Rend. Istit. Mat. Univ. Trieste
Vol. XXXI Suppl. 2, 103–127 (2000)

Critical Exponent for Wave Equation with Potential

Vladimir Georgiev, Charlotte Heiming and Hideo Kubo (*)

Summary. - *We establish a weighted L^∞ estimate for the solution of the linear wave equation with a smooth positive potential depending only on space variables.*

This estimate is similar to F.John's estimates in ([9]) and enables one to prove existence of global small data solution for the corresponding semilinear wave equation with potential.

1. Introduction

In this work we study the scalar wave equation

$$\partial_t^2 u - \Delta u + V(x)u = F(u), \tag{1}$$

(*) Authors' addresses: V. Georgiev, Università dell'Aquila, Dipartimento di Matematica pura ed applicata, Via Vetoio, I-67010 Coppito (L'Aquila), Italy, e-mail: georgiev@univaq.it
C. Heiming, Fachbereich Mathematik und Statistik, Universität Konstanz, Fach D 187, 78457 Konstanz, Germany, e-mail: Charlotte.Kerler@uni-konstanz.de
H. Kubo, Department of Applied Mathematics, Faculty of Engineering, Shizuoka University, Hamamatsu 432-8561, Japan, e-mail: tshkubo@ipc.shizuoka.ac.jp
AMS Subject Classification: 35A08, 35L15, 35L05, 35L70
Keywords: Wave equation, decay estimates, semilinear equation, global solution, wave equation with potential.
The first author was partially supported by the Programma Nazionale M.U.R.S.T. "Problemi e Metodi nella Teoria delle Equazioni Iperboliche."
The second author was partially supported by Contract TMR-Network "Hyperbolic Systems of Conservation Laws".
The third author was partially supported by Grant-in-Aid for Science Research (No.09304012), Ministry of Education, Science and Culture, Japan.

where $x \in \mathbf{R}^3$ and $F(u)$ behaves like $|u|^p$ for small values of u. This equation is one of the simplest models of nonlinear wave equation having linear part with nonconstant coefficients.

The case of constant potential $V(x) = V_0 \geq 0$ is intensively studied in the last 20 years (see [19], [6] for more complete list of references).

For the case $V = 0$ a classical result due to F.John [9] shows that the critical exponent for small amplitude solutions with compact support is

$$p_c = 1 + \sqrt{2}.$$

More precisely, this means that the solution is global for $p > p_c$ and blows up if $1 < p \leq p_c$.

The key point in the approach developed by F.John is the evaluation of the fundamental solution of the linear wave equation with constant coefficients. In this case for lower dimensional case $n = 2, 3$ one can exploit positivity of the fundamental solution as well as the explicit representation of this fundamental solution. For the higher dimensional case interpolation methods combined with Fourier representation of the solution lead to suitable a priori estimate of Strichartz type (see [21]). Despite of the fact the Strichartz estimate can be generalized for the case of wave equation with variable coefficients or for the case of wave equation with potential (see [3]), the application of this estimate to semilinear problem (1) needs a suitable refinement of this apriori estimate. More presicely, we need a weighted variant of Strichartz estimate (similar to the one established in [4]). Since the proof of this weighted Strichartz estimate requires explicit representation of the fundamental solution, we see the necessity to estimate suitably the fundamental solution of perturbed wave equation with potential.

In a recent work ([20]) W.Strauss and K.Tsutaya studied the case of a small potential $V(x)$ decaying faster than $|x|^{-2}$ at infinity. In this case, the critical exponent is the same and the smallness assumption of the potential plays crucial role in the existence part.

The main goal of this work is to obtain weighted F.John type estimate for the linear wave equation with potential without any restriction of type smallness of the potential. On the basis of this estimate we aim at showing the existence of global solution for the supercritical case $p > p_c$.

Since the decay of the linear wave equation enables one to adapt a suitable contraction argument, it is natural to assume the operator $-\Delta + V(x)$ has no point spectrum. A simple condition that guarantees this property is

$$V(x) \geq 0. \tag{2}$$

For simplicity in this work we assume also $V \in C_0^\infty(\mathbf{R}^3)$.

The main tool to establish existence of small data solution is the following F.John type apriori estimate for the solution $u = L(F)$ of the perturbed wave equation

$$\partial_t^2 u - \Delta u + Vu = F \quad \text{in } [0,\infty) \times \mathbf{R}^3, \tag{3}$$

$$u(0,x) = 0, \quad \partial_t u(0,x) = 0 \quad \text{in } \mathbf{R}^3. \tag{4}$$

THEOREM 1.1. *Let $F \in C(\mathbf{R}^3 \times [0,\infty))$ and $\tau_\pm = 1 + |t \pm |x||$.*
If $\lambda \geq 0$ and $\mu > 0$ satisfy

$$\lambda < 1, \quad \mu > 2 + \lambda, \tag{5}$$

then we have

$$\|\tau_+ \tau_-^\lambda L(F)\|_{L^\infty} \leq C_1 \|\tau_+^\mu \tau_- F\|_{L^\infty}. \tag{6}$$

As we underlined before, the main difficulty to establish such type of estimate is the absence of explicit formula of the fundamental solution for perturbed wave equation with potential. To this end we shall apply weighted estimates for the resolvent of the operator $-\Delta + V$. Resolvent estimates as well as generalized eigenfuction expansion have been intensively studied for the case of the Schrödinger operator (see [7], [1], [2]). Of special importance for us is the weighted resolvent estimate established by C.Morawetz in [16] (see estimate (28) below). Connecting the representation of the solution of the perturbed wave equation with the resolvent of $-\Delta + V$, we shall establish an asymptotic type expansion of the amplitude in the Fourier representation of the solution (see the relation (19) and Theorem 2.3 below). Combining this representation with stationary phase method, we arrive at the needed estimate (6).

Once this apriori estimate is established, we can treat the existence of small data solution for the problem (1) and establish a global existence result for the supercritical case

$$p > p_c = 1 + \sqrt{2}.$$

We shall sketch the plan of the work. In section 2 we study the asymptotic properties of the amplitude of the generalized Fourier transform for the operator $-\Delta + V$. A pointwise estimate of the fundamental solution for points far away from the caracteristic light cone is represented in section 3. A simplified proof of F.John's type estimates for the free wave equation is given in section 4. Using suitable decomposition of the solution of the wave equation, we combine the estimate from section 3 and Gronwall lemma and complete the proof (6). The application of this estimate to semilinear wave equation with potential is discussed in section 6.

2. Generalized Fourier transform

We assume that the potential V is of class $C_0^\infty(\mathbf{R}^3)$. Let $H_0 = \sqrt{-\Delta}$, $H = \sqrt{-\Delta + V}$ and let

$$R_0(z) = (-\Delta - z)^{-1} = (H_0^2 - z)^{-1}, \quad R(z) = (H^2 - z)^{-1}. \tag{7}$$

Then the generalized (distorted) Fourier transform is defined (see [5] vol.II, Def.14.6.3) by

$$\mathrm{F}_\pm(f)(\xi) = F[(I + VR_0(|\xi|^2 \pm i0))^{-1}(f)](\xi), \tag{8}$$

where $F(g)(\xi) = \int e^{-ix\xi} g(x)dx$ is the standard Fourier transform. The existence of the limits

$$R_0(|\xi|^2 \pm i0)f = \lim_{\varepsilon \downarrow 0} R_0(|\xi|^2 \pm i\varepsilon)f \tag{9}$$

is the well-known limiting absorption principle. Using the identity

$$R(z) - R_0(z) = -R_0(z)VR(z), \tag{10}$$

one can see that

$$(I + VR_0(|\xi|^2 + i0))^{-1} = I - VR(|\xi|^2 + i0) \tag{11}$$

Since $R(|\xi|^2+i0)$ is a self-adjoint operator, we have

$$F_+(f)(\xi)=\int[e^{-ix\xi}+\varphi_+(x,\xi)]f(x)dx, \tag{12}$$

where

$$\varphi_+(x,\xi)=R(|\xi|^2+i0)(V_\xi)(x), \quad V_\xi(x)=-V(x)e^{-ix\xi}. \tag{13}$$

It is well known that $\varphi_+(x,\xi)$ is the unique outgoing solution of the problem

$$(-\Delta_x+V-|\xi|^2)\varphi_+(x,\xi)=V_\xi(x) \tag{14}$$

satisfying the Sommerfeld radiation condition (see [15])

$$\varphi_+(x,\xi)=\frac{e^{i|x||\xi|}}{|x|}(a(\frac{x}{|x|})+O(\frac{1}{|x|})) \quad \text{as } |x|\to\infty \tag{15}$$

for $n=3$. Here $a(\theta):\ S^2\to\mathbf{R}$ is the so called scattering amplitude. Note that for $n=3$

$$R_0(|\xi|^2+i0)f(x)=\frac{1}{4\pi}\int\frac{e^{i|\xi||x-y|}}{|x-y|}f(y)dy. \tag{16}$$

The condition (15) suggests to consider

$$A_+(x,\xi)=e^{-i|x||\xi|}\varphi_+(x,\xi)=e^{-i|x||\xi|}R(|\xi|^2+i0)(V_\xi)(x). \tag{17}$$

Then the result in [3] shows that

PROPOSITION 2.1. *If $n=3$, $k+j=0,1,2$, then*

$$|\partial_\omega^k\partial_{|\xi|}^j A_+(x,\xi)|\le C\langle x\rangle^{-1}(1+|\xi|^{-j})\langle\xi\rangle^k \quad \text{for } x,\xi\in\mathbf{R}^3, \tag{18}$$

where C is a constant independent of x, $\xi\in\mathbf{R}^3$.

REMARK 2.2. *In the above theorem ∂_ω^k denotes differentiation of order k in the coordinates $\omega=\xi/|\xi|\in\mathbf{S}^2$. It is not difficult to see that the embedding $\mathbf{S}^2\subset\mathbf{R}^3$ enables one to replace ∂_ω^k by*

$$\Omega^\beta=\Omega_{12}^{\beta_1}\Omega_{23}^{\beta_2}\Omega_{13}^{\beta_3},$$

where $\beta=(\beta_1,\beta_2,\beta_3)$ is a multi index with $|\beta|=k$ and

$$\Omega_{ij}=\xi_i\partial_{\xi_j}-\xi_j\partial_{\xi_i}, 1\le i<j\le 3.$$

Our next step is to study more precisely the behavior of $\partial_\xi^\alpha A_+(x,\xi)$ for $|\xi| \geq 1$. It follows from the resolvent identity (10) and (17) that

$$A_+(x,\xi) = \sum_{j=0}^{N-1} A_+^{(j)}(x,\xi) + R_N(x,\xi), \tag{19}$$

where

$$A_+^{(j)}(x,\xi) = (-1)^j e^{-i|x||\xi|}[R_0(|\xi|^2+i0)V]^j R_0(|\xi|^2+i0)(V_\xi)(x) \tag{20}$$

and

$$R_N(x,\xi) = (-1)^N e^{-i|x||\xi|}[R_0(|\xi|^2+i0)V]^N R(|\xi|^2+i0)(V_\xi)(x). \tag{21}$$

Now we introduce the following definition.

Definition 1.
We say $A(x,\xi) \in S^m$ if and only if for any nonnegative integer k, there is a number $C = C(k)$ such that

$$|\partial_\xi^\alpha A(x,\xi)| \leq C\langle\xi\rangle^m \quad \text{for } |\xi| \geq 1,\ 0 \leq |\alpha| \leq k.$$

It is clear that $V_\xi(x) \in S^0$.

THEOREM 2.3. *(***Main Theorem***)*

$$\langle x\rangle A_+^{(j)}(x,\xi) \in S^{-j}, \quad \langle x\rangle R_N(x,\xi) \in S^{-N}.$$

In order to show the theorem, we prepare the following two lemmas.

LEMMA 2.4.
Let $A(x,\xi) \in S^m$. Set

$$T_0(A)(x,\xi) = e^{-i|x||\xi|}R_0(|\xi|^2+i0)(VA(\cdot,\xi))(x).$$

Then we have for any $N \in R$

$$|\partial_\xi^\alpha T_0(A)(x,\xi)| \leq C\langle x\rangle^{-1} \sum_{|\beta|\leq|\alpha|} \|\langle\cdot\rangle^{-N}\partial_\xi^\beta A(\cdot,\xi)\|_{L^2}. \tag{22}$$

Proof. Since ∂_{ξ_k} is a linear combination of $\partial_{|\xi|}$ and $|\xi|^{-1}\Omega_{ij}$ with coefficients homogeneous of degree 0 with respect to ξ, we can reduce the proof to the following estimate

$$|\partial^k_{|\xi|} T_0(A)(x,\xi)| \le C\langle x\rangle^{-1} \sum_{l=0}^{k} \|\langle\cdot\rangle^{-N} \partial^l_{|\xi|} A(\cdot,\xi)\|_{L^2}. \tag{23}$$

It follows from (16) that

$$\partial^k_{|\xi|} T_0(A)(x,\xi) = \partial^k_{|\xi|} \int \frac{e^{i|\xi|(|x-y|-|x|)}}{|x-y|} V(y)A(y,\xi)dy$$
$$= \sum_{l=0}^{k} C_l \int \frac{(i(|x-y|-|x|))^{k-l}}{|x-y|} e^{i|\xi|(|x-y|-|x|)} V(y) \partial^l_{|\xi|} A(y,\xi)dy.$$

Let $M > 3/2$. Since $||x-y|-|x|| \le |y|$, we see that

$$|\partial^k_{|\xi|} T_0(A)(x,\xi)| \le$$
$$C\|\langle\cdot\rangle^{k+M+N} V\|_{L^\infty} \sum_{l=0}^{k} C_l \int \frac{1}{|x-y|\langle y\rangle^{M+N}} |\partial^l_{|\xi|} A(y,\xi)|dy$$
$$\le \|\langle\cdot\rangle^{k+M+N} V\|_{L^\infty} \sum_{l=0}^{k} \|\langle\cdot\rangle^{-N} \partial^l_{|\xi|} A(\cdot,\xi)\|_{L^2} \left(\int \frac{dy}{|x-y|^2\langle y\rangle^{2M}}\right)^{\frac{1}{2}}.$$

Therefore (23) follows from

$$I := \int \frac{dy}{|x-y|^2\langle y\rangle^{2M}} \le C\langle x\rangle^{-2}, \tag{24}$$

where $M > 3/2$. We divide I into I_1 and I_2 as follows.

$$I_1 = \int_{|x-y|\le 1} \frac{dy}{|x-y|^2\langle y\rangle^{2M}}, \quad I_2 = \int_{|x-y|\ge 1} \frac{dy}{|x-y|^2\langle y\rangle^{2M}}.$$

It is easy to see that

$$I_1 \le C\langle x\rangle^{-2M} \int_{|x-y|\le 1} \frac{dy}{|x-y|^2} \le C\langle x\rangle^{-2M}. \tag{25}$$

While we have

$$\begin{aligned} I_2 &\le \int \frac{dy}{\langle x-y\rangle^2 \langle y\rangle^{2M}} \\ &\le C\langle x\rangle^{-2} \left(\int_{|y|\le|x|/2} \frac{dy}{\langle y\rangle^{2M}} + \int_{|y|\ge|x|/2} \frac{dy}{\langle x-y\rangle^2 \langle y\rangle^{2M-2}} \right) \\ &\le C\langle x\rangle^{-2} \left(\int \frac{dy}{\langle y\rangle^{2M}} + \int \frac{dy}{\langle x-y\rangle^{2M}} \right), \end{aligned}$$

since $A^\alpha B^\beta \le A^{\alpha+\beta} + B^{\alpha+\beta}$ for A, B, α and β are positive numbers. By the assumption $M > 3/2$, we get

$$I_2 \le C\langle x\rangle^{-2},$$

and in combination with (25) we obtain (24). This completes the proof. □

LEMMA 2.5.

Let $A(x,\xi) \in S^m$ and let $\tilde{R}(z)$ be either $R_0(z)$ or $R(z)$ defined in (7). Let k be a nonnegative integer and let $N > k + (1/2)$ and $M \in \mathbf{R}$. Suppose that $\xi \in \mathbf{R}^3$ with $|\xi| \ge 1$ and $x \in \mathbf{R}^3$. Then we have

$$\begin{aligned} \sum_{|\alpha|=k} \|\langle\cdot\rangle^{-N} \partial_\xi^\alpha [\tilde{R}(|\xi|^2 + i0)(VA(\cdot,\xi))]\|_{L^2} \le \\ C|\xi|^{-1} \sum_{|\beta|\le k} \|\langle\cdot\rangle^{-M} \partial_\xi^\beta A(\cdot,\xi)\|_{L^2}. \end{aligned} \tag{26}$$

Proof. Using polar coordinates as above we reduce the proof to the following inequality

$$\|\langle\cdot\rangle^{-N} \partial_{|\xi|}^k [\tilde{R}(|\xi|^2 + i0)(VA(\cdot,\xi))]\|_{L^2} \le C|\xi|^{-1} \sum_{l=0}^{k} \|\langle\cdot\rangle^{-M} \partial_{|\xi|}^l A(\cdot,\xi)\|_{L^2}. \tag{27}$$

We use the estimate

$$\|\langle x\rangle^{-\frac{1}{2}-\varepsilon-l} \left(\frac{d}{d|\xi|}\right)^l \tilde{R}(|\xi|^2 + i0) g\|_{L^2} \le C|\xi|^{-1} \|\langle x\rangle^{\frac{1}{2}+\varepsilon+l} g\|_{L^2}, \tag{28}$$

where $|\xi| \geq 1$, l is a nonnegative integer and $\varepsilon > 0$. For the proof, see Theorem 1.10 of [8] or Korollar 2.48 of [11]. It follows that

$$\begin{aligned} & \|\langle\cdot\rangle^{-\frac{1}{2}-\varepsilon-k}\partial^k_{|\xi|}[\tilde{R}(|\xi|^2+i0)(VA(\cdot,\xi))]\|_{L^2} \qquad (29) \\ \leq\ & C|\xi|^{-1}\sum_{l=0}^{k}\|\langle\cdot\rangle^{\frac{1}{2}+\varepsilon+k-l}V(\cdot)\partial^l_{|\xi|}A(\cdot,\xi)\|_{L^2} \\ \leq\ & C|\xi|^{-1}\|\langle\cdot\rangle^{\frac{1}{2}+\varepsilon+k+M}V\|_{L^\infty}\sum_{l=0}^{k}\|\langle\cdot\rangle^{-M}\partial^l_{|\xi|}A(\cdot,\xi)\|_{L^2}, \end{aligned}$$

which implies (27), since $N > k + (1/2)$. □

Proof of Theorem 2.1. It follows from Lemmas 2.1 and 2.2 that

$$\begin{aligned} |\partial^\alpha_\xi A^{(j)}_+(x,\xi)| &\leq C\langle x\rangle^{-1}|\xi|^{-j}\sum_{|\beta|\leq|\alpha|}\|\langle y\rangle^{-M}\partial^\beta_\xi e^{-iy\xi}\|_{L^2} \\ &\leq C\langle x\rangle^{-1}\langle\xi\rangle^{-j} \end{aligned}$$

for $|\xi| \geq 1$, provided $M > |\alpha| + (3/2)$. Therefore $\langle x\rangle A^{(j)}_+(x,\xi) \in S^{-j}$. Similarly, we see from Lemmas 2.1 and 2.2 that $\langle x\rangle R_N(x,\xi) \in S^{-N}$. □

3. An application of Generalized Fourier transform

Let $V \in C_0^\infty(\mathbf{R}^3)$ be such that $\mathrm{supp} V \subset \{|x| \leq R\}$. Let $\varphi \in C^\infty(\mathbf{R})$ such that

$$\varphi(s) = \begin{cases} 0 & \text{for} \quad s \leq 1, \\ 1 & \text{for} \quad s \geq 2 \end{cases} \qquad (30)$$

Consider the kernel $K(t,x,y)$ of the operator

$$P = \frac{\sin tH}{H}\varphi(H) - \frac{\sin tH_0}{H_0}\varphi(H_0), \qquad (31)$$

that is, $K(t,x,y)$ is a distribution satisfying

$$P(f)(t,x) = \int K(t,x,y)f(y)dy.$$

THEOREM 3.1.
Suppose that $|x| \leq R$ *and* $t - |y| \geq C^*R$ *with* $C^* \geq 28$. *Then* $K(t,x,y)$ *is a classical function satisfying*

$$|K(t,x,y)| \leq C\langle t\rangle^{-1}\langle t - |x-y|\rangle^{-1}. \tag{32}$$

Proof. Since

$$\frac{\sin tH}{H}\varphi(H)f(x) = \iint \left(e^{ix\xi} + \overline{\varphi_+(x,\xi)}\right)\frac{\sin t|\xi|}{|\xi|}\varphi(|\xi|)\left(e^{-iy\xi} + \varphi_+(y,\xi)\right)f(y)dyd\xi,$$

we see from (31) that

$$K(t,x,y) = (2i)^{-1}\sum_{j=1}^{3}\{K_j^+(t,x,y) - K_j^-(t,x,y)\}, \tag{33}$$

where we have set

$$K_1^\pm(t,x,y) = \int e^{i(x\xi \pm t|\xi|)}\varphi_+(y,\xi)\frac{\varphi(|\xi|)}{|\xi|}d\xi, \tag{34}$$

$$K_2^\pm(t,x,y) = \int e^{i(-y\xi \pm t|\xi|)}\overline{\varphi_+(x,\xi)}\frac{\varphi(|\xi|)}{|\xi|}d\xi, \tag{35}$$

$$K_3^\pm(t,x,y) = \int e^{\pm it|\xi|}\overline{\varphi_+(x,\xi)}\varphi_+(y,\xi)\frac{\varphi(|\xi|)}{|\xi|}d\xi. \tag{36}$$

Estimate of $K_1^\pm(t,x,y)$

It follows from (34), (17) and (19) that

$$K_1^\pm(t,x,y) = \sum_{j=0}^{3} K_{1,j}^\pm(t,x,y),$$

where we have set

$$K^{\pm}_{1,j}(t,x,y) = \int e^{i(x\xi \pm t|\xi| + |y||\xi|)} A^{(j)}_{+}(y,\xi) \frac{\varphi(|\xi|)}{|\xi|} d\xi, \quad 0 \le j \le 2,$$

and

$$K^{\pm}_{1,3}(t,x,y) = \int e^{i(x\xi \pm t|\xi| + |y||\xi|)} R_3(y,\xi) \frac{\varphi(|\xi|)}{|\xi|} d\xi.$$

First consider $K^{\pm}_{1,0}(t,x,y)$. From (20) and (16) we have

$$\begin{aligned} K^{\pm}_{1,0}(t,x,y) &= \int e^{i(x\xi \pm t|\xi|)} \frac{\varphi(|\xi|)}{|\xi|} R_0(|\xi|^2 + i0)(V_\xi)(y) d\xi \quad (37) \\ &= \int K^{\pm}(t,x,y,z) \frac{V(z)}{|y-z|} dz, \end{aligned}$$

where we have set

$$K^{\pm}(t,x,y,z) = -\int e^{i(x\xi \pm t|\xi| + |\xi||y-z| - z\xi)} \frac{\varphi(|\xi|)}{|\xi|} d\xi. \qquad (38)$$

When $z \in \operatorname{supp} V$, we see from the assumption that

$$|x\hat{\xi} \pm t + |y-z| - z\hat{\xi}| \ge t - |y| - 3R \ge (t - |y|)/2, \quad \hat{\xi} = \xi/|\xi|.$$

Therefore, by the integration by parts with respect to $|\xi|$, we get

$$|K^{\pm}(t,x,y,z)| \le C\langle t - |y| \rangle^{-M}$$

for any $M > 0$. Moreover, we see from the proof of (24) that for $M' > 3$

$$\int \frac{V(z)}{|y-z|} dz \le C\langle y \rangle^{-1} \|\langle z \rangle^{M'} V\|_{L^\infty}. \qquad (39)$$

Therefore, we get

$$|K^{\pm}_{1,0}(t,x,y)| \le C\langle t - |y| \rangle^{-M} \langle y \rangle^{-1} \|\langle z \rangle^{M'} V\|_{L^\infty},$$

which yields the desired estimate. Analogously, we can deal with $K^{\pm}_{1,j}(t,x,y)$ $(j = 1,2)$.

Next consider $K^{\pm}_{1,3}(t,x,y)$. Notice that

$$|x\hat{\xi} \pm t + |y|| \ge t - |y| - R \ge (t - |y|)/2.$$

It follows from the integration by parts and Theorem 1 that

$$|K_{1,3}^{\pm}(t,x,y)| \le C\langle t-|y|\rangle^{-M}\langle y\rangle^{-1}$$

for any $M>0$. We thus obtained (32) for $K_1^{\pm}(t,x,y)$.

Estimate of $K_2^{\pm}(t,x,y)$

If $|y|<t/2$ then we can use the integration by parts with respect to $|\xi|$ as above in the estimate of the term $K_1^{\pm}(t,x,y)$. For $|y|\ge t/2$ we choose the coordinates so that $y=(0,0,|y|)$. It follows from (35), (17) and (19) that

$$K_2^{\pm}(t,x,y)=\sum_{j=0}^{3}K_{2,j}^{\pm}(t,x,y),$$

where we have set

$$K_{2,j}^{\pm}(t,x,y)=\overline{K_{1,j}^{\mp}(t,y,x)} \quad 0\le j\le 3.$$

For simplicity we shall treat only the term $K_{2,0}^{\pm}(t,x,y)$.

From (37) we have

$$K_{2,0}^{\pm}(t,x,y)=\int \overline{K^{\mp}(t,y,x,z)}\frac{V(z)}{|y-z|}dz.$$

Introduce polar coordinates

$$\begin{aligned}\xi_1&=\rho\cos\theta_1\sin\theta_2,\\ \xi_2&=\rho\sin\theta_1\sin\theta_2\\ \xi_3&=\rho\cos\theta_2.\end{aligned} \tag{40}$$

where $\rho=|\xi|\ge 0$, $\theta_1\in[0,2\pi]$, $\theta_2\in[0,\pi]$. Then we see from (38) that

$$\overline{K^{\mp}(t,y,x,z)}=$$
$$-\int_0^\infty e^{i(\pm t\rho-\rho|x-z|)}\rho\varphi(\rho)d\rho\int_0^{2\pi}d\theta_1\int_0^\pi e^{-i|y|\rho\cos\theta_2}e^{iz\xi}\sin\theta_2 d\theta_2,$$

where $\xi = \xi(\rho, \theta_1, \theta_2)$ is defined by (40). By integration by parts, we see that

$$\begin{aligned} \theta_2 - \text{integral} &= \frac{1}{i|y|\rho}(e^{i(|y|\rho - z_3\rho)} - e^{i(-|y|\rho + z_3\rho)}) \\ &\quad -\frac{1}{|y|}\int_0^\pi e^{-i|y|\rho\cos\theta_2 + iz\xi}\partial_{\theta_2}(z\hat{\xi})d\theta_2, \end{aligned}$$

where $\hat{\xi} = \xi(\rho, \theta_1, \theta_2)/\rho$. Noting that

$$|\partial_{\theta_2}(z\hat{\xi})| \le |z| \le R \quad \text{on supp} V$$

and integrating by parts with respect to ρ, we arrive at

$$|K^{\pm}_{2,0}(t,x,y)| \le C|y|^{-1}\langle t - |y|\rangle^{-M}$$

for any $M > 0$, which implies (32) for $K^{\pm}_{2,0}(t,x,y)$ when $|y| \ge t/2$. In a similar way we proceed for the other terms $K^{\pm}_{2,j}(t,x,y)$, $j = 1,2,3$.

Estimate of $K_3^{\pm}(t,x,y)$

It follows from (36), (17) and (19) that

$$K_3^{\pm}(t,x,y) = \sum_{j=0}^{3} K^{\pm}_{3,j}(t,x,y),$$

where we have set

$$K^{\pm}_{3,j}(t,x,y) = \int e^{i(\pm t|\xi| - |x||\xi| + |y||\xi|)}\overline{A_+^{(j)}(x,\xi)}A_+^{(j)}(y,\xi)\frac{\varphi(|\xi|)}{|\xi|}d\xi, \quad 0 \le j \le 2,$$

and

$$K^{\pm}_{3,3}(t,x,y) = \int e^{i(\pm t|\xi| - |x||\xi| + |y||\xi|)}\overline{R_3(y,\xi)}R_3(y,\xi)\frac{\varphi(|\xi|)}{|\xi|}d\xi.$$

First consider $K^{\pm}_{3,0}(t,x,y)$. From (20) and (16) we have

$$\begin{aligned} K^{\pm}_{3,0}(t,x,y) &= \int e^{\pm it|\xi|}\frac{\varphi(|\xi|)}{|\xi|}\overline{R_0(|\xi|^2 + i0)(V_\xi)(x)}R_0(|\xi|^2 + i0)(V_\xi)(y)d\xi \\ &= \iint K^{\pm}(t,x,y,z_1,z_2)\frac{V(z_1)}{|y - z_1|}\frac{V(z_2)}{|y - z_2|}dz_1dz_2, \end{aligned}$$

where we have set

$$K^{\pm}(t,x,y,z_1,z_2) = \int e^{i(\pm t|\xi| - |\xi||x-z_1| + z_1\xi + |\xi||y-z_2| - z_2\xi)} \frac{\varphi(|\xi|)}{|\xi|} d\xi.$$

When z_1, $z_2 \in \mathrm{supp} V$, we have

$$|\pm t - |x-z_1| + z_1\hat{\xi} + |y-z_2| - z_2\hat{\xi}| \geq t - |y| - 5R \geq (t-|y|)/2, \quad \hat{\xi} = \xi/|\xi|,$$

hence we get

$$|K^{\pm}(t,x,y,z)| \leq C\langle t - |y|\rangle^{-M}$$

for any $M > 0$. By (39) we see that for $M' > 3$

$$|K^{\pm}_{3,0}(t,x,y)| \leq C\langle t-|y|\rangle^{-M}\langle x\rangle^{-1}\langle y\rangle^{-1}\|\langle z\rangle^{M'}V\|^2_{L^\infty},$$

which yields the desired estimates. Analogously, we can deal with $K^{\pm}_{3,j}(t,x,y)$ $(j = 1,2)$.

Next consider $K^{\pm}_{3,3}(t,x,y)$. Notice that

$$| \pm t - |x| + |y|| \geq t - |y| - R \geq (t-|y|)/2.$$

It follows from the integration by parts and Theorem 1 that

$$|K^{\pm}_{3,3}(t,x,y)| \leq C\langle t-|y|\rangle^{-M}\langle x\rangle^{-1}\langle y\rangle^{-1}$$

for any $M > 0$. We thus obtained (32) for $K^{\pm}_3(t,x,y)$. This completes the proof. □

Finally, we turn to the following

THEOREM 3.2.

Suppose that $|x| \leq R$ *and* $t - |y| \geq C^*R$ *with* $C^* \geq 28$. *Then the kernel* $K(t,x,y)$ *of the operator*

$$\frac{\sin tH}{H}, \tag{41}$$

is a classical function satisfying

$$|K(t,x,y)| \leq \frac{C\ln(2+t)}{\langle t\rangle\langle t-|y|\rangle}. \tag{42}$$

Proof. Let $\varphi(s)$ be the function defined in (30).Then we make the decomposition

$$\frac{\sin tH}{H} = P_1 + P_2 + P_3,$$

where

$$P_1 = \frac{\sin tH}{H}\varphi(H) - \frac{\sin tH_0}{H_0}\varphi(H_0),$$

$$P_2 = \frac{\sin tH}{H}(1-\varphi(H)) - \frac{\sin tH_0}{H_0}(1-\varphi(H_0)),$$

$$P_3 = \frac{\sin tH_0}{H_0}.$$

Let $K_j(t,x,y), j = 1,2,3$, be the kernel that corresponds to P_j. From Theorem 3.1 we know that

$$|K_1(t,x,y)| \le C\langle t\rangle^{-1}\langle t-|y|\rangle^{-1}$$

provided the asumptions $|x| \le R$ and $t-|y| \ge 2C^*R$ with $C^* \ge 14$ are satisfied. If we replace in the proof of the previous Theorem 3.1 the application of Theorem 2.3 (Main Theorem) with Proposition 2.1, then we arrive at the same estimate for the kernel K_2 i.e.

$$|K_2(t,x,y)| \le C\frac{\ln(2+t)}{\langle t\rangle\langle t-|y|\rangle}.$$

For completeness Finally finite dependence domain argument shows that in 3−dimensional case the kernel $K_3(t,x,y)$ is zero for $|x| \le R$ and $t-|y| \ge 2C^*R$ with $C^* \ge 14$. Hence, we arrive at the estimate (42) and this completes the proof. □

4. L^∞ estimate for the unperturbed wave equation

Let $u = L_0(F)$ be the solution of the unpertabed wave equation

$$\Box u \equiv \partial_t^2 u - \Delta u = F \quad \text{in } [0,\infty)\times \mathbf{R}^3, \tag{43}$$

$$u(0,x) = 0, \quad \partial_t u(0,x) = 0 \quad \text{in } \mathbf{R}^3. \tag{44}$$

PROPOSITION 4.1. *Let* $F \in C(\mathbf{R}^3 \times [0,\infty))$. *Set* $\tau_\pm = \tau_\pm(t,x) = 1 + |t \pm |x||$.

If $\lambda \geq 0$ *and* $\mu > 0$ *satisfy*

$$\lambda < 1, \quad \mu > 2 + \lambda, \tag{45}$$

then we have

$$\|\tau_+ \tau_-^\lambda L_0(F)\|_{L^\infty} \leq C_1 \|\tau_+^\mu \tau_- F\|_{L^\infty}. \tag{46}$$

If $m > 2$ *and* $0 < \nu \leq m - 1$, *we have*

$$\|\tau_+ \tau_-^\nu L_0(F)\|_{L^\infty} \leq C_2 \|\langle y\rangle^m \tau_+ \tau_-^\nu F\|_{L^\infty}. \tag{47}$$

The proof of the above stimates are based on the following representation of the solution of the unperturbed wave equation $\Box u = F$ with zero initial data

$$u = L_0(F)(t,x) = \frac{1}{4\pi}\int_0^t (t-s) \int_{|\omega|=1} F(s, x - (t-s)\omega) d\omega ds. \tag{48}$$

In essence, (46) and (47) are shown in [9] and [20], respectively. For the sake of completeness, we shall give a proof only for (47) (see also Proposition 3.1 of [13]).

Proof. It follows from (48) that

$$|L_0(F)(t,x)| \leq \frac{1}{4\pi}\|\langle y\rangle^m \tau_+ \tau_-^\nu F\|_{L^\infty} \int_0^t (t-s) \int_{|\omega|=1} \frac{d\omega ds}{\langle y\rangle^m \tau_+ \tau_-^\nu},$$

where $y = x - (t-s)\omega$. We shall use the following identity

$$\int_{|\omega|=1} b(|x + \rho\omega|) d\omega = \frac{2\pi}{r\rho}\int_{|\rho - r|}^{\rho + r} \lambda b(\lambda) d\lambda \tag{49}$$

for $b(\lambda)$ is a continuous function of $\lambda \in [0,\infty)$, $\rho > 0$ and $x \in \mathbf{R}^3$ with $r = |x| > 0$. (For the proof, see e.g.[18] or [14], Lemma 2.3). Then we get

$$|L_0(F)(t,x)| \leq \|\langle y\rangle^m \tau_+ \tau_-^\nu F\|_{L^\infty} \times$$
$$\times \frac{1}{2r}\int_0^t \int_{|t-s-r|}^{t-s+r} \frac{d\lambda ds}{(1+\lambda)^{m-1}(1+s+\lambda)(1+|s-\lambda|)^\nu}.$$

Note that

$$
\begin{aligned}
&\frac{1}{(1+\lambda)^{m-1}(1+s+\lambda)(1+|s-\lambda|)^{\nu}} \\
\leq\ &\frac{C}{(1+s+\lambda)^{1+\nu}}\left(\frac{1}{(1+|s-\lambda|)^{m-1}}+\frac{1}{(1+\lambda)^{m-1}}\right)
\end{aligned} \tag{50}
$$

for $0<s<t$ and $|t-s-r|<\lambda<t-s+r$. Indeed, if $0<s\leq 2\lambda$, we have

$$
\begin{aligned}
(1+\lambda)^{m-1}(1+s+\lambda)(1+|s-\lambda|)^{\nu} &\geq C(1+s+\lambda)^{m}(1+|s-\lambda|)^{\nu} \\
&\geq C(1+s+\lambda)^{1+\nu}(1+|s-\lambda|)^{m-1},
\end{aligned}
$$

since $m\geq 1+\nu$. On the other hand, if $s\geq 2\lambda$, we have

$$
(1+\lambda)^{m-1}(1+s+\lambda)(1+|s-\lambda|)^{\nu}\geq C(1+s+\lambda)^{1+\nu}(1+\lambda)^{m-1}.
$$

We thus get (50). Therefore, changing the variables as

$$
\alpha=s+\lambda, \quad \beta=\lambda-s,
$$

and noting that $m>2$, we arrive at

$$
\begin{aligned}
&\frac{1}{r}\int_0^t ds\int_{|t-s-r|}^{t-s+r}\frac{d\lambda}{(1+\lambda)^{m-1}(1+s+\lambda)(1+|s-\lambda|)^{\nu}}\leq \\
&\frac{1}{r}\int_{|t-r|}^{t+r}\frac{d\alpha}{(1+\alpha)^{1+\nu}}.
\end{aligned} \tag{51}
$$

When $t\geq 2r$, we estimate the right hand side by $C(1+|t-r|)^{-1-\nu}$. While $0<t\leq 2r$, we use the bound $Cr^{-1}(1+|t-r|)^{-\nu}$. In conclusion, we find from (4) and (51) that (47) holds. □

5. L^∞ estimate for the perturbed wave equation

Let $u=L(F)$ be the solution of the perturbed wave equation

$$
\partial_t^2 u-\Delta u+Vu=F \quad \text{in } [0,\infty)\times\mathbf{R}^3, \tag{52}
$$

$$
u(0,x)=0, \quad \partial_t u(0,x)=0 \quad \text{in } \mathbf{R}^3, \tag{53}
$$

where $V \in C_0^\infty(\mathbf{R}^3)$.

First of all, we state a uniqueness lemma for (52). For the proof, see e.g. F. John [10].

LEMMA 5.1. *Let $\phi(t,x,u)$ be a function of class C^2 in its arguments and satisfy*

$$\phi(t,x,u) = 0 \quad \text{for all } t, x. \tag{54}$$

Let $u(t,x)$ be a classical solution of the equation

$$\Box u = \phi(t,x,u) \tag{55}$$

in the cone

$$\Gamma(t_0,x_0) = \{(t,x) \in [0,t_0] \times \mathbf{R}^3 : \ t - t_0 + |x - x_0| \leq 0\} \tag{56}$$

for certain $(t_0,x_0) \in [0,\infty) \times \mathbf{R}^3$. If

$$u(0,x) = \partial_t u(0,x) = 0 \quad \text{for } |x - x_0| \leq t_0,$$

then $u(t,x) \equiv 0$ in $\Gamma(t_0,x_0)$.

Then the main result of this section is the following estimate.

THEOREM 5.2.

Suppose that $F \in L^\infty_{loc}([0,\infty) \times \mathbf{R}^3)$ and

$$suppF \subset \{(x,t) \in [0,\infty) \times \mathbf{R}^3 : \ |x| \leq t + R\}$$

for some $R > 0$. Let $\lambda \geq 0$ and $\mu > 0$ satisfy (45). Then there is $C_3 = C_3(R) > 0$ such that

$$\|\tau_+ \tau_-^\lambda L(F)\|_{L^\infty} \leq C_3 \|\tau_+^\mu \tau_- F\|_{L^\infty}. \tag{57}$$

Proof. Without loss of generality we can assume $\mu < 3$ due to the assumption (45). Setting $u = L(F)$, we have

$$u(t,x) = L_0(F)(t,x) - L_0(Vu)(t,x). \tag{58}$$

From the estimate of the unperturbed operator L_0 from the previous section we have

$$\|\tau_+ \tau_-^\lambda u\|_{L^\infty} \leq C_1 \|\tau_+^\mu \tau_- F\|_{L^\infty} + C_2 \|\tau_+ \tau_-^\lambda V_m u\|_{L^\infty},$$

where $V_m(x) = \langle x\rangle^m V(x)$. This estimate shows that it is sufficient to establish the estimate

$$\langle t_0\rangle^{1+\lambda}\|u(t_0,.)\|_{L^\infty(|x|\le R)} \le C\|\tau_+^\mu\tau_- F\|_{L^\infty}. \tag{59}$$

Here $t_0 \ge 0$, while $R > 0$ is chosen so that the support of the potential V is inside the ball of radius R centered at 0.

First we consider the case when t_0 is bounded, say $0 \le t_0 \le 2T$. Using the representation of the fundamental solution in (48), we take advantage of the fact that V is supported in $\{|x| \le R\}$ and find from (48) that

$$|L_0(Vu)(t,x)| \le \|V\|_{L_\infty}\int_0^t (t-s)\sup_{|y|\le R}|u(s,y)|ds. \tag{60}$$

If we set $h(s) = \sup_{|y|\le R}|u(s,y)|$, then we get from (58)

$$h(t) \le C_1\|\tau_+^\mu\tau_- F\|_{L^\infty} + 2T\|V\|_{L^\infty}\int_0^t h(s)ds.$$

Then Gronwall inequality implies

$$h(t) \le C_1\|\tau_+^\mu\tau_- F\|_{L^\infty}\mathrm{e}^{2T\|V\|_{L^\infty}t}$$

for $0 \le t \le 2T$, so with $t = t_0$ we arrive at (59).

Next we consider the case when t_0 is large enough, i.e.

$$t_0 \ge 2T. \tag{61}$$

Let

$$F_+(s,y) = \begin{cases} F(s,y) & \text{if} \quad t_0 - s - |y| \le T - R, \\ 0 & \text{otherwise} \end{cases}$$

and $F_- = F - F_+$. Then it suffices to show (59) with F replaced by $F_\pm$.

We start with the evaluation of $v(t,x) = L(F_+)(t,x)$. It follows from Lemma 5.1 that

$$v(s,y) = 0$$

if $t_0 - s - |y| \geq T - R$. If we set $T_0 = t_0 - T$, then the inequalities $s \leq T_0$ and $t_0 - s - |y| \leq T - R$ imply $|y| \geq R$. Therefore, we see that

$$V(y)v(s,y) = 0 \quad \text{if} \quad s \leq T_0 \quad \text{or} \quad |y| \geq R,$$

hence

$$|L_0(Vv)(t,x)| \leq \|V\|_{L^\infty} \int_{T_0}^t (t-s) \sup_{|y|\leq R} |v(s,y)| ds,$$

in view of the representation formula (48) applied for $L_0(Vv)(t,x)$. For $T_0 \leq t \leq t_0$ and $|x| \leq R$ we have

$$\begin{aligned} |v(t,x)| &\leq |L_0(F_+)(t,x)| + |L_0(Vv)(t,x)| \\ &\leq C_1 t^{-1-\lambda} \|\tau_+^\mu \tau_- F_+\|_{L^\infty} + T\|V\|_{L^\infty} \int_{T_0}^t \sup_{|y|\leq R} |v(s,y)| ds. \end{aligned}$$

Since s is equivalent to t, when $T_0 \leq s \leq t \leq t_0$, we arrive at

$$t^{1+\lambda}|v(t,x)| \leq C_1 \|\tau_+^\mu \tau_- F_+\|_{L^\infty} + C' \int_{T_0}^t s^{1+\lambda} \sup_{|y|\leq R} |v(s,y)| ds,$$

where $C' = 2^{1+\lambda} T \|V\|_{L^\infty}$. Let us denote

$$h(\tau) = (\tau + T_0)^{1+\lambda} \sup_{|x|\leq R} |v(\tau + T_0, x)|$$

for $0 \leq \tau \leq T$. Then we get

$$h(\tau) \leq C_1 \|\tau_+^\mu \tau_- F_+\|_{L^\infty} + C' \int_0^\tau h(s) ds$$

and this gives

$$h(\tau) \leq C_1 \mathrm{e}^{C'\tau} \|\tau_+^\mu \tau_- F_+\|_{L^\infty}$$

for $0 \leq \tau \leq T$. Taking $\tau = T$, we obtain (59) with $F = F_+$.

To establish the estimate (59) with F replaced by F_-, we note that

$$\mathrm{supp} F_-(s,y) \subseteq \{(s,y); t_0 - s - |y| \geq T - R\}.$$

Then $L(F_-)$ can be represented in the form

$$L(F_-)(t,.) = \int_0^t \frac{\sin(t-s)H}{H} F_-(s,.) ds$$

Taking T such that $T \geq (2C^*R+1)R$ and applying the estimate from Theorem 3.2, we see that

$$|L(F_-)(t_0,x)| \leq CI(t_0)\|\tau_+^{\mu}\tau_- F_-\|_{L^\infty}, \tag{62}$$

for $t_0 \geq 2T$ and $|x| \leq R$, where we have set

$$I(t) = \int_0^t \int_{|y|\leq (t-s)-(T-R)} \frac{\ln(2+t-s)}{\langle t-s\rangle\langle t-s-|y|\rangle\langle s+|y|\rangle^{\mu}\langle s-|y|\rangle} dsdy. \tag{63}$$

Introduce polar coordinates $\rho = |y|, \omega = y/|y| \in \mathbf{S}^2$ and find

$$I(t) \leq C\int_0^t \int_{\rho\leq t-s} \frac{\ln(2+t-s)\rho^2}{\langle t-s\rangle\langle t-s-\rho\rangle\langle s+\rho\rangle^{\mu}\langle s-\rho\rangle} dsd\rho.$$

Further,we split the integral in the right side of this inequality into I_+ and I_-, where

$$I_{\pm}(t) = \int_{K_{\pm}(t)} \frac{\ln(2+t-s)\rho^2 dsd\rho}{\langle t-s\rangle\langle t-s-\rho\rangle\langle s+\rho\rangle^{\mu}\langle s-\rho\rangle}$$

with

$$\begin{aligned} K_+(t) &= \{(s,\rho) \in \mathbf{R}_+ \times \mathbf{R}_+; t/2 \leq s+\rho \leq t\}, \\ K_-(t) &= \{(s,\rho) \in \mathbf{R}_+ \times \mathbf{R}_+; t/2 \geq s+\rho\}. \end{aligned}$$

Consider $I_+(t)$. Since on K_+ we have $\rho \leq \langle t-s\rangle$ and $\rho \leq C\langle s+\rho\rangle$, we conclude that

$$I_+(t) \leq C\int_{K_+(t)} \frac{dsd\rho}{\langle t-s-\rho\rangle\langle s+\rho\rangle^{\mu-1-\sigma}\langle s-\rho\rangle}$$

for sufficiently small positive σ. Changing the variables

$$\alpha = s+\rho\ ,\ \beta = s-\rho, \tag{64}$$

we get

$$\begin{aligned} I_+(t) &\leq C\int_{t/2}^t \langle t-\alpha\rangle^{-1}\langle\alpha\rangle^{1-\mu+\sigma}\int_{-\alpha}^t \langle\beta\rangle^{-1} d\beta d\alpha \\ &\leq C\langle t\rangle^{1-\mu+\sigma}\log^2(2+t). \end{aligned}$$

Choosing $\sigma > 0$ small enough, from $\mu > 2 + \lambda$, we see that

$$\langle t \rangle^{1+\lambda} I_+(t) \leq C.$$

Next we study $I_-(t)$. Since on $K_-(t)$ we have $t - s \geq t - s - \rho \geq t/2$, we conclude that

$$I_-(t) \leq \frac{C \ln(2+t)}{\langle t \rangle^2} \int_{K_-(t)} \frac{ds d\rho}{\langle s + \rho \rangle^{\mu-2} \langle s - \rho \rangle}.$$

Making the change of variables (64), we arrive at

$$I_-(t) \leq \frac{C}{\langle t \rangle^{2-\sigma}} \int_0^{t/2} \langle \alpha \rangle^{2-\mu} \int_{-\alpha}^{t} \langle \beta \rangle^{-1} d\beta d\alpha.$$

Since we have assumed $\mu < 3$, we have $\mu - 2 < 1$ and

$$I_-(t) \leq \frac{C}{\langle t \rangle^{\mu-1-\sigma}} \log(2+t).$$

Again with small $\sigma > 0$ from the assumption $\mu > 2 + \lambda$ we obtain

$$\langle t \rangle^{1+\lambda} I_-(t) \leq C.$$

The above observation leads to the estimate

$$\langle t \rangle^{1+\lambda} I(t) \leq C \tag{65}$$

where $I(t)$ is the integral defined in (63).

From this estimate and (62) we see that the proof is complete.
□

6. Application to semilinear wave equations

In this section, we consider the initial value problem with small data for the semilinear wave equations of the form

$$\partial_t^2 u - \Delta u + Vu = F(u) \qquad \text{in } [0,\infty) \times \mathbf{R}^3, \tag{66}$$

$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x) \quad \text{in } \mathbf{R}^3. \tag{67}$$

We shall assume that $V \in C_0^\infty(\mathbf{R}^3)$ satisfies

$$V(x) \geq 0. \tag{68}$$

Further we suppose that $F \in C^2(\mathbf{R})$ satisfies

$$F(0) = F'(0) = F''(0) = 0, \tag{69}$$

and that there are $p > 2$ and $A > 0$ such that for $|u_1| \leq 1$, $|u_2| \leq 1$

$$|F''(u_1) - F''(u_2)| \leq \begin{cases} Ap(p-1)|u_1 - u_2|^{p-2} & \text{if} \quad 2 < p \leq 3, \\ Ap(p-1)|u_1 - u_2|(|u_1| + |u_2|)^{p-3} & \text{if} \quad p > 3, \end{cases} \tag{70}$$

Typical examples are $F = |u|^p$ and $F = |u|^{p-1}u$. As for the initial data, we assume that $f, g \in C_0^\infty(\mathbf{R}^3)$ satisfy

$$\sup_{x \in \mathbf{R}^3} \{ \sum_{|\alpha| \leq 1} |\partial_x^\alpha f(x)| + |g(x)| \} \leq \varepsilon, \tag{71}$$

where ε is a small positive parameter.

Next we introduce a function space X, in which we will look for a solution of (66)-(67), defined as follows

$$X = \{u \in C([0,\infty) \times \mathbf{R}^3) : \|u\| < +\infty\}, \tag{72}$$

with the norm

$$\|u\| = \|\tau_+ \tau_-^\kappa u\|_{L^\infty},$$

where κ is a real number satisfying

$$1/p < \kappa < p - 2, \quad \kappa < 1. \tag{73}$$

Then we have the following theorem.

THEOREM 6.1. *Assume that* V, F, f *and* g *satisfy* (68) *through* (71) *with* $p > p_c = 1 + \sqrt{2}$. *Then there exists* $\varepsilon_0 > 0$ *such that for any* ε *with* $0 < \varepsilon \leq \varepsilon_0$, *the problem* (66)-(67) *has a unique classical global solution belonging to* X.

Thanks to Theorem 1.1, we can achieve the proof of Theorem 6.1 by a standard application of the contraction mapping principle. For the argument, the following lemma plays a crucial role. For further details, see for instance [9] (or Section 7 of [12]).

LEMMA 6.2. *Assume that* $p > p_c = 1 + \sqrt{2}$ *and that* κ *is chosen such that* (73) *holds. Then we have for all* $u \in X$

$$\|\!|L(|u|^p)\|\!| \leq C \|\!|u\|\!|^p \tag{74}$$

with C *independent of* u.

Proof. It is clear that we can choose κ such that (73) holds. Moreover, (73) ensures that the existence of a positive number μ such that

$$2 + \kappa < \mu < p.$$

Therefore we have

$$\|(1 + t + |x|)^\mu (1 + |t - |x||)|u|^p\|_{L^\infty} \leq \|\!|u\|\!|^p.$$

Combining this with Theorem 1.1 with $\lambda = \kappa$, we obtain (74). □

REFERENCES

[1] S. AGMON, *Spectral properties of Schrödinger operators*, Actes Congr. Int. Math. Nice, no. 2, 1970, pp. 679–683.

[2] S. AGMON, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa (1975), no. 4, 151–218.

[3] M. BEALS AND W. STRAUSS, *L^p estimates for the wave equation with a potential*, Comm. Part. Diff. Eq. **18** (1993), no. 7, 8, 1365–1397.

[4] V. GEORGIEV, H. LINDBLAD, AND C. SOGGE, *Weighted Strichartz estimates and global existence for semilinear wave equations*, Amer. J. Math. **119** (1997), no. 6, 1291–1319.

[5] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operatora II: Differential Operators with Constant Coefficients*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

[6] L. HÖRMANDER, *Lectures on Nonlinear Hyperbolic Differential Equations*, Mathematiques et Applications, vol. 26, Springer, Berlin, 1997.

[7] I. IKEBE, *Eigenfunction expansions associated with the Scrödinger operator and their applications to scattering theory*, Arch. Rational Mech. Anal. **5** (1960), 1–34.

[8] H. Isozaki, *Differentiability of generalized Fourier transforms associated with Schrödinger operators*, J. Math Kyoto Univ. **25** (1985), no. 4, 789–806.

[9] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. **28** (1979), 235–268.

[10] F. John, *Blow-up of quasi-linear wave equations in three space dimensions*, Comm. Pure Appl. Math. **34** (1981), 29–51.

[11] C. Kerler, *Perturbations of the Laplacian with variable coefficients in exterior domains and differentiability of the resolvent*, Asymptotic Analysis **19** (1999), 209–232.

[12] H. Kubo and K. Kubota, *Asymptotic behavior of classical solutions to a system of semilinier wave equations in low space dimensions*, Preprint, Hokkaido University, 1999.

[13] H. Kubo and M. Ohta, *The life span of classical solutions to a system of semilinear wave equations in three space dimensions*, Preprint, Università di L'Aquila, 1999.

[14] K. Kubota, *Existence of a global solutions to a semi-linear wave equation with initial data of non-compact support in low space dimensions*, Hokkaido Math. J. **22** (1993), 123–180.

[15] P. Lax and R. Philips, *Scattering Theory*, Academic Press, New York, 1967.

[16] C. Morawetz, *Notes on time decay and scattering for some hyperbolic problems*, Society for Industrial and Applied Mathematics, Philadelphia, 1975.

[17] G. Perla Menzala, *Decay of expanding spheres as $t \to \infty$ of the solutions of semilinear wave equation*, Annali di Matematica pura ed applicata **CLII** (1988), 387–399.

[18] J. Schaeffer, *Wave equation with positive nonlinearities*, Ph.D. thesis, Indiana Univ., 1983.

[19] W. Strauss, *Nonlinear scattering theory at low energy*, J. Funct. Anal. **41** (1981), 110–133.

[20] W. Strauss and K. Tsutaya, *Existence and blow up of small amplitude nonlinear waves with a negative potential*, Discrete and Cont. Dynam. Systems **3** (1997), no. 2, 175–188.

[21] R. Strichartz, *A priori estimates for the wave equation and some applications*, J. Funct. Anal. **5** (1970), 218–235.

[22] N. Weck and K. J. Witsch, *Complete low frequency analysis for the reduced wave equation with variable coefficients in three dimensions*, Commun. Partial Differ. Equations **17** (1992), no. 9/10, 1619–1663, Correction 18, No.3/4, 729, 1993.

Received March 1, 2000.