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Constructing Weak Solutions in a Direct Variational Method and an Application of Varifold Theory

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Summary. - *A weak solution to a fourth order parabolic equation is constructed by the method of discretization in time and minimizing variational functionals. The convergence of nonlinear terms is obtained by the use of varifold theory.*

1. Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^n$ with $\partial\Omega$ sufficiently smooth. Let $\mathcal{V}$ be a reflexive Banach space and $\mathcal{V}_0$ be a closed subspace of $\mathcal{V}$. Suppose that $\mathcal{V}$ and $\mathcal{V}_0$ are continuously imbedded in $W^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$, respectively. For $w \in \mathcal{V}$ we define $\mathcal{W}_w \subset \mathcal{V}$ by $\{v \in W^{1,2}(\Omega); v - w \in \mathcal{V}_0\}$. Given a functional $J : \mathcal{V} \to [0, \infty)$, we suppose that i) J is Gâteaux differentiable on $\mathcal{V}$, ii) J is weakly lower semicontinuous, iii) there exist constants c_0 and c_1 such that $J(v) \geq c_0 \| v \|_{\mathcal{V}} - c_1$ for each $v \in \mathcal{W}_w$, and iv) there exists a constant μ_0 such that $J(v) \leq \mu_0 \| v \|_{\mathcal{V}}$ for each $v \in \mathcal{V}$.

Here we consider the following evolution equation:

$$\frac{\partial u}{\partial t} + \operatorname{grad} J(u) = 0 \quad (t > 0), \tag{1}$$

$$u(0) = u_0 \in \mathcal{W}_w, \tag{2}$$

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This talk is based on [8].

$$u(t) \in \mathcal{W}_w \quad (t > 0). \tag{3}$$

We say that a function u is a weak solution to (1)–(3) if u satisfies $u \in L^\infty((0,\infty);\mathcal{V})$, $\dfrac{\partial u}{\partial t} \in \bigcup_{T>0} L^2((0,T);\mathcal{V}')$, (1) holds in $\bigcup_{T>0} L^2((0,T);\mathcal{V}')$, $\text{s-}\lim_{t\searrow 0} u(t,x) = u_0(x)$ in $L^2(\Omega)$, and $u(t) \in \mathcal{W}_w$ for $\mathcal{L}^1$-a.e. t.

Our objective is an approximate solution to (1)–(3) constructed by the method of discretization in time and minimizing variational functionals. This approximating method is firstly applied to constructing weak solutions to linear parabolic equations ([10]). In [9] N. Kikuchi has independently rediscovered this method, and after [9] there are many works in applying this method to constructing weak solutions to nonlinear partial differential equations ([2], [3], and references cited in [8]). Let h be a positive number. First we construct a sequence $\{u_\ell\} \subset \mathcal{W}_w$ in the following way: we let u_0 be as in (2) and for $\ell \geq 1$ we define u_ℓ as the minimizer of the functional

$$\mathcal{F}_\ell(v) = \frac{1}{2}\int_\Omega \frac{|v - u_{\ell-1}|^2}{h} dx + J(v)$$

in the class $\mathcal{W}_w$. The existence of a minimizer of $\mathcal{F}_\ell$ is assured by assumptions iii) and iv) on J. Approximate solutions $u^h(t)$ and $\overline{u}^h(t)$ for $t \in (0,\infty)$ are also defined as follows : for $(\ell-1)h < t \leq \ell h$

$$\begin{cases} u^h(t) = \dfrac{t-(\ell-1)h}{h} u_\ell + \dfrac{\ell h - t}{h} u_{\ell-1} \\[2ex] \overline{u}^h(t) = u_\ell. \end{cases} \tag{4}$$

Then the following facts hold (see, for example, [2]).

THEOREM 1.1. *We have*

1, $\{\| \dfrac{\partial u^h}{\partial t} \|_{L^2((0,\infty)\times\Omega)}\}$ *is uniformly bounded with respect to* h

2, $\{\| \overline{u}^h \|_{L^\infty((0,\infty);\mathcal{V})}\}$ *is uniformly bounded with respect to* h

3, $\{\| u^h \|_{L^\infty((0,\infty);\mathcal{V})}\}$ *is uniformly bounded with respect to* h

4, *for any* $T > 0$, $\{\| u^h \|_{W^{1,2}((0,T)\times\Omega)}\}$ *is uniformly bounded with respect to* h.

Then there exist a sequence $\{h_j\}$ *with* $h_j \to 0$ *as* $j \to \infty$ *and a function* $u \in L^\infty((0,\infty);\mathcal{V}) \cap \bigcup_{T>0} W^{1,2}((0,T)\times\Omega)$ *such that*

5, $\overline{u}^{h_j}$ *converges to* u *as* $j \to \infty$ *weakly star in* $L^\infty((0,\infty);\mathcal{V})$

6, *for any* $T > 0$, u^{h_j} *converges to* u *as* $j \to \infty$ *weakly in* $W^{1,2}((0,T)\times\Omega)$

7, u^{h_j} *converges to* u *as* $j \to \infty$ *strongly in* $L^2((0,T)\times\Omega)$

8, $\overline{u}^{h_j}$ *converges to* u *as* $j \to \infty$ *strongly in* $L^2((0,T)\times\Omega)$

9, $s\text{-}\lim_{t\searrow 0} u(t) = u_0$ *in* $L^2(\Omega)$.

Our problem is *whether the limit* u *in Theorem* 1.1 *is really a weak solution to* (1)–(3). Theorem 1.1 9) means that u satisfies (2) in the weak sense. Theorem 1.1 5) implies that u satisfies (3) in the weak sense since $\overline{u}^h - w \in L^\infty((0,\infty);\mathcal{V}_0)$ for each h. Thus the problem is whether u satisfies (1). Since u_ℓ is the minimizer of $\mathcal{F}_\ell(v)$, we have

$$\operatorname{grad}\mathcal{F}_\ell(u_\ell) = \frac{u_\ell - u_{\ell-1}}{h} + \operatorname{grad} J(u_\ell) = 0$$

in $\mathcal{V}'$. Noting that, for $(\ell-1)h < t < \ell h$, $\dfrac{\partial u^h}{\partial t} = \dfrac{u_\ell - u_{\ell-1}}{h}$, we have, for each $h > 0$,

$$\frac{\partial u^h}{\partial t}(t) + \operatorname{grad} J(\overline{u}^h(t)) = 0 \tag{5}$$

in $\bigcup_{T>0} L^2((0,T);\mathcal{V}')$. Now we put $f^h = \operatorname{grad} J(\overline{u}^h)$. For any $T > 0$ we see by Theorem 1.1 3) that f^{h_j} belongs to $L^2((0,T)\times\Omega)$, converges weakly to an f in $L^2((0,T)\times\Omega)$, and satisfies $\dfrac{\partial u}{\partial t} + f = 0$ in $L^2((0,T)\times\Omega)$. Thus (1) follows when we obtain

$$f = \operatorname{grad} J(u) \quad \text{in} \quad L^2((0,T);\mathcal{V}') \tag{6}$$

for each $T > 0$.

In this note we present one way of obtaining (6) under a little more concrete setting. The basic idea is to employ the topology of the space of “varifolds”, which are a kind of generalized surfaces.

2. Fourth order parabolic equations

Suppose that $F = F(x, y, p)$ is a real valued function in $C^1(\Omega \times \boldsymbol{R} \times \boldsymbol{R}^n)$. For a function $v : \Omega \to \boldsymbol{R}$ we define a functional J as

$$J(v) = \int_\Omega \{|\triangle v|^2/2 + F(x, v, \nabla v)\}dx. \tag{7}$$

Now J is a functional defined on $\mathcal{V} := W^{2,2}(\Omega)$, and we consider that $\mathcal{V}_0 := W_0^{2,2}(\Omega)$. Let $w \in \mathcal{V} = W^{2,2}(\Omega)$. Then $\mathcal{W}_w = \{v \in W^{2,2}(\Omega); v - w \in W_0^{2,2}(\Omega)\}$.

Assumption We assume that there exists a positive constant μ_0 such that

$$\begin{cases} 0 \le F(x, y, p) \le \mu_0(1 + |y|^{q_0} + |p|^{r_0}) \\ |F_p| \le \mu_0(1 + |y|^{q_1} + |p|^{r_1}) \\ |F_y| \le \mu_0(1 + |y|^{q_2} + |p|^{r_2}), \end{cases}$$

where

$$0 < q_0 < \frac{2n}{n-4},\ 0 < q_1 < \frac{n+2}{n-4},\ 0 < q_2 < \frac{n+4}{n-4},$$

$$0 < r_0 < \frac{2n}{n-2},\ 0 < r_1 < \frac{n+2}{n-2},\ 0 < r_2 < \frac{n+4}{n-2}$$

when $n \ge 5$,

$$q_0 > 0,\ q_1 > 0,\ q_2 > 0,$$

$$0 < r_0 < \frac{2n}{n-2},\ 0 < r_1 < \frac{n+2}{n-2},\ 0 < r_2 < \frac{n+4}{n-2}$$

when $n = 3$, 4, and

$$q_0 > 0,\ q_1 > 0,\ q_2 > 0,\ r_0 > 0,\ r_1 > 0,\ r_2 > 0$$

when $n = 1$, 2.

REMARK 2.1. *1. It follows from Assumption and Sobolev's imbedding theorem that J is Gâteaux differentiable on $W^{2,2}(\Omega)$.*

2. It also follows that J is lower semicontinuous.

3. By Assumption and Poincaré's inequality we see that there exist const ants c_0 and c_1 depending on Ω and w such that

$$J(v) \geq c_0 \| v \|_{W^{2,2}(\Omega)} - c_1$$

for each $v \in \mathcal{W}_w$.

Now our equation (1)–(3) is

$$\frac{\partial u}{\partial t}(t,x) + \triangle^2 u - \operatorname{div}\{F_p(x, u(t,x), \nabla u(t,x))\} \qquad (8)$$
$$+F_y(x, u(t,x), \nabla u(t,x)) = 0, \quad x \in \Omega,$$

$$u(0,x) = u_0(x), \qquad x \in \Omega, \qquad (9)$$

$$u(t,x) = w(x), \quad \nabla u(t,x) = \nabla w(x), \qquad x \in \partial\Omega, \qquad (10)$$

where u_0 and w are functions in $W^{2,2}(\Omega)$ with $u_0 - w \in W_0^{2,2}(\Omega)$. Throughout this note ∇ and $\triangle$ are used for differentiations with respect to only x variables. We construct weak solutions u^h and $\overline{u}^h$ to (8)–(10), then Theorem 1.1 holds, and in particular u^h and $\overline{u}^h$ converge to a function u . Now our purpose here is to show

THEOREM 2.2 (MAIN THEOREM). *u is a weak solution to* (8)–(10).

REMARK 2.3. *Theorem 1.1 1) implies $\{\overline{u}^h\}$ is uniformly boun ded in $L^\infty((0,\infty); W^{2,2}(\Omega))$. But we do not have any estimates for the derivative of $\overline{u}^h$ with respect to t, and th us we cannot obtain the strong convergence of $\nabla\overline{u}^h$.*

3. Varifolds

Our basic tool is the varifold theory, and here we briefly review this theory (compare to, for example, [11, Chapter 8]). Let U be an open set of $\boldsymbol{R}^{n+1}$, and let $G = G(n+1, n)$ be the collection of all n-dimensional vector subspaces of $\boldsymbol{R}^{n+1}$, equipped with the metric $d(S,T) = (\sum_{i,j=1}^{n+1} (p_S^{ij} - p_T^{ij})^2)^{\frac{1}{2}}$, where p_S, p_T denote the orthogonal projections of $\boldsymbol{R}^{n+1}$ onto S, T, respectively, and (p_S^{ij}), (p_T^{ij}) are corresponding matrices with respect to the standard orthonormal basis

$\{e_1, \dots, e_{n+1}\}$ for $\boldsymbol{R}^{n+1}$. A Radon measure on $U \times G$ is said to be an *n-varifold* in U.

Suppose that M is a countably n-rectifiable set in U (refer to [11, Chapter 3] for the definition and basic properties of an n-rectifiable set) and that θ is a locally $\mathcal{H}^n$-integrable function on M, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. We define a continuous linear functional on $C_0^0(U \times G)$ by

$$L(\varphi) = \int_M \varphi(z, T_z M)\theta(z) d\mathcal{H}^n \qquad (\varphi \in C_0^0(U \times G)),$$

where $T_z M$ denotes the approximate tangent space of M at z. It follows from the Riesz representation theorem (see, for example, Theorem 4.1 of [11]) that there exists a Radon measure V on $U \times G$ (thus a varifold V in U) such that

$$L(\varphi) = \int_{U \times G} \varphi(z, S) dV(z, S).$$

Such a varifold is called an n-rectifiable varifold, and it is denoted by $\boldsymbol{v}(M, \theta)$. We call θ a multiplicity function. When θ is positive integer valued, we call $\boldsymbol{v}(M, \theta)$ an n-integral varifold. When $\theta \equiv 1$, it is simply denoted by $\boldsymbol{v}(M)$.

Let V be an n-varifold in U. For each Borel set $A \subset U$ we define

$$\mu_V(A) = V(\pi^{-1}(A)) \qquad (\pi : U \times G \ni (z, S) \longmapsto z \in U, \quad A \subset U).$$

Clearly μ_V is a Radon measure on U. It is called the weight of V.

We define for $S \in G(n+1, n)$ and $X = (X^1, \dots, X^{n+1}) \in C_0^1(U; \boldsymbol{R}^{n+1})$

$$\operatorname{div}_S X = \sum_{i,j=1}^{n+1} p_S^{ij} \frac{\partial X^i}{\partial z^j}. \tag{11}$$

The first variation δV for an n-varifold V on U is given by

$$\delta V(X) = \int_{U \times G} \operatorname{div}_S X(z) dV(z, S). \tag{12}$$

We say that V has locally bounded first variation in U if for each $W \subset\subset U$ and each $X \in C_0^1(U; \boldsymbol{R}^{n+1})$ with spt $X \subset W$ there exists a constant $C > 0$ such that

$$|\delta V(X)| \leq C \sup_U |X|.$$

For $z \in U$ we define the upper and lower densities $\Theta^{*n}(\mu_V, z)$ and $\Theta^n_*(\mu_V, z)$ by

$$\Theta^{*n}(\mu_V, z) = \limsup_{\rho \to 0} \frac{\mu_V(\overline{B_\rho(z)})}{\omega_n \rho^n}$$

and

$$\Theta^n_*(\mu_V, z) = \liminf_{\rho \to 0} \frac{\mu_V(\overline{B_\rho(z)})}{\omega_n \rho^n},$$

where $B_\rho(z)$ denotes the open ball with center at z and radius ρ and where ω_n is the volume of the n-dimensional unit ball. If $\Theta^{*n}(\mu_V, z) = \Theta^n_*(\mu_V, z)$, this common value is denoted by $\Theta^n(\mu_V, z)$ and it is called the n-dimensional density of μ_V at z. Now Allard's rectifiability theorem is as follows (Theorem 42.4 of [11]):

Theorem 3.1. *Suppose that V has locally bounded first variation in U and $\Theta^n(\mu_V, z) > 0$ for μ_V-a.e. $z \in U$. Then V is n-rectifiable.*

For $S \in G$ we let $\nu(S) = (\nu_1(S), \ldots, \nu_n(S), \nu_{n+1}(S))$ denote the unit normal to S with $\nu_{n+1}(S) \geq 0$. It is uniquely determined except for the case that $\nu_{n+1}(S) = 0$. For $S \in G$ with ω unit normal to S we have $p_S = I_{n+1} - \omega \otimes \omega$, where $\boldsymbol{a} \otimes \boldsymbol{a}$ denotes $\boldsymbol{a}^t \boldsymbol{a}$ for a column vector $\boldsymbol{a}$. Thus we easily see that ν is a homeomorphism from $G \setminus \mathrm{irr}(G)$ to $S^n_+ = \{\omega \in S^n; \omega_{n+1} > 0\}$, where $\mathrm{irr}(G)$ denotes the set $\{S \in G; \nu_{n+1}(S) = 0\}$ and where S^n denotes the n dimensional unit sphere, and that ν_{n+1} is a continuous function on G.

We are going to use the varifold theory for the case that $U = \Omega \times \boldsymbol{R}$. We use notations x and $z = (x, y)$ for variables in Ω and $U = \Omega \times \boldsymbol{R}$, respectively. The following theorem is a special case of Theorems 4 and 5 of [6, I Section 3.1.5].

Theorem 3.2. *Let v be a function in $W^{1,q}(\Omega)$, $q \geq 1$.*

1) The graph G_v is countably n-rectifiable.

2) $\mathcal{H}^n(G_v) = \int_\Omega \sqrt{1 + |\nabla v(x)|^2} dx$.

3) For $\mathcal{L}^n$-a.e. $x \in \Omega$, the approximate tangent space $T_{(x,v(x))} G_v$ exists and the vector

$$\frac{1}{\sqrt{1 + |\nabla v|^2}} {}^t\Big(- \frac{\partial v}{\partial x^1}, \ldots, - \frac{\partial v}{\partial x^n}, 1\Big)$$

is normal to $T_{(x,v(x))}G_v$.

Theorem 3.2 2) implies that for each Borel set $C \subset \Omega$

$$\mathcal{H}^n((p_x)^{-1}(C) \cap G_v) = \int_C \sqrt{1 + |\nabla v(x)|^2} dx, \tag{13}$$

where p_x is the projection $U \ni z = (x, y) \mapsto x \in \Omega$. Especially, if $C \subset \Omega$ is an $\mathcal{L}^n$ null set, then $(p_x)^{-1}(C) \cap G_v$ is an $\mathcal{H}^n$ null set. Hence Theorem 3.2 3) holds for $\mathcal{H}^n$-a.e. $z = (x, v(x)) \in G_v$. Using the notation above, we see that Theorem 3.2 3) means, for $\mathcal{H}^n$-a.e. $z \in G_v$,

$$\nu(T_z G_v) = \frac{1}{\sqrt{1 + |\nabla v(p_x(z))|^2}} {}^t\Big(- \frac{\partial v}{\partial x^1}(p_x(z)), \ldots, - \frac{\partial v}{\partial x^n}(p_x(z)), 1\Big). \tag{14}$$

By (13) we have, for each measurable function g on Ω,

$$\int_\Omega g(x) \sqrt{1 + |\nabla v(x)|^2} dx = \int_{G_v} g(p_x(z)) d\mathcal{H}^n(z). \tag{15}$$

By the use of (15) we have the following lemma.

Lemma 3.3. *([8, Lemma 2.4]) Put* $V = \boldsymbol{v}(G_v)$ *for* $v \in W^{1,q}(\Omega)$, $q \geq 1$. *Then it holds that*

$$\int_\Omega f(x, v(x), \nabla v(x)) dx = \int_{U \times G} f\Big(z, - \frac{\nu'(S)}{\nu_{n+1}(S)}\Big) \nu_{n+1}(S) dV(z, S)$$

for each continuous function f *on* $U \times \boldsymbol{R}^n$.

Suppose that $u \in L^\infty((0, \infty); W^{2,2}(\Omega)) \cap \bigcup_{T>0} W^{1,2}((0,T) \times \Omega)$ is a weak solution to (8), that is,

$$\begin{gathered} \textstyle\int_0^\infty \psi(t) \int_\Omega \{ \frac{\partial u}{\partial t}(t, x) \phi(x) \\ + \triangle u(t, x) \triangle \phi(x) + F_p(x, u(t, x), \nabla u(t, x)) \cdot \nabla \phi(x) \\ + F_y(x, u(t, x), \nabla u(t, x)) \phi(x) \} dx dt = 0 \end{gathered} \tag{16}$$

for each $\psi(t) \in C_0^\infty(0, \infty)$ and $\phi(x) \in C_0^\infty(\Omega)$. By Theorem 3.2 1) there is a rectifiable varifold $\boldsymbol{v}(G_{u(t,\cdot)})$ in U for $\mathcal{L}^1$-a.e. t. Hence by

(16) and Lemma 3.3 we have

$$\int_0^\infty \psi(t)\{\int_\Omega (\frac{\partial u}{\partial t}(t,x)\phi(x) + \triangle u(t,x)\triangle\phi(x))dx + \int_{U\times G}\{F_p(z,-\frac{\nu'(S)}{\nu_{n+1}(S)})\cdot\nabla\phi(x)\nu_{n+1}(S) + F_y(z,-\frac{\nu'(S)}{\nu_{n+1}(S)})\phi(x)\nu_{n+1}(S)\}dV_t(z,S)\}dt = 0, \tag{17}$$

where $V_t = \boldsymbol{v}(G_{u(t,\cdot)})$ and $\nu'(S) = (\nu_1(S),\ldots,\nu_n(S))$.

Conversely suppose that a function u and a one parameter family of general varifolds V_t for $t\in(0,\infty)$ satisfy (17). Then u is a weak solution to (8) if

$$V_t = \boldsymbol{v}(G_{u(t,\cdot)}) \quad \text{for} \quad \mathcal{L}^1\text{-a.e. } t. \tag{18}$$

4. Outline of the proof of Main Theorem

Let $u^h(t,x)$ and $\overline{u}^h(t,x)$ be approximate solutions. Now we put

$$V_t^h = \boldsymbol{v}(G_{\overline{u}^h(t,\cdot)}).$$

The following theorem can be obtained in the same way as in the proof of Proposition 4.3 of [4].

LEMMA 4.1. *([8, Theorem 3.1]) There exists a subsequence of $\{V_t^h\}$ (still denoted by $\{V_t^h\}$) and a one parameter family of varifolds V_t in $U = \Omega\times\boldsymbol{R}$, for $t\in(0,\infty)$, such that, for each $\psi(t)\in L^1(0,\infty)$ and $\varphi(z,S)\in C_0^0(U\times G)$,*

$$\lim_{h\to 0}\int_0^\infty \psi(t)\int_{U\times G}\varphi(z,S)dV_t^h(z,S)dt = \int_0^\infty \psi(t)\int_{U\times G}\varphi(z,S)dV_t(z,S)dt.$$

By (5) we have

$$\int_0^\infty \psi(t)\int_\Omega\{\frac{\partial u^h}{\partial t}(t,x)\phi(x) + \triangle\overline{u}^h(t,x)\triangle\phi(x) + F_p(x,\overline{u}^h(t,x),\nabla\overline{u}^h(t,x))\cdot\nabla\phi(x) + F_y(x,\overline{u}^h(t,x),\nabla\overline{u}^h(t,x))\phi(x)\}dxdt = 0 \tag{19}$$

for any $\phi \in W_0^{2,2}(\Omega)$ and any $\psi \in C_0^\infty(0,\infty)$. Then we have by Lemma 3.3 that for each $\psi(t) \in C_0^\infty(0,\infty)$ and $\phi(x) \in C_0^\infty(\Omega)$

$$\int_0^\infty \psi(t)\{\int_\Omega (\frac{\partial u^h}{\partial t}(t,x)\phi(x) + \triangle\overline{u}^h(t,x)\triangle\phi(x))dx \tag{20}$$
$$+\int_{U\times G}\{F_p(z,-\frac{\nu'(S)}{\nu_{n+1}(S)})\cdot\nabla\phi(x)\nu_{n+1}(S)$$
$$+F_y(z,-\frac{\nu'(S)}{\nu_{n+1}(S)})\phi(x)\nu_{n+1}(S)\}dV_t^h(z,S)\}dt = 0.$$

LEMMA 4.2. *([8, Lemma 3.5]) Let $f(z,p)$ be a continuous function on $U \times \boldsymbol{R}^n$ and let q and r satisfy $0 < q < 2n/(n-4)$ and $0 < r < 2n/(n-2)$. Suppose that the set $\{x; f(x,y,p) \neq 0$ for some $(y,p)\}$ is contained in a compact subset of Ω and that for each $z = (x,y) \in U$ and each $p \in \boldsymbol{R}^n$*

$$|f(z,p)| \leq \mu_1(1 + |y|^q + |p|^r)$$

holds with a constant μ_1. Then, if $\{V_t^h\}$ and V_t are as in Lemma 4.1, for each $\psi(t) \in L^1(0,\infty)$ we have

$$\lim_{h\to 0}\int_0^\infty \psi(t)\int_{U\times G} f(z,\frac{\nu'(S)}{\nu_{n+1}(S)})\nu_{n+1}(S)dV_t^h(z,S)dt$$
$$= \int_0^\infty \psi(t)\int_{U\times G} f(z,\frac{\nu'(S)}{\nu_{n+1}(S)})\nu_{n+1}(S)dV_t(z,S)dt.$$

Applying this lemma to (20), we find

LEMMA 4.3. *([8, Theorem 3.2]) The function u of Theorem 1.1 and V_t of Lemma 4.1 satisfy (17).*

Lemma 4.3 implies

$$\text{Theorem 2.2} \Leftrightarrow (18).$$

Thereby our problem is reduced to proving (18). Now there are three steps in proving (18):

Step 1. μ_{V_t} and $\mathcal{H}^n \llcorner G_{u(t,\cdot)}$ are mutually absolutely continuous for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$.

(This implies in particular that spt $\mu_{V_t} = \text{spt } \mathcal{H}^n \llcorner G_{u(t,\cdot)}$.)

Step 2. V_t is an n-rectifiable varifold $\boldsymbol{v}(G_{u(t,\cdot)}, \theta_t)$ for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$.

Step 3. $\theta_t(z) = 1$ for $\mathcal{H}^n$-a.e. $z \in G_{u(t,\cdot)}$, for $\mathcal{L}^1$-a.e. t.

Step 1 is of course proved by showing both that $\mu_{V_t}(A) = 0$ implies $(\mathcal{H}^n \llcorner G_{u(t,\cdot)})(A) = 0$ and that $(\mathcal{H}^n \llcorner G_{u(t,\cdot)})(A) = 0$ implies $\mu_{V_t}(A) = 0$. In the proof of the latter part Lemma 4.2 is essentially used ([8, Theorem 4.2]).

Theorem 3.1 implies that Step 2 immediately follows from following two facts.

LEMMA 4.4. *([8, Theorem 4.4]) V_t has locally finite first variation for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$.*

LEMMA 4.5. *([8, Theorem 4.5]) $\Theta^n(\mu_{V_t}, z) \geq 1$ for μ_{V_t}-a.e. $z \in U$, for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$.*

In the proof of Lemma 4.4 we use that

$$(\delta V_t^h)(X) = -\int_\Omega \operatorname{div} \frac{\nabla \overline{u}^h}{\sqrt{1+|\nabla \overline{u}^h|^2}} \cdot (-\nabla \overline{u}^h \cdot X'(x,\overline{u}^h) + X^{n+1}(x,\overline{u}^h))dx$$

and that by Theorem 1.1 1) $\| \overline{u}^h \|_{L^\infty((0,\infty);W^{2,2}(\Omega))}$ is uniformly bounded with respect to h (compare to the proof of [8, Theorem 4.4]).

LEMMA 4.6. *([8, Lemma 4.1]) Put $\mathcal{A} = \{(f,g) \in C_0^0(\Omega) \times C_0^0(\Omega; \boldsymbol{R}^n);$ $\sqrt{f(x)^2 + |g(x)|^2} \leq 1$ for $x \in \Omega\}$. For each $\psi \in L^1(0,\infty)$, $\phi \in C_0^0(U)$, and $(f,g) \in \mathcal{A}$ we have*

$$\int_0^\infty \psi(t) \int_\Omega \phi(x,u(t,x))(f(x) + g(x) \cdot \nabla u(t,x))dxdt$$
$$= \int_0^\infty \psi(t) \int_{U \times G} \phi(z)(f(x)\nu_{n+1}(S) - g(x) \cdot \nu'(S))dV_t(z,S)dt.$$

Since $|f(x)\nu_{n+1}(S) - g(x)\cdot\nu'(S)| \le 1$ for $(f,g)\in\mathcal{A}$, we have by Lemma 4.6, for each nonnegative functions $\psi\in L^1(0,\infty)$ and $\phi\in C_0^0(U)$,

$$\int_0^\infty \psi(t)\int_U \phi(z)d(\mathcal{H}^n\llcorner G_{u(t,\cdot)})dt \le \int_0^\infty \psi(t)\int_U \phi(z)d\mu_{V_t}dt.$$

Thus, approximating the characteristic function of $B_\rho(z_0)$ from above, we obtain

$$(\mathcal{H}^n\llcorner G_{u(t,\cdot)})(B_\rho(z_0)) \le \mu_{V_t}(\overline{B_\rho(z_0)})$$

for $\mathcal{L}^1$-a.e. $t\in(0,\infty)$. This implies Lemma 4.5.

Finally we show Step 3, the end of the proof of which is at the same time the end of the proof of Theorem 2.2. When $f\equiv 1$ on $p_x(\text{spt }\phi)$ and $g\equiv 0$ in Lemma 4.6, we have

$$\int_0^\infty \psi(t)\int_\Omega \phi(x,u(t,x))dxdt = \int_0^\infty \psi(t)\int_{U\times G}\phi(z)\nu_{n+1}(S)dV_t(z,S)dt. \tag{21}$$

By the definition of an n -rectifiable varifold we have, for $\mathcal{L}^1$-a.e. $t\in(0,\infty)$,

$$\int_{U\times G}\phi(z)\nu_{n+1}(S)dV_t(z,S) = \int_U \phi(z)\nu_{n+1}(T_zG_{u(t,\cdot)})\theta_t(z)d(\mathcal{H}^n\llcorner G_{u(t,\cdot)}). \tag{22}$$

The right hand side of (22) coincides with

$$\int_\Omega \phi(x,u(t,x))\theta_t(x,u(t,x))dx$$

by (14) and (15). On the other hand we have by (21) that for $\mathcal{L}^1$-a.e. $t\in(0,\infty)$ the left hand side of (22) coincides with $\int_\Omega \phi(x,u(t,x))dx$. Then the conclusion follows.

5. Notes

1. Second order quasilinear parabolic equations

Our method here is also available for second order quasilinear parabolic equations. More precisely, if J has the form

$$J(v) = \int_\Omega F(x, v, \nabla v)dx \tag{23}$$

and F satisfies

$$\begin{cases} \lambda|p|^2 \le F(x,u,p) \le \mu(1+|u|^2+|p|^2) \\ |F_p|,\ |F_{p,x}|,\ |F_u|,\ |F_{x,u}| \le \mu(1+|u|+|p|) \\ |F_{p,u}|,\ |F_{u,u}| \le \mu \\ m|\xi|^2 \le \displaystyle\sum_{\alpha,\beta=1}^{n} F_{p_\alpha p_\beta}(x,u,p)\xi_\alpha\xi_\beta \le M|\xi|^2 \qquad \text{for any} \quad \xi \end{cases} \tag{24}$$

with some positive constants $\mu,\ \lambda,\ m,\ M$, then in the same way as in ours we can obtain our main theorem for this functional. The point is to obtain uniform estimate of second derivatives with respect to space variables of approximate solutions $\overline{u}^h$. In fact by the use of (24) we have

PROPOSITION 5.1. *For any $\Omega' \subset\subset \Omega$ and for any $T > 0$ the set $\{\| \nabla^2 \overline{u}^h \|_{L^2((0,T)\times\Omega')}\}$ is uniformly bounded with respect to h.*

This proposition can be proved in the same way as in the proof of Theorem 1.1 of [5, Chapter II]. However condition (24) is somewhat unnatural. For example, a function of the form $F(x,u,p) = \sum_{\alpha,\beta=1}^{n} a^{\alpha\beta}(x,u)p_\alpha p_\beta$ is not admitted. A natural example which satisfies (24) is a function F which is independent of u. But in this case the functional J is convex, and, if J is convex, we can obtain our main theorem in a more simple way (see Appendix of [8]).

2. Vectorial cases

So far we restrict our discussions to scalar cases. Here we briefly comment on extending our method to vectorial cases.

For a function $v : \Omega \to \boldsymbol{R}^N$ let us consider a functional J as in (7) with $F \in C^1(\Omega \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN})$. As a result we can say that, if

$$n \le 3 \qquad \text{or} \qquad N \le 2, \tag{25}$$

then our main theorem holds. Condition (25) is required in a geometrical reason.

As has mentioned above, our method is also available for J as in (23). In the scalar case (24) forces F to be convex with respect to p. But in the vectorial case (24) can be slightly changed and under this changed condition F is not necessarily convex but quasiconvex with respect to p. As a result, if (25) is satisfied, our main theorem holds for some quasiconvex functionals. Details about this fact will be discussed in [7].

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