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The Lifespan of Classical Solutions to Systems of Nonlinear Wave Equations

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Summary. - *Any results in this talk are based on a joint paper with R. Agemi & Y. Kurokawa [1]. The existence of the critical curve for p-q systems of nonlinear wave equations was already established by D. Del Santo & V. Georgiev & E. Mitidieri [3] except for the critical case. Our main purpose is to prove a blow-up theorem for which the nonlinearity (p,q) is just on the critical curve in three space dimensions. Moreover, the lower and upper bounds of the lifespan of solutions are precisely estimated including the sub-critical case.*

1. Introduction

We are concerned with the Cauchy problem for p-q systems of nonlinear wave equations

$$\begin{cases} \Box u = |v|^p, \\ \Box v = |u|^q, \end{cases} \quad \text{in} \quad \mathbf{R}^n \times [0,\infty), \tag{1}$$

where $\Box = \partial^2/\partial_t^2 - \sum_{j=1}^n \partial^2/\partial x_j^2$ is a usual d'Alembertian in $\mathbf{R}^{n+1}$ and $p,q>1$. The initial data takes the following form.

$$\begin{cases} u(x,0) = \varepsilon f_1(x), \ (\partial u/\partial t)(x,0) = \varepsilon g_1(x), \\ v(x,0) = \varepsilon f_2(x), \ (\partial v/\partial t)(x,0) = \varepsilon g_2(x), \end{cases} \tag{2}$$

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where $f_i,\ g_i\ (i=1,2)$ are smooth functions of compact support and ε is a small positive parameter which measures the smallness of the amplitude of solutions.

The problem (1) sometimes arises from the comparison with Lane-Emden system and its associated parabolic version in which $\Box$ in (1) is replaced by $-\Delta$ or $\partial_t-\Delta$. See [3] for details and further references.

Recently, D. Del Santo & V. Georgiev & E. Mitidieri [3] proved in any space dimensions $n\geq 2$ that there exists a critical curve in (p,q)-plane which divides the plane into two pieces. One is a range where we can show the global in time existence of small amplitude solution. Another is a range where we can give an example of the nonexistence of the global in time solution. We note that the critical curve is determined by cubic relation between p and q, and has a cusp at $p=q$.

More precisely, defining

$$F(p,q)\equiv\max\left\{\frac{p+2+q^{-1}}{pq-1},\frac{q+2+p^{-1}}{pq-1}\right\}-\frac{n-1}{2}, \tag{3}$$

they proved the following fact. If $F(p,q)<0$, the system (1) with any data (2) admits a unique global solution provided ε is sufficiently small. Remark that, in general, the solution must be weak whenever p or q is less than 2 because of the regularity of nonlinearities. By this reason, the classical solution can be obtained only in the case $n=2,\ 3$, or $n=4$ at the cusp only. Conversely, if $F(p,q)>0$, (1) with some positive data (2) has no global solution. The critical case $F(p,q)=0$ was investigated by D. Del Santo & E. Mitidieri [4] for $n=3$ in which nonexistence of global solutions for some positive data was proved.

Our aim in this article is to clarify the lifespan, the maximal existence time, of the solution in three space dimensions without any positivity on data by local in time existence and nonexistence in long time of solutions. Here, we restrict our attension to a classical sense so that the lifespan $T(\varepsilon)$ is defined by

$$T(\varepsilon) = \sup\{T \in (0,\infty] \ : \text{There exists a unique solution } (u,v) \in \{C^2(\mathbf{R}^n \times [0,T))\}^2 \text{ of (1) with any data (2).}\}. \tag{4}$$

By virtue of well-known uniqueness theorem, one has $\lim_{\varepsilon\to 0} T(\varepsilon) = \infty$. For example, see Appendix1 in F. John [10]. We will prove the following theorem.

THEOREM 1.1. *Let $n = 3$ and $p, q \geq 2$. Suppose that both $f_i \in C_0^4(\mathbf{R}^3)$ and $g_i \in C_0^3(\mathbf{R}^3)$ do not identically vanish for each $i = 1, 2$. Then there exists a positive constant ε_0 such that, for any ε with $0 < \varepsilon \leq \varepsilon_0$, the lifespan $T(\varepsilon)$ of the classical solution (u,v) of (1),(2) satisfies*

$$T(\varepsilon) = \infty \tag{5}$$

provided $F(p,q) < 0$,

$$\exp\left(c\varepsilon^{-l}\right) \leq T(\varepsilon) \leq \exp\left(C\varepsilon^{-l}\right), \tag{6}$$

where $l = \min\{p(pq-1), q(pq-1)\}$, provided $F(p,q) = 0$ with $p \neq q$,

$$\exp\left(c\varepsilon^{-p(p-1)}\right) \leq T(\varepsilon) \leq \exp\left(C\varepsilon^{-p(p-1)}\right) \tag{7}$$

provided $F(p,q) = 0$ with $p = q$, and

$$c\varepsilon^{-F(p,q)^{-1}} \leq T(\varepsilon) \leq C\varepsilon^{-F(p,q)^{-1}} \tag{8}$$

provided $F(p,q) > 0$, where c and C are positive constants independent of ε.

REMARK 1.2. *The restriction $2 < p, q \leq 3$ in three space dimensions of the global existence theorem of [3] was relaxed by D. Del Santo [2]. Actually he used a weighted L^∞ estimate originally introduced by F. John [9] as conjectured in Remark1.1 of [3]. See also our proof.*

REMARK 1.3. *In the blow-up part of Theorem1.1, there is no requirement of the positivity of initial data (cf. Theorem3 in [3]). D. Del*

Santo [2] also proved the sub-critical blow-up without any positivity on data. He employed some technique by F. John [10]. But it cannot be applicable to estimating the lifespan. By making use of the local existence, we will succeed to remove the positivity. Such an argument can be found in F. John [9] in which the lifespan is estimated for a single equation with a quadratic nonlinearity.

REMARK 1.4. *At the cusp (p,p) on the critical curve $F(p,p) = 0$, p must be a number $p_0(3) = 1 + \sqrt{2}$ which is the critical power of the single case. See (14) below. Moreover, the lifespan at the cusp coincides with the one for single case. See also (18) below.*

REMARK 1.5. *For $1 < p, q < 2$, we cannot expect any existence of classical solutions by lack of the differentiablity of the nonlinearity. But we may obtain the same lifespan of C^1-solution of the integral equation associated to (1), (2).*

As in [3], it is interesting to compare the result of p-q system with the one of the single equation

$$\begin{cases} \Box u = |u|^p \quad \text{in} \quad \mathbf{R}^n \times [0,\infty) \\ u(x,0) = \varepsilon f(x), \quad (\partial u/\partial t)(x,0) = \varepsilon g(x). \end{cases} \tag{9}$$

It is well-known, as Strauss' conjecture [16], that the lifespan $T(\varepsilon)$ of a solution of (9) satisfies $T(\varepsilon) = \infty$ for small ε if

$$p > p_0(n), \tag{10}$$

where $p_0(n)$ is a positive root of the quadratic equation

$$\gamma(p,n) \equiv 2 + (n+1)p - (n-1)p^2 = 0, \tag{11}$$

and $T(\varepsilon) < \infty$ for some special data with a positivity if

$$1 < p \le p_0(n) \tag{12}$$

which can be rewritten as

$$\frac{1+p^{-1}}{p-1} > \frac{n-1}{2} \quad \text{and} \quad p > 1. \tag{13}$$

In this sense, $p_0(n)$ is a critical value of (9). One can find that

$$p_0(n) = \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}, \quad n \geq 2. \tag{14}$$

We note that $p_0(n)$ is monotonously decreasing in n and $p_0(4) = 2$. Therefore, we have to consider the weaker solution rather than C^2 if p is in the neighborhood of $p_0(n)$ in higher dimensions $n \geq 4$.

This conjecture was verified by F. John [9] for $n = 3$ and by R. T. Glassey [7] [6] for $n = 2$ except for the critical case. The critical case was proved by J. Schaeffer [14] for $n = 2, 3$. The blow-up part in higher dimensions was verified by T. C. Sideris [15] except for the critical case. For the global existence part, there were many partial results. A complete result was given by V. Georgiev & H. Lindblad & C. Sogge [5] in which we can find references on history. The open problem is the case $p = p_0(n)$ for $n \geq 4$.

As for the order of $T(\varepsilon)$, we have a few results. In the case $n = 2, 3$, H. Lindblad [13] proved that

$$\lim_{\varepsilon \to 0} \varepsilon^{2p(p-1)/\gamma(p,n)} T(\varepsilon) > 0 \quad \text{exists for} \quad 4-n < p < p_0(n). \tag{15}$$

and, for $n = 2$, $p = 2$,

$$\begin{aligned} &\lim_{\varepsilon \to 0} a^{-1}(\varepsilon) T(\varepsilon) > 0 && \text{exists if} \quad \int_{\mathbf{R}^2} f(x)dx \neq 0 \\ &\lim_{\varepsilon \to 0} \varepsilon T(\varepsilon) > 0 && \text{exists if} \quad \int_{\mathbf{R}^2} f(x)dx = 0, \end{aligned} \tag{16}$$

where $a = a(\varepsilon)$ satisfies

$$\varepsilon^2 a^2 \log(1+a) = 1. \tag{17}$$

REMARK 1.6. *Making use of H. Lindblad's methods, we may have a limit of the lifespan in the sub-critical case of Theorem1.1. But this is another story.*

Zhou Yi [19] [20] proved that there exist positive constants c, C independent of ε such that

$$\exp(c\varepsilon^{-p(p-1)}) \leq T(\varepsilon) \leq \exp(C\varepsilon^{-p(p-1)}) \quad \text{for} \quad p = p_0(n),\ n = 2, 3. \tag{18}$$

By making use of L^2-frame work, Li Ta-Tsien & Zhou Yi [17] showed that, in the case $n = 4$, there exists a positive constant c independent of ε such that

$$T(\varepsilon) \geq \exp(c\varepsilon^{-2}) \quad \text{for} \quad p = p_0(4) = 2. \tag{19}$$

REMARK 1.7. *The proof of the blow-up result for single equation (9) with a sub-critical power is essentially due to T. Kato's blow-up theorem [11] for 2nd order ordinary differential inequality. Such an inequality can be applicable to the sub-critical case of our system (1) by iteration argument. See [3]. The critical case for single equation is due to Zhou Yi's blow-up theorem for 2nd order ordinary differential equations. We note that his theorem cannot be directly applicable to our system. Because we have to make a comparison argument with a system of 2nd order ordinary differential equations which is difficult to solve. Our success of the blow-up result on the critical curve is due to a logarithmic term in the iteration argument which is made by our new slicing method.*

After this work was completed, we were informed a result of H. Kubo & M. Ohta [12]. They have proved the blow-up part of Theorem1.1 for $n = 2, 3$, in which data must have positivity, by comparison argument with a system of integral equations.

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2. Lower bound of the lifespan

We shall start from well-known integral representation formula of the solution. Namely, solutions (u, v) of (1), (2) have to satisfy the following integral equations.

$$\begin{cases} u(x,t) = \varepsilon u^0(x,t) + L(|v|^p)(x,t), \\ v(x,t) = \varepsilon v^0(x,t) + L(|u|^q)(x,t), \end{cases} \tag{20}$$

where u^0 and v^0 satisfy $\Box u^0 = \Box v^0 = 0$ with the same initial data to u/ε and v/ε respectively. Moreover $L(w)$ satisfies

$$\begin{cases} \Box L(w) = w \quad \text{in} \quad \mathbf{R}^n \times [0,\infty), \\ L(w)(x,0) = (\partial L(w)/\partial t)(x,0) = 0, \end{cases} \tag{21}$$

where $w = |v|^p$ or $|u|^q$.

Then we have the following two lemmas on the dependence domain of the solution.

LEMMA 2.1. *Assume that*

$$supp\{f_i(x), g_i(x)\} \subset \{|x| \le k\}, \tag{22}$$

where $k > 0$ and $i = 1,2$. Then the classical solutions (u,v) of (1), (2) have to satisfy the following support property.

$$supp\{u(x,t), v(x,t)\} \subset \{|x| \le t + k\} \tag{23}$$

LEMMA 2.2. *Let $n = 3$. Assume a support property (22). Then,*

$$u^0(x,t) \equiv v^0(x,t) \equiv 0 \quad for \quad t - |x| \ge k. \tag{24}$$

The lower bound of the lifespan is estimated by proving the following proposition.

PROPOSITION 2.3. *Let $n = 3$. Assume (22). Under the same assumption as Theorem1.1, there exists a positive constant ε_0 such that (20) admit a unique solution $(u,v) \in \{C^2(\mathbf{R}^3 \times [0,T))\}^2$, as far as T satisfies*

$$T \le \begin{cases} \infty & if \quad F(p,q) < 0, \\ \exp\left(c\varepsilon^{-\min\{p(pq-1),q(pq-1)\}}\right) & if \quad F(p,q) = 0 \text{ with } p \ne q, \\ \exp\left(c\varepsilon^{-p(p-1)}\right) & if \quad F(p,q) = 0 \text{ with } p = q, \\ c\varepsilon^{-F(p,q)^{-1}} & if \quad F(p,q) > 0 \end{cases} \tag{25}$$

for $0 < \varepsilon \le \varepsilon_0$ and some positive constant c independent of ε.

We will solve (20) by classical iteration method in suitable function space. Here and hereafter, it is sufficient to consider the case $p \leq q$. Because, due to the symmetricity of the equation, nothing new will come by switching p and q to each other. Let us define sequences of functions $\{u_m\}, \{v_m\}$ by

$$\left\{\begin{array}{l} u_m = u_0 + L(|v_{m-1}|^p), \\ v_m = v_0 + L(|u_{m-1}|^q), \end{array}\right. \quad \text{for } m \geq 1 \text{ and } \quad \left\{\begin{array}{l} u_0 = \varepsilon u^0, \\ v_0 = \varepsilon v^0. \end{array}\right. \tag{26}$$

In order to solve this, we shall follow F. John [9]. Denote a weighted L^∞-norm of u by

$$\|u\|_j = \sup_{(x,t)\in \mathbf{R}^3\times[0,T)} \{w_j(|x|,t)|u(x,t)|\}, \ (j=1,2), \tag{27}$$

with the weight function

$$w_1(r,t) = \left\{\begin{array}{l} \dfrac{t+r+2k}{k}\left(\dfrac{t-r+2k}{k}\right)^{p-2} \\ \quad \text{when } p>2, \\ \dfrac{t+r+2k}{k}\left(\log 4\dfrac{t+r+2k}{|t-r|+2k}\right)^{-1} \\ \quad \text{when } p=2,\ q>2, \\ \dfrac{t+r+2k}{k}\chi_1 + \dfrac{r}{k}\left(\log\dfrac{t+r+2k}{t-r+2k}\right)^{-1}\chi_2 \\ \quad \text{when } p=q=2 \end{array}\right. \tag{28}$$

and

$$w_2(r,t) = \left\{\begin{array}{l} \dfrac{t+r+2k}{k}\chi_1 + \dfrac{r}{k}\left(\log\dfrac{t+r+2k}{t-r+2k}\right)^{-1}\chi_2 \\ \quad \text{when } p=q=2, \\ \dfrac{t+r+2k}{k}\left(\dfrac{t-r+2k}{k}\right)^{1/p}\left(\log\dfrac{t-r+3k}{k}\right)^{\nu} \\ \quad \text{when } F(p,q)=0 \text{ with } p\neq q, \\ \dfrac{t+r+2k}{k}\left(\dfrac{t-r+2k}{k}\right)^{\mu} \\ \quad \text{otherwise,} \end{array}\right. \tag{29}$$

where μ and ν are defined by

$$\begin{aligned} \mu &= \frac{pq-p-q}{p} + \frac{2(p-q)}{p(pq-1)}, \\ \nu &= \frac{q(p-1)}{p(pq-1)} \end{aligned} \tag{30}$$

and k is the one in Lemma2.1. χ_1 is a characteristic function of a set

$$S_1 = \{(x,t) \in \mathbf{R}^3 \times [0,T) \ : \ -k \le t - |x| \le k\} \tag{31}$$

and χ_2 is a characteristic function of a set

$$S_2 = \{(x,t) \in \mathbf{R}^3 \times [0,T) \ : \ k \le t - |x|\}. \tag{32}$$

REMARK 2.4. *We note that* $p \ge 2$ *implies* $\mu \ge 0$.

Proposition2.3 is proved by the following two *a priori* estimates.

LEMMA 2.5. *Let* $n = 3$. *Suppose that* $2 \le p \le q$. *Let* (u, v) *be a solution of (20). Then there exists a positive constant* C *independent of* ε, k *and* T *such that*

$$\begin{aligned} &\|\chi_1 L(|v|^p)\|_1 \le Ck^2 \|\chi_1 v\|_2^p D(T), \\ &\|\chi_1 L(|u|^q)\|_2 \le Ck^2 \|\chi_1 u\|_1^q D(T) \end{aligned} \tag{33}$$

for any $T > 0$, *where* D *is defined by*

$$D(T) = \begin{cases} \log \dfrac{2T+3k}{k} & \text{if} \quad p = q = 2, \\ 1 & \text{otherwise.} \end{cases} \tag{34}$$

LEMMA 2.6. *Let* $n = 3$. *Suppose that* $2 \le p \le q$. *Let* (u, v) *be a solution of (20). Then there exists a positive constant* C *independent of* ε, k *and* T *such that, for any* $T > 0$,

$$\begin{aligned} &\|\chi_2 L(|v|^p)\|_1 \le Ck^2 \{\|\chi_1 v\|_2^p + \|\chi_2 v\|_2^p E_1(T)\}, \\ &\|\chi_2 L(|u|^q)\|_2 \le Ck^2 \{\|\chi_1 u\|_1^q + \|\chi_2 u\|_1^q E_2(T)\}, \end{aligned} \tag{35}$$

where E_1 *and* E_2 *are defined by*

$$
\begin{aligned}
&E_1(T) = E_2(T) = 1 \\
&\qquad \textit{if } F(p,q) < 0, \\
&E_1(T) = \left(\log \frac{T+3k}{k}\right)^{1-p\nu}, \; E_2(T) = \left(\log \frac{T+3k}{k}\right)^{\nu} \\
&\qquad \textit{if } F(p,q) = 0 \textit{ with } p \neq q, \\
&E_1(T) = E_2(T) = \log \frac{T+2k}{k} \\
&\qquad \textit{if } F(p,q) = 0 \textit{ with } p = q, \\
&E_1(T) = \left(\frac{T+2k}{k}\right)^{p(q-1)F(p,q)}, \; E_2(T) = \left(\frac{T+2k}{k}\right)^{q(p-1)F(p,q)} \\
&\qquad \textit{if } F(p,q) > 0.
\end{aligned}
\tag{36}
$$

Remark 2.7. *We note that*

$$1 - p\nu = \frac{q-1}{pq-1} > 0. \tag{37}$$

Here we introduce a function spaces X defined by

$$
\begin{aligned}
X = \{(u,v) \in \{C^2(\mathbf{R}^3 \times [0,T))\}^2 \; : \\
\operatorname{supp}(u,v) \subset \{|x| \leq t+k\}, \; \|(u,v)\|_X < \infty\},
\end{aligned}
\tag{38}
$$

where

$$\|(u,v)\|_X = \sum_{|\alpha| \leq 2} (\|\nabla_x^\alpha u\|_1 + \|\nabla_x^\alpha v\|_2). \tag{39}$$

Remark that $\partial u/\partial_t$ and $\partial v/\partial_t$ are expressed by $\nabla_x u$ and $\nabla_x v$ in view of the representation formula of the solution. So, it is sufficient to consider the spatial derivatives only. We also note that X is a Banach space for any fixed $T > 0$. Our purpose is to construct a unique solution in X of the equivalent integral equation (20) which must be a classical solution of the original p-q systems.

In order to see this, putting

$$M = \max_{|\alpha| \leq 2} \{\|\nabla_x^\alpha u^0\|_1, \|\nabla_x^\alpha v^0\|_2\} > 0, \tag{40}$$

we also define a closed subspace Y of X by

$$\begin{aligned} Y = \ & \{(u,v) \in X \ : \ \|\chi_1 \nabla_x^\alpha u\|_1, \|\chi_1 \nabla_x^\alpha v\|_2 \le 2M\varepsilon, \\ & \|\chi_2 \nabla_x^\alpha u\|_1 \le N\varepsilon^p, \ \|\chi_2 \nabla_x^\alpha v\|_2 \le N\varepsilon^q \quad (|\alpha| \le 2)\}, \end{aligned} \tag{41}$$

where N is defined by

$$N = 2Ck^2 \max\{p^2(2M)^p, q^2(2M)^q\} \tag{42}$$

and C is the one in *a priori* estimate (33), (35). We note that $M < \infty$.

The solution will be constructed as a limit of a sequence (u_m, v_m) in Y if ε is suitably small. After two solutions are constructed in each domains, S_1 and S_2, we will know that one must coinside with another on the intersection of both domains by uniqueness of the solution.

3. Upper bound of the lifespan (critical case)

Denoting a spherical mean of $h(x) \in C(\mathbf{R}^3)$ at the origin with radius $r = |x|$ by

$$\overline{h}(r) = \frac{1}{4\pi} \int_{|\omega|=1} h(r\omega) dS_\omega. \tag{43}$$

and following John's iteration argument together with a new slicing method, we shall estimate the upper bound of the lifespan in the critical case. The sub-critical case is much easier than this.

Proposition 3.1. *Let $n = 3$. Assume that (u,v) be a classical solution of (1), (2) in the domain $\mathbf{R}^3 \times [0,T)$ under the same assumption as Theorem1.1. Then, for sufficiently small ε and some positive constant C independent of ε, T cannot be taken as*

$$T > \begin{cases} \exp\left(C\varepsilon^{-\min\{p(pq-1),q(pq-1)\}}\right) \\ \quad \textit{provided} \quad F(p,q) = 0 \ \textit{with} \ p \ne q, \\ \exp\left(C\varepsilon^{-p(p-1)}\right) \\ \quad \textit{provided} \quad F(p,q) = 0 \ \textit{with} \ p = q = p_0(3). \end{cases} \tag{44}$$

In order to prove the blow-up result, we have to make an iteration frame.

Lemma 3.2. *Let (u,v) be a classical solution of (1), (2) with the support condition (22). Assume that $2 \le p \le q$. Then there exist a positive constant M independent of ε such that*

$$\begin{cases} \overline{u}(r,t) \ge \dfrac{1}{t+r}\displaystyle\int\int_{R(r,t)} \lambda|\overline{v}(\lambda,\tau)|^p d\lambda d\tau + \dfrac{M\varepsilon^p}{(t+r)(t-r)^{p-2}}, \\ \overline{v}(r,t) \ge \dfrac{1}{t+r}\displaystyle\int\int_{R(r,t)} \lambda|\overline{u}(\lambda,\tau)|^q d\lambda d\tau \end{cases} \tag{45}$$

in Σ_0, for sufficiently small ε,where

$$\begin{aligned} &\Sigma_0 = \{(r,t) : k \le t-r \le r\}, \\ &R(r,t) = \{(\lambda,\tau) : t-r \le \lambda, \tau+\lambda \le t+r, k \le \tau-\lambda \le t-r\}. \end{aligned} \tag{46}$$

Remark 3.3. *If we assume that the initial data is positive in some sense, Lemma3.2 becomes an easy application of the single case in H. Takamura [18]. For example, one may assume that*

$$f_2(x) \equiv 0 \quad \textit{and} \quad g_2(x) \ge 0 (\not\equiv 0) \tag{47}$$

while f_1 and g_1 can be arbitrary. In order to remove the positivity on the initial data, we have to use the local existence of solutions which gives us a restriction $p,q \ge 2$. As a consequence, once we get a local solution of associated integral equation (20), Lemma3.2 will be valid for $p,q>1$ without the positivity on the initial data.

Proof of Proposition3.1.
Throughout this section we assume that $2 \le p \le q$. In this case, we note that $p \le p_0(3) = 1+\sqrt{2} \le q$ for

$$F(p,q) \equiv \frac{q+2+p^{-1}}{pq-1} - 1 = 0. \tag{48}$$

The opposite case is proved by replacing u, p by v, q, respectively.

Let (u,v) be a classical solution of (1), (2) in $\mathbf{R}^3 \times [0,T)$. Let us define the blow-up domain. For $j \ge 1$,

$$\Sigma_j = \{(r,t) \in \mathbf{R}_+ \times [0,T) \ : \ l_j k \le t-r \le r\}, \tag{49}$$

where $l_j = 1+2^{-1}+\cdots+2^{-j}$. We will use the fact that a sequence $\{l_j\}$ is monotonously increasing and bounded, $1 < l_j < 2$, so $\Sigma_{j+1} \subset \Sigma_j$. This is the *slicing* of the blow-up set.

Assume an estimate of the form

$$\overline{u}(r,t) \geq \frac{C_j}{(t+r)(t-r)^{p-2}} \left(\log \frac{t-r}{l_{2j}k}\right)^{a_j} \quad \text{in} \quad \Sigma_{2j}, \tag{50}$$

where $a_j \geq 0$ and $C_j > 0$. Putting (50) into the second inequality of (3.2) and noticing that $t+r \geq 3(t-r)$, we get an estimate of $\overline{v}$ in Σ_{2j+1} such as

$$\begin{aligned}\overline{v}(r,t) \geq & \frac{C_j^q}{2 \cdot 3^q (t+r)(t-r)^{q-1}} \\ & \int_{l_{2j}k}^{t-r} \beta^{-q(p-2)}(t-r-\beta)\left(\log \frac{\beta}{l_{2j}k}\right)^{qa_j} d\beta.\end{aligned} \tag{51}$$

At this stage, the proof must be divided into two cases.
Case $p \neq q$.

It follows from

$$1 - q(p-2) > 0 \quad \text{for} \quad F(p,q) = 0 \text{ with } p \neq q \tag{52}$$

and

$$1 - \frac{l_{2j}}{l_{2j+1}} > \frac{1}{2^{2j+2}} \tag{53}$$

that β-integral is greater than

$$\begin{aligned}& \int_{(l_{2j}/l_{2j+1})(t-r)}^{t-r} \beta^{1-q(p-2)-1}(t-r-\beta)\left(\log \frac{\beta}{l_{2j}k}\right)^{qa_j} d\beta \\ & \geq 2^{q(p-2)-6} \frac{1}{16^j}(t-r)^{2-q(p-2)} \left(\log \frac{t-r}{l_{2j+1}k}\right)^{qa_j}.\end{aligned} \tag{54}$$

Hence we obtain an estimate for $\overline{v}$ such that

$$\overline{v}(r,t) \geq \frac{D_j}{(t+r)(t-r)^{pq-q-3}} \left(\log \frac{t-r}{l_{2j+1}k}\right)^{qa_j} \quad \text{in} \quad \Sigma_{2j+1}, \tag{55}$$

where we put

$$D_j = 3^{-q} 2^{q(p-2)-7} \frac{C_j^q}{16^j}. \tag{56}$$

Similarly, putting (55) into the first inequality of Lemma3.2, we have a new estimate for $\overline{u}$ in Σ_{2j+2} as follows.

$$\overline{u}(r,t) \geq \frac{D_j^p}{2 \cdot 3^p (t+r)(t-r)^{p-1}} \int_{l_{2j+1}k}^{t-r} \frac{t-r-\beta}{\beta} \left(\log \frac{\beta}{l_{2j+1}k} \right)^{pqa_j} d\beta \tag{57}$$

because

$$p(pq-q-3) = 1 \quad \text{when} \quad F(p,q) = 0. \tag{58}$$

The integration by parts yields that the β-integral is equal to

$$\frac{1}{pqa_j + 1} \int_{l_{2j+1}k}^{t-r} \left(\log \frac{\beta}{l_{2j+1}k} \right)^{pqa_j+1} d\beta. \tag{59}$$

Hence, for $(r,t) \in \Sigma_{2j+2}$, the β-integral is greater than

$$\begin{aligned} &\frac{1}{pqa_j + 1} \int_{(l_{2j+1}/l_{2j+2})(t-r)}^{t-r} \left(\log \frac{\beta}{l_{2j+1}k} \right)^{pqa_j+1} d\beta \\ &\geq \frac{1}{2^2 4^j (pqa_j + 1)} \left(\log \frac{t-r}{l_{2j+2}k} \right)^{pqa_j+1} (t-r). \end{aligned} \tag{60}$$

Therefore we finally obtain

$$\overline{u}(r,t) \geq \frac{C_{j+1}}{(t+r)(t-r)^{p-2}} \left(\log \frac{t-r}{l_{2j+2}k} \right)^{pqa_j+1} \quad \text{in} \quad \Sigma_{2j+2}, \tag{61}$$

where we put

$$C_{j+1} = 2^{-3} 3^{-p} \frac{D_j^p}{4^j (pqa_j + 1)}. \tag{62}$$

Now, we are in a position to define sequences in the iteration. In view of (3.2), the original estimate is

$$\overline{u}(r,t) \geq \frac{M\varepsilon^p}{(t+r)(t-r)^{p-2}} \quad \text{in} \quad \Sigma_0 \tag{63}$$

so that, with the help of (50), (55) and (61), a sequence $\{a_j\}$ must be defined by

$$\begin{cases} a_{j+1} &= pqa_j + 1, \quad j \geq 1, \\ a_0 &= 0. \end{cases} \tag{64}$$

Another sequence $\{C_j\}$ is determined by

$$\begin{cases} C_{j+1} &= 2^{-3}3^{-p}\dfrac{D_j^p}{4^j(pqa_j+1)}, \quad j \geq 1, \\ D_j &= 3^{-q}2^{q(p-2)-7}\dfrac{C_j^q}{16^j}, \quad j \geq 1, \\ C_0 &= M\varepsilon^p. \end{cases} \tag{65}$$

One can readily check that

$$a_j = \frac{1}{pq-1}\left\{(pq)^j - 1\right\}, \quad j \geq 1 \tag{66}$$

which gives

$$\frac{1}{pqa_j+1} \geq \frac{pq-1}{pq}(pq)^{-j}. \tag{67}$$

Hence one can find that

$$C_{j+1} \geq E\frac{C_j^{pq}}{F^j}, \quad j \geq 1, \tag{68}$$

where E and F are positive constants defined by

$$E = \frac{2^{pq(p-2)-7p-3}(pq-1)}{3^{p(q+1)}pq}, \ F = 4^{2p+1}pq. \tag{69}$$

Repeating this inequality j-times, we get

$$\log C_j \geq \\ (pq)^j\left(\log C_0 + \sum_{m=1}^{j}\frac{(pq)^{m-1}\log E - (m-1)(pq)^{j-m}\log F}{(pq)^j}\right). \tag{70}$$

The sum part of the above inequality converges as $j \to \infty$ by d'Alembert' criterion. It follows that there exists a constant S independent of j such that

$$C_j \geq \exp\{(pq)^j(\log C_0 + S)\}, \quad j \geq 1. \tag{71}$$

Combining all estimates and using the monotonicity of Σ_j, we can reach the final inequality

$$\overline{u}(r,t) \geq \frac{\exp\left\{(pq)^j I(r,t)\right\}}{(t+r)(t-r)^{p-2}} \left(\log \frac{t-r}{2k}\right)^{-(pq-1)^{-1}} \tag{72}$$

for $(r,t) \in \Sigma_\infty = \{(r,t) \in \mathbf{R}_+ \times [0,T) \ : \ 2k < t-r \leq r\}$, where we put

$$I(r,t) = \log\left\{Me^S\varepsilon^p \left(\log \frac{t-r}{2k}\right)^{(pq-1)^{-1}}\right\}. \tag{73}$$

At this stage, it is clear that there exists a point $(t_0/2, t_0) \in \Sigma_\infty$ such that $I(t_0/2, t_0) > 0$ provided

$$T > 4k \exp\left\{(Me^S)^{1-pq}\varepsilon^{-p(pq-1)}\right\}. \tag{74}$$

Taking $j \to \infty$, we get a desired result $\overline{u}(t_0/2, t_0) \to \infty$ which contradicts to the assumption that u is a classical solution in $\mathbf{R}^3 \times [0,T)$. So, the proof of the critical case for $p \neq q$ is completed.

Case $p = q = p_0(3)$.

This case will proved in the almost same way as previous case. In fact, nothing changed must appear before the inequality (51). From now on, we denote q by p.

It follows from the definition of $p_0(3) = 1 + \sqrt{2}$ and related quadratic equation

$$\gamma(p,3) \equiv 2 + 4p - 2p^2 = 0 \quad \text{with} \quad p = p_0(3) \tag{75}$$

that

$$1 - p(p-2) = 0 \quad \text{when} \quad F(p,p) = 0 \text{ with } p = p_0(3). \tag{76}$$

Therefore, β-integral in (51) is equal to

$$\int_{l_{2j}}^{t-r} \frac{t-r-\beta}{\beta} \left(\log \frac{\beta}{l_{2j}k} \right)^{pa_j} d\beta. \tag{77}$$

So, the integration by parts yields that β-integral must be

$$\frac{1}{pa_j+1} \int_{l_{2j}}^{t-r} \left(\log \frac{\beta}{l_{2j}k} \right)^{pa_j+1} d\beta. \tag{78}$$

Hence, by slicing again, we obtain the following estimate for $\overline{v}$ instead of (55).

$$\overline{v}(r,t) \geq \frac{D_j'}{(t+r)(t-r)^{p-2}} \left(\log \frac{t-r}{l_{2j+1}k} \right)^{pa_j+1} \quad \text{in} \quad \Sigma_{2j+1}, \tag{79}$$

where we put

$$D_j' = 2^{-3}3^{-p} \frac{C_j^p}{4^j(pa_j+1)}. \tag{80}$$

Similarly to the previous case, we have a new estimate for $\overline{u}$ in Σ_{2j+2} as follows.

$$\begin{aligned} \overline{u}(r,t) \geq & \frac{(D_j')^p}{2 \cdot 3^p (t+r)(t-r)^{p-1}} \\ & \int_{l_{2j+1}k}^{t-r} \frac{t-r-\beta}{\beta} \left(\log \frac{\beta}{l_{2j+1}k} \right)^{p(pa_j+1)} d\beta. \end{aligned} \tag{81}$$

Hence the same treatment on the β-integral implies an new estimate

$$\overline{u}(r,t) \geq \frac{C_{j+1}}{(t+r)(t-r)^{p-2}} \left(\log \frac{t-r}{l_{2j+2}k} \right)^{p(pa_j+1)+1} \quad \text{in} \quad \Sigma_{2j+2}, \tag{82}$$

where we put

$$C_{j+1} = 2^{-3}3^{-p} \frac{(D_j')^p}{4^j(p(pa_j+1)+1)}. \tag{83}$$

along with the same procedure, we can reach the final inequality

$$\overline{u}(r,t) \geq \frac{\exp\left\{p^{2j}I'(r,t)\right\}}{(t+r)(t-r)^{p-2}}\left(\log\frac{t-r}{2k}\right)^{-(p-1)^{-1}} \tag{84}$$

for $(r,t) \in \Sigma_\infty = \{(r,t) \in \mathbf{R}_+ \times [0,T) \ : \ 2k < t-r \leq r\}$, where we put

$$I'(r,t) = \log\left\{Me^S\varepsilon^p\left(\log\frac{t-r}{2k}\right)^{(p-1)^{-1}}\right\}. \tag{85}$$

Hence there exists a point $(t_0/2, t_0) \in \Sigma_\infty$ such that $I(t_0/2, t_0) > 0$ provided

$$T > 4k\exp\left\{(Me^S)^{1-p}\varepsilon^{-p(p-1)}\right\}. \tag{86}$$

Taking $j \to \infty$, we get a desired contradiction. The proof is now completed.

Remark 3.4. *We note that the above proof never require that* $p \geq 2$ *except for the assumption in Lemma3.2. This means that the result may be still valid for* C^1*-solution of associated integral equations (20) with a low power* $1 < p < 2$. *See Remark3.3.*

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