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Global Existence of Nonlinear Elastic Waves

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Summary. - *We prove the global existence of the solutions to the Cauchy problem for a nonlinear hyperbolic system describing the motion for the displacement of an isotropic, homogeneous, hyperelastic material. The result is obtained under a null condition which is the complement of genuine nonlinearity condition given by John [4].*

1. Introduction

The motion for the displacement of an isotropic, homogeneous, hyperelastic material satisfies a quasilinear hyperbolic system described in Section 2. Sideris [8] has proved under some restricted null condition that this system has a smooth global solution with small initial data. We derive the null condition reflected the special features of the system from the John-Shatah observation ([6]) on the Klainerman's null condition. We will also prove that the null condition is precisely the complement of genuine nonlinearity condition given by John [4] and guarantees global existence of smooth solution to the system with small initial data.

The plane of this paper is as follows. We introduce the null condition in Section 3 and characterize the nonlinear terms by the null condition in Section 4 and state the main results in Section 5. The rest of the paper is devoted to the outline of proof of results, based on energy and weighted $L^\infty - L^2$ estimates.

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2. The equations of motion for the displacement

Let $\varphi(t,x), x \in \mathbb{R}^3$, be a smooth deformation of the material evolving with time. The unknown of the problem is the displacement $u(t,x) = \varphi(t,x) - x$ from reference configuration. The displacement gradient is then the matrix $G = \nabla u$ with components $G_{ie} = \partial_\ell u^i$, where the spatial gradient will be denoted by ∇ or grad. For the materials under consideration, the potential energy density is characterized by a stored energy function $\sigma = \sigma(\jmath_1, \jmath_2, \jmath_3)$, where $\jmath_1, \jmath_2, \jmath_3$ are principal invariants of the strain matrix $C = G + {}^tG + G^tG$. Thus the motion for the displacement is governed by the nonlinear system

$$\partial_t^2 u - \operatorname{div}\frac{\partial \sigma}{\partial G} = 0, \tag{1}$$

that is,

$$\partial_t^2 u^i - \sum_{\ell=1}^{3} \frac{\partial}{\partial x_\ell}\frac{\partial \sigma}{\partial G_{i\ell}} = 0 \quad (i = 1, 2, 3).$$

(see [2]).

Since we will consider only small displacement, it is enough to truncate (1) at third order in u. Then the relevant terms in the Taylor expansion of σ about $\jmath_k = 0$ are

$$\sigma = \sigma_0 + \sigma_1 \jmath_1 + \sigma_2 \jmath_2 + \sigma_3 \jmath_3 + \frac{1}{2}\sigma_{11}\jmath_1^2 + \sigma_{12}\jmath_1\jmath_2 + \frac{1}{6}\sigma_{111}\jmath_1^3 + \cdots .$$

We make use of the following formula for principal invariants:

$$\begin{aligned}
\jmath_1 &= \operatorname{tr} C \\
\jmath_2 &= \frac{1}{2}\{(\operatorname{tr} C)^2 - \operatorname{tr} C^2\} \\
\jmath_3 &= \frac{1}{6}\{(\operatorname{tr} C)^3 - 3(\operatorname{tr} C)(\operatorname{tr} C^2) + 2\operatorname{tr} C^3\}.
\end{aligned}$$

Putting $C = G + {}^tG + G^tG$ and using the relation

$$\frac{\partial \sigma}{\partial G} = \sum_{k=1}^{3} \frac{\partial \sigma}{\partial \jmath_k}\frac{\partial \jmath_k}{\partial G},$$

we obtain

$$
\begin{aligned}
\frac{\partial\sigma}{\partial G} =& 2\sigma_1(I+G)+4(\sigma_{11}+\sigma_2)(\mathrm{tr}\ G)I-2\sigma_2(G+{}^tG)\\
&+4(\sigma_{111}+3\sigma_{12}+\sigma_3)(\mathrm{tr}\ G)^2I\\
&+2(\sigma_{11}-\sigma_{12}+\sigma_2-\sigma_3)\{2(\mathrm{tr}\ G)G+\mathrm{tr}\ (G^tG)I\}\\
&-2(\sigma_{12}+\sigma_3)\{2(\mathrm{tr}\ G)^tG+(\mathrm{tr}\ G^2)I\}\\
&-2(\sigma_2-\sigma_3)(G^2+G^tG+{}^tGG)+2\sigma_3{}^tG^tG\\
&+\cdots .
\end{aligned} \tag{2}
$$

For details see [8].

We impose the condition $\sigma_1 = 0$, which implies the reference configuration is a stress-free state. The Lamé constants $\lambda = 4(\sigma_{11} + \sigma_2)$ and $\mu = -2\sigma_2$ are assumed to be positive. Then it follows from (1) and (2) that the linear part of (1) becomes the following hyperbolic linear operator

$$Lu = \partial_t^2 u - c_2^2\Delta u - (c_1^2 - c_2^2)\mathrm{grad\ div}\ u, \tag{3}$$

where $c_1 = (\lambda + 2\mu)^{1/2}$ and $c_2 = \mu^{1/2}$ correspond to the speeds of spherical and rotational waves, respectively. Thus the truncated equations of (1) are formulated by

$$Lu = \mathrm{div}\ H = F(\nabla u, \nabla^2 u), \tag{4}$$

where H stands for the quadratic term in (2) and the last equality is definition of F.

We will show that the nonlinear term has the energy symmetry. To this end, we rewrite i-th component of nonlinear term

$$F^i(\nabla u, \nabla^2 u) = \sum_{j\ell m=1}^{3} C_{ij}^{\ell m}(\nabla u)\partial_\ell\partial_m u^j \tag{5}$$

where

$$C_{ij}^{\ell m}(\nabla u) = \sum_{k,n=1}^{3} C_{ijk}^{\ell mn}\partial_n u^k. \tag{6}$$

Then we have the following proposition which has proved in [8].

Proposition 2.1.

$$C_{ij}^{\ell m}(\nabla u) = C_{ji}^{\ell m}(\nabla u) = C_{ij}^{m\ell}(\nabla u).$$

3. The null condition

In this section we apply John-Shatah's observations ([6]) on Klainerman null condition to nonlinear elastic waves (3)-(5).

We introduce new unknowns $v(t,x) = (\partial_t u(t,x), \nabla u(t,x))$. The vector $v \in \mathbb{R}^{12}$ satisfies the quasilinear system of first order which is hyperbolic near $v = 0$:

$$a_0(v)\partial_t v + \sum_{i=1}^{3} a_i(v)\partial_i v = 0. \tag{7}$$

We next consider the plane wave solution w of (7) in the form

$$v(t,x) = w(t,s), \quad s = \zeta \cdot x, \tag{8}$$

where $\zeta \cdot x$ stands for inner product of $\zeta, x \in \mathbb{R}^3$. Then we find from (7) and (8) that w satisfies the following system in one space dimension

$$a_0(w)\partial_t w + \sum_{i=1}^{3} \zeta_i a_i(w)\partial_s w = 0. \tag{9}$$

Making use of the methods in [6] we can prove the following

PROPOSITION 3.1. *The quasilinear system (9) is not genuinely nonlinear for any $\zeta \neq 0$ if and only if*

$$\sum_{ijk\ell mn=1}^{3} C_{ijk}^{\ell mn} X_i X_j X_k X_\ell X_m X_n = 0 \quad \textit{for } X \in \mathbb{R}^3 \tag{N$_1$}$$

and

$$\begin{aligned} &\sum_{ijk\ell mn=1}^{3} C_{ijk}^{\ell mn}(|X|^2 - X_i^2)\zeta_k X_\ell X_m X_n \\ &- \sum_{i\neq j,k\ell mn=1}^{3} C_{ijk}^{\ell mn} X_i X_j \zeta_k X_\ell X_m X_n = 0 \\ &\textit{for } \zeta, X \in \mathbb{R}^3 \textit{ satisfying } \zeta \cdot X = 0, \end{aligned} \tag{N$_2$}$$

where constants $C_{ijk}^{\ell mn}$ are defined in (6).

We call the condition, $(\mathrm{N})_1$ and $(\mathrm{N})_2$, on the nonlinear term F the *null condition* for nonlinear elastic waves. We will list typical nonlinear terms in F satisfying the condition $(\mathrm{N})_1$ or $(\mathrm{N})_2$.

LEMMA 3.2. *(i)* $Q_{\ell m}(\partial_n u^j, u^k) = \partial_\ell \partial_n u^j \partial_m u^k - \partial_m \partial_n u^j \partial_\ell u^k$ *in* F^i *satisfy the null condition. More precisely,* $Q_{\ell m}(\partial_n u^j, u^k)$ *satisfy*

$$\sum_{\ell m n} C_{ijk}^{\ell m n} X_\ell X_m X_n = 0 \quad \textit{for any } i, j, k. \tag{10}$$

(ii) The components of $\partial_\ell u^j \partial_m \operatorname{rot} u$ *and* $\partial_\ell \partial_m u^j \operatorname{rot} u$ *in* F^i *satisfy the condition* $(\mathrm{N})_1$*, where* $\operatorname{rot} u = \nabla \wedge u$*. (iii)* $\partial_\ell u^j \partial_m \operatorname{div} u$*,* $\partial_\ell \partial_m u^j \operatorname{div} u$ *in* F^i *and* $F = \partial_\ell u^j \operatorname{grad}(\partial_m u^k)$ *satisfy the condition* $(\mathrm{N})_2$*.*

REMARK 3.3. *In [8] Sideris has called (10) the null condition.*

4. The characterization of nonlinear term by the null condition

We first rewrite the coefficients of σ_2 and σ_3 in F the sum of null forms of type (i) in Lemma 3.2

LEMMA 4.1. *The i-th component,* $Q_1^i(u, \nabla u)$ *of nonlinear term involving* σ_2 *and* σ_3 *can be expressed by*

$$\begin{aligned} &2\sigma_2 \sum_{j,k} (2Q_{jk}(\partial_j u^i, u^k) + Q_{jk}(\partial_k u^j, u^i) + Q_{ij}(\partial_j u^k, u^k)) \\ &+ 2\sigma_3 \{\sum_{j,k} (2Q_{ij}(\partial_k u^k, u^j) + Q_{ji}(\partial_k u^j, u^k)) - (\textit{coefficients of } 2\sigma_2)\}. \end{aligned}$$

We find from (2) and Lemma 4.1 that

$$\begin{aligned} &F^i(\nabla u, \nabla^2 u) = 4(\sigma_{111} + 3\sigma_{12})\partial_i(\operatorname{div} u)^2 \\ &+ 2(\sigma_{11} - \sigma_{12})(2(\operatorname{div} u \, \Delta u^i + \nabla \operatorname{div} u \cdot \nabla u^i) + \partial_i |\nabla u|^2) \\ &- 2\sigma_{12}(\partial_i(\operatorname{div} u)^2 + 2\sum_k \partial_k \operatorname{div} u \, \partial_i u^k + \sum_{j \cdot k} \partial_i(\partial_j u^k \partial_k u^j)) \\ &+ Q_1^i(u, \nabla u) \quad (i = 1, 2, 3) \end{aligned} \tag{11}$$

Sideris has imposed in [8] the conditions $\sigma_{11} - \sigma_{12} = 0$ and $2(2\sigma_{111} + 6\sigma_{12} - 3\sigma_{12}) = 0$, having the coefficients of $4(\sigma_{111} + 3\sigma_{12})$

and $-2\sigma_{12}$ the same nonlinearity condition. Thus Sideris' null condition is

$$\sigma_{11} - \sigma_{12} = 0 \quad \text{and} \quad 2\sigma_{111} + 3\sigma_{12} = 0.$$

Next we rewrite the nonlinear term F the form involving div u and rot u. To this end we make use of the following fundamental identities of vector fields.

$$\begin{aligned}
&\Delta u = \text{grad div } u - \text{rot rot } u, \\
&|\nabla u|^2 = |\text{rot } u|^2 + \sum_{j\cdot k} \partial_j u^k \partial_k u^j, \\
&\partial_i(\text{div } u)^2 + 2\sum_{j\cdot k} Q_{jk}(\partial_i u^k, u^j) = \partial_i(\sum_{j\cdot k} \partial_j u^k \partial_k u^j), \\
&\sum_k \partial_k \text{ div } u \ \partial_i u^k = \text{div } u \ \partial_i(\text{div } u) + \sum_{j\cdot k} Q_{ik}(u^k, \partial_j u^j).
\end{aligned}$$

Thus we get from them and (11) that

$$\begin{aligned}
F(\partial u, \partial^2 u) =&2(2\sigma_{111} + 3\sigma_{11})\text{grad}(\text{div } u)^2 \\
&+ 2(\sigma_{11} - \sigma_{12})(\text{grad}|\text{rot } u|^2 - 2 \text{ rot}(\text{div } u \text{ rot } u)) \\
&+ Q(u, \nabla u)
\end{aligned} \tag{12}$$

where $Q = Q_1 + Q_2$ and

$$Q_2^i(u, \partial u) = 4(\sigma_{11} - 2\sigma_{12}) \sum_{j\cdot k} (Q_{ik}(u^k, \partial_j u^j) + Q_{jk}(\partial_i u^k, u^j)).$$

Now we ready to state the characterization of nonlinear term by the null condition.

PROPOSITION 4.2. *The nonlinear term F satisfies the null condition if and only if*

$$2\sigma_{111} + 3\sigma_{11} = 0. \tag{13}$$

Proof. We find from (12) and Lemma 3.2 that nonlinear terms except for grad(div $u)^2$ satisfy the null condition. Since the coefficient of $\partial_i u^i \partial_i^2 u^i$ in F^i corresponds one to one to the one of X_i^6 in $(\text{N})_1$, it follows from the condition $(\text{N})_1$ that $2\sigma_{111} + 3\sigma_{11} = 0$. □

REMARK 4.3. *John has proved in [4] that if* $2\sigma_{111} + 3\sigma_{11} \neq 0$ *then radial solutions to (3) blow up.*

REMARK 4.4. *The nonlinear term grad (div u)*^{2} *has the energy symmetry.*

5. Statement of the main result

Assume that the nonlinear term F satisfies the null condition. Then, from (3), (12) and (13), we can formulate the initial value problem for nonlinear elastic waves as follows:

$$\begin{aligned}
&\partial_t^2 u - c_2^2 \Delta u - (c_1^2 - c_2^2)\text{grad div } u \\
&= 2(\sigma_{11} - \sigma_{12})(\text{grad}|\text{rot } u|^2 - 2\text{rot}(\text{div } u \text{ rot } u)) \\
&+ Q(u, \nabla u) \quad t > 0, x \in \mathbb{R}^3, \\
&u = \varepsilon f(x), \ \partial_t u = \varepsilon g(x) \quad t = 0, x \in \mathbb{R}^3,
\end{aligned} \tag{14}$$

where $f \cdot g \in C_0^\infty(\mathbb{R}^3)$ and ε is a small positive parameter. The main aim of this paper is to prove the following

THEOREM 5.1. *There exists a positive constant* ε_0 *such that the initial value problem (5.1) has a unique global in time* C^∞*- solution* u *for any* $\varepsilon (0 < \varepsilon \leq \varepsilon_0)$.

6. Notation

The space-time gradient will be denoted by

$$\partial = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_0, \nabla),$$

where

$$\partial_0 = \partial_t = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i} \quad (i = 1, 2, 3)$$

The angular momentum operators are the vector fields

$$\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla.$$

Then the spatial derivatives can be decomposed into radial and angular components

$$\nabla = \frac{x}{r}\partial_r - \frac{x}{r^2} \wedge \Omega, \quad \text{where } r = |x|, \ \partial_r = \frac{x}{r} \cdot \nabla. \tag{15}$$

We also use the vector fields

$$\tilde{\Omega} = \Omega I + U$$

where

$$U^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \ U^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ U^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The seven vector fields will be written as

$$\Gamma = (\Gamma_0, \cdots, \Gamma_6) = (\partial, \Omega),$$
$$\tilde{\Gamma} = (\tilde{\Gamma}_0, \cdots, \tilde{\Gamma}_6) = (\partial I, \tilde{\Omega}).$$

The linear hyperbolic operator L in (3) commutes with any $\tilde{\Gamma}$. The following commutation relations play a crucial role for handling the nonlinear term grad $|\mathrm{rot} u|^2$.

$$\tilde{\Omega} \ \mathrm{grad} \ f = \mathrm{grad} \ \Omega f, \ \mathrm{div} \ \tilde{\Omega} u = \Omega \ \mathrm{div} \ u. \tag{16}$$

In order to obtain weighted $L^\infty - L^2$ estimates we adopt the following weight functions.

$$w_i(t, r) = (1 + r)(1 + |c_i t - r|) \quad (i = 1, 2),$$
$$w(t, r) = \min_{i=1,2} w_i(t, r), \quad r = |x|.$$

We also use the following norms.

$$[u]_{k,t} = \sum_{|a| \le k} \sup_{0 \le s \le t} \sup_{x \in \mathbb{R}^3} \{ \sum_{\alpha=0}^{3} |w(s, |x|) \Gamma^a \partial_\alpha u(s, x)|$$
$$+ |w_1(s, |x|) \Gamma^a \ \mathrm{div} \ u(s, x)| + |w_2(s, |x|) \Gamma^a \ \mathrm{rot} \ u(s, x)| \},$$
$$||\partial u(s)||_k = \sum_{|a| \le k} \sum_{\alpha=0}^{3} ||\Gamma^a \partial_\alpha u(s, \cdot)||_{L^2(\mathbb{R}^3)},$$
$$||\partial u||_{k,t} = \sup_{0 \le s \le t} ||\partial u(s)||_k.$$

7. $L^\infty - L^\infty$ estimates

In order to obtain the weighted L^∞-estimates for solutions u of (14), we make use of concrete expressions of solution to the honogeneous linear problem

$$\begin{aligned} &Lv = \partial_t^2 v - c_2^2 \Delta v - (c_1^2 - c_2^2) \text{ grad div } v = 0, \\ &v(0,x) = 0, \ \partial_t v(0,x) = g(x). \end{aligned} \tag{17}$$

The solution of (17) is expressed in two manners:

$$\begin{aligned} v^i(t,x) = &\frac{t}{4\pi} \int_{|\omega|=1} g^i(x + c_2 t\omega) dS_\omega \\ + &\frac{t}{4\pi} \sum_{j=1}^{2} (-1)^{j-1} \int_{|\omega|=1} \omega_i \sum_{k=1}^{3} \omega_k g^k(x + c_j t\omega) dS_\omega \\ - &\frac{t}{4\pi} \int_{c_2 t}^{c_1 t} \tau^{-1} d\tau \int_{|\omega|=1} \sum_{k=1}^{3} (\delta_{ik} - 3\omega_i\omega_k) g^k(x + \tau\omega) dS_\omega \end{aligned} \tag{18}$$

and

$$\begin{aligned} v(t,x) = &\frac{t}{4\pi} \int_{|\omega|=1} g(x + c_1 t\omega) dS_\omega \\ &+ \frac{t}{4\pi} \int_{c_2 t \le |y| \le c_1 t} |y|^{-3} y \wedge (\text{rot } g)(x+y) dy. \end{aligned} \tag{19}$$

The expression (18) is standard (for instant see [5]). The new expression (19) will be used to get a good decay of the nonlinear term grad $|\text{rot } u|^2$. To describe the weighted L^∞-estimates, we introduce some notations:

$$\begin{aligned} &z_{\mu,\nu}^{(j)}(s,\lambda) = (1 + |c_j s - \lambda|)^\mu (1 + s + \lambda)^\nu \\ &c_0 = 0, \ j = 0,1,2. \\ &M_{\mu,\nu,k}^{(j)}(F) = \sum_{|a| \le k} \sup_{0 \le s \le t} \sup_{y \in \mathbb{R}^3} |y| z_{\mu,\nu}^{(j)}(s,|y|) |\Gamma^a F(s,y)|. \end{aligned}$$

Making use of the standard expression (18) and Duhamel's principle, we can prove the following

PROPOSITION 7.1. *Let u be the solution to the problem*

$$\begin{aligned} Lu(t,x) &= F(t,x), \\ u(0,x) &= \partial_t u(0,x) = 0. \end{aligned} \tag{20}$$

Then there exist a positive constant C such that

$$|u(t,x)| \le C(1+t+|x|)^{-1}(log(2+t))^2 M^{(j)}_{1,1,0}(F) \tag{21}$$

and

$$|\partial u(t,x)| \le Cw(t,|x|)^{-1}(log(2+t))^2 M^{(j)}_{1,1,1}(F) \tag{22}$$

$$|\partial u(t,x)| \le Cw(t,|x|)^{-1} M^{(j)}_{\mu,\mu,1}(F) \tag{23}$$

$$\textit{for } \mu > 1 \textit{ and } j = 0,1,2.$$

Next, applying div and rot to (14), we get

$$\begin{aligned} &\partial_t^2 \text{ div } u - c_1^2 \Delta \text{div } u \\ &= 2(\sigma_{11} - \sigma_{12})\Delta |\text{rot } u|^2 + \text{div } Q(u, \nabla u), \end{aligned} \tag{24}$$

$$\begin{aligned} &\partial_t^2 \text{ rot } u - c_2^2 \Delta \text{rot } u \\ &= -4(\sigma_{11} - \sigma_{12})(\text{rot})^2(\text{div } u \text{ rot } u) + \text{rot } Q(u, \nabla u). \end{aligned} \tag{25}$$

To obtain the weighted L^∞-estimates for div u and rot u, we will use the results in [9].

PROPOSITION 7.2. *Let $v_i (i = 1,2)$ be*

$$v_i(t,x) = \frac{1}{4\pi}\int_0^t (t-s)ds \int_{|\omega|=1} F(s, x + c_i(t-s)\omega) dS_\omega. \tag{26}$$

Then there exists a positive constant C such that

$$|\partial v_i(t,x)| \le Cw_i(t,|x|)^{-1} \ \log(2+t) \ M^{(j)}_{1,1,1}(F), \tag{27}$$

$$|\partial v_i(t,x)| \le C(1+|x|)^{-1}(1+|c_i t - |x||)^{-\nu} M^{(j)}_{\mu,\nu,1}(F) \tag{28}$$

$$\text{for } \mu > 1, \ \nu > 0, \ j \ne i,$$

and

$$|\partial v_i(t,x)| \le Cw_i(t,|x|)^{-1} M^{(j)}_{\mu,\mu,1}(F) \quad \textit{for } \mu > 1. \tag{29}$$

8. Weighted $L^\infty - L^2$ estimates

PROPOSITION 8.1. *Let u be the solution to the initial value problem (14). Then*

$$[\partial u]_{N,t} \le C_N(\varepsilon + ||\nabla u||^4_{N+7}), \tag{30}$$

provided $\varepsilon < 1$ and $[\nabla u]_{[(N+5)/2],t} < 1$.

Proof. Let u_0 be the solution of the homogeneous equation

$$\begin{aligned} &Lu_0 = \partial_t^2 u_0 - c_2^2 \Delta u_0 - (c_1^2 - c_2^2)\text{grad div } u_0 = 0, \\ &u_0(0,x) = \varepsilon f(x),\ \partial_t u_0(0,x) = \varepsilon g(x). \end{aligned} \tag{31}$$

Applying div and rot to (31), we have

$$\begin{aligned} &\partial_t^2 \text{ div } u_0 - c_1^2 \Delta \text{ div } u_0 = 0 \\ &\text{div } u_0(0,x) = \varepsilon \text{ div } f(x),\ \partial_t \text{ div } u_0(0,x) = \varepsilon \text{ div } g(x). \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\partial_t^2 \text{ rot } u_0 - c_2^2 \Delta \text{ rot } u_0 = 0 \\ &\text{rot } u_0(0,x) = \varepsilon \text{ rot } f(x),\ \partial_t \text{ rot } u_0(0,x) = \varepsilon \text{ rot } g(x). \end{aligned} \tag{33}$$

Since L commutes with $\tilde{\Gamma}$ and $\partial_t^2 - c_i^2\Delta (i = 1, 2)$ commute with Γ, we find that

$$\begin{aligned} &|\Gamma^a u_0(t,x)| \le C_N \varepsilon w(t,|x|)^{-1}, \\ &|\Gamma^a \text{ div } u_0(t,x)| \le C_N \varepsilon w_1(t,|x|)^{-1}, \\ &|\Gamma^a \text{ rot } u_0(t,x)| \le C_N \varepsilon w_2(t,|x|)^{-1} \end{aligned} \tag{34}$$

for $|a| \le N$.

Set

$$u_1 = u - u_0 \tag{35}$$

and apply $\tilde{\Gamma}^a$ to (14). Then, $\Gamma^a u_1$ satisfy the equation in the form

$$\begin{aligned} &L\Gamma^a u_1 = \sum_{b \le a} C_{ab} \Gamma^b F \\ &\Gamma^a u_1(0,x) = \partial_t \Gamma^a u_1(0,x) = 0. \end{aligned} \tag{36}$$

Here we denote again by $F = F(\nabla u, \nabla^2 u)$ the nonlinear term in (14). We define the weight function $z(s,\lambda)$ by

$$z(s,\lambda)^{-1} = \sum_{j=0}^{2} z_{1,1}^{(j)}(s,\lambda)^{-1}. \tag{37}$$

Then it follows from (36), (37) and Proposition 7.1 that

$$\begin{aligned} &|\Gamma^a u_1(t,x)| \le C_N(1+t+|x|)^{-1}(\log(2+t))^2 M_N(F), \\ &|\partial\Gamma^a \ u_1(t,x)| \le C_N w(t,|x|)(\log(2+t))^2 M_{N+1}(F), \end{aligned} \tag{38}$$

for $|a| \le N$ where

$$M_k(F) = \sum_{|a|\le k} \sup_{0\le s\le t} \sup_{y\in\mathbb{R}^3} |y| z(s,|y|) |\Gamma^a \ F(\nabla u, \nabla^2 u)(s,y)|.$$

Making use of Sobolev inequality

$$|y| \ |f(y)| \le C \left(\sum_{|a|\le 2} ||\Omega^a f||_{L^2(\mathbb{R}^3)} + \sum_{|a|\le 1} ||\partial_r \Omega^a f||_{L^2(\mathbb{R}^3)} \right)$$

and the fact that $z(s,|y|) \le Cw(s,|y|)$ we have

$$\begin{aligned} &|y| z(s,|y|) |\nabla\Gamma^b u^i(s,y)| \ |\nabla\Gamma^c u^i(s,y)| \\ &\le c_k [\nabla u]_{[k/2],t} ||\nabla u||_{k+2,t} \end{aligned}$$

for $|b| + |c| \le k, 0 \le s < t$, which implies

$$M_k(F) \le C_k [\nabla u]_{[(k+1)/2],t} \ ||\nabla u||_{k+3,t}. \tag{39}$$

Thus, it follows from (34), (35), (38) and (39) that, for $|a| \le N$,

$$\begin{aligned} &|\Gamma^a u(t,x)| \\ &\le C_N(1+t+|x|)^{-1}(\log(2+t))^2 \left(\varepsilon + [\nabla u]_{[(N+1)/2],t} ||\nabla u||_{N+3,t}\right), \end{aligned} \tag{40}$$

$$\begin{aligned} &|\partial\Gamma^a u(t,x)| \\ &\le C_N w(t,|x|)^{-1}(\log(2+t))^2 \left(\varepsilon + [\nabla u]_{[(N+2)/2],t} ||\nabla u||_{N+4,t}\right), \end{aligned} \tag{41}$$

Similarly, apply div and rot to (36), we find from (27) in Proposition 7.2 that

$$\begin{aligned} &|\Gamma^a \text{ div } u(t,x)| \\ &\leq C_N w(t,x)^{-1} \log(2+t) \left(\varepsilon + [\nabla u]_{[(N+2)/2],t} ||\nabla u||_{N+4,t}\right), \end{aligned} \tag{42}$$

$$\begin{aligned} &|\Gamma^a \text{ rot } u(t,x)| \\ &\leq C_N w_2(t,x)^{-1} \log(2+t) \left(\varepsilon + [\nabla u]_{[(N+2)/2],t} ||\nabla u||_{N+4,t}\right), \end{aligned} \tag{43}$$

In order to remove log terms from the inequalities above, it is necessary to further analyze the nonlinear terms. The following pointwise estimates follows from (15).

$$|Q_{\ell m}(\partial_n u^i, u^k)| \leq C r^{-1}(|\nabla \Omega u^i| \; |\nabla u^k| + |\nabla^2 u| \; |\Omega u^k|). \tag{44}$$

Hence, it follows from (40), (41), (44) and Lemma 2.1 in [7] that, for $|c_i t - |x|| < c_i t/2$,

$$\begin{aligned} &|\Gamma^a Q(u, \nabla u)| \\ &\leq C_N (1+t+|x|)^{-3} \min_{i=1,2} (1+|c_i t - |x||)^{-1} (\log(2+t))^4 \times \\ &\times \left(\varepsilon + [\nabla u]_{[(N+3)/2],t} ||\nabla u||^2_{N+5,t}\right) \end{aligned} \tag{45}$$

In the case where $|c_i t - |x|| > c_i t/2$, the estimate (41) yields

$$\begin{aligned} &|\Gamma^a Q(u, \nabla u)| \\ &\leq C_N (1+|x|)^{-2} (1+t+|x|)^{-2} (\log(2+t))^4 \times \\ &\times \left(\varepsilon + [\nabla u]_{[(N+3)/2],t} ||\nabla u||^2_{N+5,t}\right) \end{aligned} \tag{46}$$

Furthermore, the estimates (42) and (43) yield

$$\begin{aligned} &|\Gamma^a \text{ rot}(\text{div } u \text{ rot } u)| \\ &\leq C_N w_1(t,|x|)^{-1} w_2(t,|x|)^{-1} (\log(2+t))^2 \times \\ &\times \left(\varepsilon + [\nabla u]_{[(N+3)/2],t} ||\nabla u||^2_{N+5,t}\right) \end{aligned} \tag{47}$$

Therefore, it follows from (45)-(47) that

$$\begin{aligned} &|\Gamma^a Q(u, \nabla u)| + |\Gamma^a \text{ rot}(\text{div } u \text{ rot } u)| \\ &\leq C_N \{(1+t+|x|)^{-1} (z^{(1)}_{\mu,\mu}(t,|x|) + z^{(2)}_{\mu,\mu}(t,|x|)) + \\ &+ (1+|x|)^{-1} z^{(0)}_{\mu,\mu}(t,|x|)\} \left(\varepsilon + [\nabla u]_{[(N+3)/2],t} ||\nabla u||^2_{N+5,t}\right) \\ &\quad \text{for } |a| \leq N \quad \text{and for some } \mu > 1. \end{aligned} \tag{48}$$

Since (48) gives the estimate of nonlinear terms in (25), it follows from (29) that

$$|\Gamma^a \text{ rot } u| \\ \le C_N w_2(t,|x|)^{-1}\left(\varepsilon + [\nabla u]_{[(N+4)/2],t}||\nabla u||^2_{N+6,t}\right), \tag{49}$$

which implies

$$|\Gamma^a \text{ grad}|\text{rot } u|^2| \\ \le C_N\{(1+|x|)^{-1}z^{(0)}_{\mu,\mu}(t,|x|) + (1+t+|x|)^{-1}z^{(2)}_{\mu,1}(t,|x|)\} \times \\ \times \left(\varepsilon + [\nabla u]_{[(N+4)/2],t}||\nabla u||^4_{N+6,t}\right) \\ \text{for } |a| \le N \quad \text{and } \mu > 1. \tag{50}$$

Since (48) and (50) give the estimates of nonlinear term in (24), we also find from (27), (28) and (34) that, for $|a| \le N$,

$$|\Gamma^a \text{ div } u| \\ \le C_N w_1(t,|x|)^{-1}\left(\varepsilon + [\nabla u]_{[(N+5)/2],t}||\nabla u||^4_{N+7,t}\right) \tag{51}$$

The commutation relation (16) yields

$$L\tilde{\Gamma}^a u = 2(\sigma_{11}-\sigma_{12})(\text{grad } \Gamma^a|\text{rot } u|^2 - 2\tilde{\Gamma}^a \text{ rot}(\text{div } u \text{ rot } u)) \\ + \tilde{\Gamma}^a Q(u,\nabla u). \tag{52}$$

Making use of the expression (19) for the first term of right hand side in (52), we find finally from (23), (28), (34), (48) and (50) that, for $|a| \le N$,

$$|\partial\Gamma^a u(t,x)| \\ \le C_N w(t,|x|)^{-1}\left(\varepsilon + [\nabla u]_{[(N+5)/2],t}||\nabla u||^4_{N+7,t}\right) \tag{53}$$

The estimate (30) follows from (49), (51) and (53). □

9. Energy estimates

Let u be the solution to the initial value problem (14). The nonlinear term in (14) satisfies the energy symmetric condition, because of Proposition 2.1 and Remark 4.4 in Section 4. Therefore, we can prove the following energy estimate (see [5]).

LEMMA 9.1. *There exist positive numbers* λ *and* c_N *such that*

$$||\partial u||_{N,t}^2 \le C_N \varepsilon^2 (1+t)^{C_N [\partial u]_{[(N+1)/2],t}} \tag{54}$$

provided $|\partial u|_{0,t} \le \lambda$.

Next we will prove the following proposition which guarantees together with Proposition 8.1 the global existence of solutions with small data.

PROPOSITION 9.2. *Let* u *be the solution of (14). Then there exist positive constants* $\lambda_N \le 1$ *and* C_N *such that*

$$||\partial u||_{N,t} \le C_N \varepsilon \tag{55}$$

provided

$$[\partial u]_{[(N+9)/2],t} \le \lambda_N. \tag{56}$$

Proof. Applying $\tilde\Gamma^a$ to (14), we get

$$\begin{aligned} L\tilde\Gamma^a u &= \tilde\Gamma^a Q(\nabla u, u) + \\ &+ 2(\sigma_{11}-\sigma_{12})\{\tilde\Gamma^a \text{ grad}|\text{rot } u|^2 + \tilde\Gamma^a \text{ rot}(\text{div } u \text{rot } u)\}. \end{aligned}$$

Integrating the inner product of $\partial_t \tilde\Gamma^a u$ and this equation, we find

$$\begin{aligned} E(\partial\tilde\Gamma^a u)(t) &= E(\partial\tilde\Gamma^a u)(0) + 2\int_0^t ds \int_{\mathbb{R}^3} \tilde\Gamma^a Q(\nabla u, u)\cdot \partial_t \tilde\Gamma^a u \, dx \\ &+ 2(\sigma_{11}-\sigma_{12}) \int_0^t ds \int_{\mathbb{R}^3} (\tilde\Gamma^a \text{ grad}|\text{rot } u|^2 \\ &+ \tilde\Gamma^a \text{ rot}(\text{div } u\text{rot } u)\cdot \partial_t \tilde\Gamma^a u \, dx \end{aligned} \tag{57}$$

where

$$E(\partial u)(s) = \int_{\mathbb{R}^3} (|\partial_t u|^2 + c_2^2 |\nabla u| + (c_1^2 - c_2^2)|\text{div } u|^2)(s,x) dx.$$

The commutation relation (16) and integration by parts yield

$$\begin{aligned} &\int_{\mathbb{R}^3} \tilde\Gamma^a \text{ grad}|\text{rot } u|^2 \partial_t \tilde\Gamma^a u \, dx \\ &= (-1)^{|a|} \int_{\mathbb{R}^3} \Gamma^a |\text{rot } u|^2 \cdot \partial_t \Gamma^a \text{div } u \, dx \end{aligned} \tag{58}$$

Therefore, making use of (48), (49), (51) and (53), we find from (57) and (58) that the integrands in the right hand side of (57) are estimated by

$$C_N(1+s)^{-1-\kappa}|x|^{-2}(\sum_{j=0}^{2}(1+|c_js-|x||)^{-1-\kappa}\times$$
$$\times(\varepsilon^2+[\nabla u]_{(N+6)/2],s}||\nabla u||^8_{N+8,s})$$

for $\kappa>0$. Using (54) in Lemma 9.1, we know that the integrals over $\mathbb{R}^3$ in (57) are also estimated by

$$C_N\varepsilon^2(1+[\partial u]_{(N+6)/2],s})(1+s)^{-1-\kappa+C_N[\partial u]_{[(N+9)/2],s}} \tag{59}$$

Note that $E(\partial u)(t)$ is equivalent to $||\partial u(t)||_0^2$. Consequently, taking λ_N as

$$\lambda_N<\min(1,\lambda,2^{-1}c_N^{-1}\kappa),$$

we conclude from (57) and (59) that (55) holds. □

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