY 5.1. *For $2 \leq g \leq 7$, all the Pontrjagin numbers vanish, in particular,*

$$\widehat{A}_{2g-3}(\mathcal{M}_g) = 0.$$

Furthermore, the Chern classes

$$c_{4g-6}(\mathcal{M}_g) = c_{4g-7}(\mathcal{M}_g) = 0,$$

the Euler characteristic of $\mathcal{M}_g$ vanishes

$$\chi(\mathcal{M}_g) = 0,$$

and

$$\chi(\mathcal{M}_g, \mathcal{O}(T^{1,0}\mathcal{M}_g)) = \begin{cases} -1 & \text{if } g = 2, \\ -6 & \text{if } g = 3, \\ -2g+1 & \text{if } g \geq 4, \end{cases}$$

$$\chi(\mathcal{M}_g, \mathcal{O}(T^{0,1}\mathcal{M}_g)) = \begin{cases} -1 & \text{if } g \neq 3, \\ 2 & \text{if } g = 3. \end{cases}$$

Proof. The Chern class vanishings and the holomorphic Euler characteristics are computed by using the expression (3) and the intersection numbers in Theorem 5.1. For instance,

$$\begin{aligned} c_{10}(\mathcal{M}_4) &= 252u^5 + 2652v^5 + 26380v^4u + 51384u^2v^3 \\ &\quad +5100u^4v + 28920u^3v^2 = 0. \end{aligned}$$

□

REMARK 5.2. *The characteristic class vanishings and the Euler characteristics constitute a generalization to $\mathcal{M}_g$ of the Newstead conjectures [8, Conjectures (a),(b),(c)], for $2 \le g \le 7$. In fact, the vanishings for $\mathcal{M}_2$ are due to the triviality of $TJ(\Sigma_2)$ and the vanishings for $\mathcal{M}_3$ were first proved by Newstead [8].*

Since $K_{\mathcal{M}_g}$ is isomorphic to $L^{-2(g-2)}$,

$$H^i(\mathcal{M}_g, \mathcal{O}(k)) = 0 \quad \text{for all } i > 0 \text{ and } k > -2g-4,$$

i.e. $d_k = \chi(\mathcal{M}_g, \mathcal{O}(k)) = h^0(\mathcal{M}_g, \mathcal{O}(k))$ for all $k > -2g-4$.

THEOREM 5.3.

$$d_{K-g+2} = \begin{cases} 16k^2 & \text{if } g = 2, \\ \frac{1}{45}k^2(14k^4+20k^2+11) & \text{if } g = 3, \\ \frac{1}{37800}k^2(k^2-1)(22k^6+82k^4+103k^2+18) & \text{if } g = 4, \\ \frac{1}{19051200}k^2(k^2-1)^2(k^2-4)\times & \\ \quad \times(5k^6+30k^4+58k^2+18) & \text{if } g = 5, \\ \frac{1}{452656512000}k^2(k^2-1)^2(k^2-4)^2(k^2-9)\times & \\ \quad \times(19k^6+157k^4+409k^2+180) & \text{if } g = 6, \\ \frac{1}{15535171491840000}k^2(k^2-1)^2(k^2-4)^2(k^2-9)^2\times & \\ \quad \times(k^2-16)(46k^6+484k^4+1585k^2+900) & \text{if } g = 7. \end{cases}$$

Proof. By the Riemann-Roch Theorem and (5)

$$\begin{aligned} h^0(\mathcal{M}_g, \mathcal{O}(L^{k-(g-2)})) &= \chi(\mathcal{M}_g, \mathcal{O}(L^{k-(g-2)})) \\ &= \langle e^{l(k-(g-2))}\mathrm{td}(\mathcal{M}_g), [\mathcal{M}_g]\rangle \\ &= \langle e^{lk}\widehat{A}(\mathcal{M}_g), [\mathcal{M}_g]\rangle. \end{aligned} \tag{6}$$

The top-dimensional component gives a polynomial in k, whose coefficients involve the intersection numbers computed in Proposition 5.1. Hence the result. □

These polynomials agree with the ones computed in [5, Section 4.2].

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