Rend. Istit. Mat. Univ. Trieste
Vol. XXXI Suppl. 1, 235–252 (2000)

On Topological Smallness

Washek F. Pfeffer [(*)]

Summary. - *We discuss a topological concepts of smallness and show that the field of real numbers contains a small uncountable subfield*

Introduction

These notes concern topological concept of smallness introduced by Karel Prikry ten years ago. All topics discussed here, and considerably more, are contained in the principle reference [11]. However, in contrast to [11] where we strived for the utmost generality, here we shall concentrate on elucidating the main ideas in their simplest form. Whenever a choice between generality and sanity is called for, we specialize.

Throughout, an ordinal is identified with the set of all smaller ordinals, and cardinals are initial ordinals. Thus if α and β are ordinals, then $\alpha < \beta$ and $\alpha \leq \beta$ are equivalent to $\alpha \in \beta$ and $\alpha \subset \beta$, respectively. As common, ω and ω_1 denote the first infinite and first uncountable cardinal, respectively. Finite and countably infinite sets are called countable. The sets of all real and rational numbers with their usual topology are denoted by $\mathbb{R}$ and $\mathbb{Q}$, respectively.

By a space we always mean a Hausdorff topological space. The family of all closed subsets of a space X is denoted by $\mathcal{F}(X)$; when no confusion can arise, we write merely $\mathcal{F}$ instead of $\mathcal{F}(X)$.

(*) Author's adress: Department of Mathematics, University of California, Davis, CA 95616, USA
This work, supported in part by the Italian Research Council, was presented at the 1997 School on Measure Theory and Real Analysis in Grado, Italy.

1. Small spaces

The intuitive reasoning leading to the concept of a small space X proceeds as follows. We look at the closed subsets of X and try to distinguish between those which are small and those which are large. If no such distinction can be made, we conclude that either all closed sets are small or all closed sets are large. It stands to reason, however, the empty set cannot be large, and so all closed sets, including X, must be small. The next definition formalizes our intuition.

Definition 1.1. A *discrimination* in a space X is a family $\mathcal{D} \subset \mathcal{F}$ that satisfies the following conditions.

1. Each disjoint family $\mathcal{F}^* \subset \mathcal{F} - \mathcal{D}$ is countable.
2. If a family $\mathcal{D}^* \subset \mathcal{D}$ is countable, then $X - \bigcup \mathcal{D}^*$ is uncountable.

A space X is called *small* when no discrimination in X exists.

We use the name discrimination to indicate that the family $\mathcal{D}$ discriminates between "large" and "small" closed subsets of X. Indeed, when "large" and "small" are interpreted relatively to the size of X, condition 1 states the closed sets not in $\mathcal{D}$ are "large", and condition 2 states the closed sets in $\mathcal{D}$ are "small." The absolute notion of smallness emerges from these relative concepts. Obviously, our intuition is not exact: the empty set, for instance, may belong to $\mathcal{D}$ or $\mathcal{F} - \mathcal{D}$.

The idea of discrimination is related to that of *σ-saturated ideal* [5, Section 27], which motivates the following terminology: a family $\mathcal{S} \subset \mathcal{F}(X)$ is called *saturated* (or more precisely, saturated in X) whenever it satisfies condition 1 of Definition 1.1. This terminology will simplify the language, and thus enhance the clarity of our exposition.

Proposition 1.2. *Let X and Y be spaces, and let $f : X \to Y$ be a continuous injection.*

1. *If Y is small, then so is X. In particular, a subspace and a homeomorphic image of a small space is small.*

2. *If X and Y are small, then so is $X \times Y$.*

3. *If X is a countable union of* ***closed*** *small subspaces, then X is small. In particular, each countable space is small.*

Proof. 1. If $\mathcal{D}_X$ is a discrimination in X, then

$$\mathcal{D}_Y = \left\{F \in \mathcal{F}(Y) : f^{-1}(F) \in \mathcal{D}_X \text{ or } f^{-1}(F) = \emptyset\right\}$$

is a discrimination in Y.

2. Let $\mathcal{D}$ be a discrimination in $X \times Y$. Replacing $\mathcal{D}$ by the family of all finite unions of elements of $\mathcal{D}$, we may assume $\mathcal{D}$ is closed with respect to finite unions. The families

$$\begin{aligned} \mathcal{D}_X &= \left\{F \in \mathcal{F}(X) : F \times Y \in \mathcal{D}\right\}, \\ \mathcal{D}_Y &= \left\{H \in \mathcal{F}(Y) : X \times H \in \mathcal{D}\right\} \end{aligned}$$

are saturated in X and Y, respectively. As neither $\mathcal{D}_X$ nor $\mathcal{D}_Y$ is a discrimination, there are sequences $\{F_i\}$ in $\mathcal{D}_X$ and $\{H_j\}$ in $\mathcal{D}_Y$ such that the sets $X - \bigcup_{i=1}^{\infty} F_i$ and $Y - \bigcup_{j=1}^{\infty} H_j$ are countable. By our assumption all sets $(F_i \times Y) \cup (X \times H_j)$ belong to $\mathcal{D}$. Since the set

$$X \times Y - \bigcup_{i,j=1}^{\infty} \left[(F_i \times Y) \cup (X \times H_j)\right] = \left(X - \bigcup_{i=1}^{\infty} F_i\right) \times \left(Y - \bigcup_{j=1}^{\infty} H_j\right)$$

is countable, we have a contradiction.

3. Let $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is a closed small subspace of X, and suppose there is a discrimination $\mathcal{D}$ in X. As $\mathcal{F}(X_n) \subset \mathcal{F}(X)$, the family $\mathcal{D}_n = \{F \in \mathcal{F}(X) : F \subset X_n\}$ is saturated in X_n. Since X_n is small, there is a sequence $\{F_{n,1}, F_{n,2}, \dots\}$ in $\mathcal{D}_n$ such that $X_n - \bigcup_{i=1}^{\infty} F_{n,i}$ is a countable set. Thus

$$X - \bigcup_{n,i=1}^{\infty} F_{n,i} = \bigcup_{n=1}^{\infty} \left(X_n - \bigcup_{i=1}^{\infty} F_{n,i}\right)$$

is a countable set, a contradiction. □

Theorem 3.9 below shows that the third claim of Proposition 1.2 is false for small subspaces of X that are not closed in X.

In $\mathbb{R}$, concepts of "smallness" different from ours were considered previously by many authors. For comparison, we mention three examples.

- A set $X \subset \mathbb{R}$ has *universal measure zero* if for every finite diffused Borel measure μ in $\mathbb{R}$ there is a Borel set $B \subset \mathbb{R}$ with $X \subset B$ and $\mu(B) = 0$. Recall a Borel measure μ in $\mathbb{R}$ is *diffused* whenever $\mu(\{x\}) = 0$ for each $x \in \mathbb{R}$.
- A set $X \subset \mathbb{R}$ is *perfectly meager* if $X \cap P$ is meager in P for every perfect set $P \subset \mathbb{R}$. Recall a set $P \subset \mathbb{R}$ is *perfect* whenever P is nonempty, closed, and contains no isolated points; a set $M \subset P$ is *meager* in P whenever M can be covered by countably many closed subsets of P whose relative interiors in P are empty.
- A set $X \subset \mathbb{R}$ is a λ-*set* if each countable subset of X is a relative G_δ set. Recall a G_δ set is the countable intersection of open sets, and an F_σ set is the countable union of closed sets.

While sets having universal measure zero and perfectly meager sets are intuitively "small," a priori this is not clear for λ-sets. However, the next proposition shows that λ-sets are, in fact, "smaller" than perfectly meager sets.

Proposition 1.3. *Each λ-set $X \subset \mathbb{R}$ is perfectly meager.*

Proof. As each subset of a λ-set is again a λ-set, it suffices to choose a perfect set $P \subset \mathbb{R}$ containing a λ-set X, and show X is meager in P. Select a dense countable set $D \subset X$, and let $H = P - D^-$ where D^- denotes the closure of D in P. By our assumption, there is a relative G_δ set $G \subset P$ with $D = X \cap G$. Since $G \cup H$ is a dense relative G_δ subset of P, the set $P - (G \cup H)$ is meager in P. Now X is the union of the sets

$$X - (G \cup H) \subset P - (G \cup H) \quad \text{and} \quad X \cap (G \cup H) = X \cap G = D$$

which are meager in P. □

Proposition 1.4. *Let X be a small subspace of $\mathbb{R}$. Then X has universal measure zero, and it is a λ-set.*

Proof. Let $\mathcal{B}$ be the Borel σ-algebra in $\mathbb{R}$, and let μ be a finite diffused measure on $\mathcal{B}$. For any set $E \subset \mathbb{R}$, let

$$\mu^*(E) = \inf\bigl\{\mu(B) : B \in \mathcal{B} \text{ and } E \subset B\bigr\}.$$

If $\mu^*(X) > 0$, the basic facts of measure theory [10, Exercises (12-6)] imply $\mathcal{D} = \bigl\{F \in \mathcal{F}(X) : \mu^*(F) = 0\bigr\}$ is a discrimination in X. Thus $\mu^*(X) = 0$, and we conclude X has universal measure zero.

Select a countable set $C \subset X$, and observe the family $\langle \mathcal{S} = \{F \in \mathcal{F}(X) : F \cap C = \emptyset\}\rangle$ is saturated in X. As $\mathcal{S}$ is not a discrimination in X, there is a relative F_σ set $H \subset X$ disjoint from C and such that $X - H$ is a countable set. Thus $X - H$ is a relative G_δ set containing C, and C differs from $X - H$ by a countable set $N = (X - H) - C$. Since countable sets are F_σ, the set $C = (X - H) - N$ is G_δ. □

In Section 3 below we shall present some surprising connections between small spaces and the spaces that have universal measure zero, or are perfectly meager, or both.

Our main tool for deciding whether a space is small is a combinatorial gadget called Ulam matrix. An *Ulam matrix* [15] in a set A is a collection $\{A_{n,\alpha} : n \in \omega,\ \alpha \in \omega_1\}$ of subsets of A that satisfy the following conditions:

1. $A_{n,\alpha} \cap A_{n,\beta} = \emptyset$ for each $n \in \omega$ and each $\alpha, \beta \in \omega_1$ with $\alpha \neq \beta$;
2. $A - \bigcup_{n\in\omega} A_{n,\alpha}$ is a countable set for each $\alpha \in \omega_1$.

Thus each Ulam matrix in A is a transfinite matrix with ω rows and ω_1 columns, whose entries are subsets of A. Moreover, each row consists of disjoint sets, and the union of each column differs from A by a countable set. The next lemma relates Ulam matrices to small spaces.

LEMMA 1.5. *A space X is small whenever there is an Ulam matrix in X whose entries are closed sets.*

Proof. Let $\{F_{n,\alpha}\} \subset \mathcal{F}$ be an Ulam matrix in X, and suppose there is a discrimination $\mathcal{D}$ in X. As $\mathcal{D}$ is saturated, $\mathcal{F} - \mathcal{D}$ contains only countably many elements of each row of $\{F_{n,\alpha}\}$. Explicitly, for each $n \in \omega$, there is an $\alpha_n \in \omega_1$ such that $F_{n,\beta} \in \mathcal{D}$ whenever $\alpha_n \leq \beta < \omega_1$. Since $\alpha = \sup \alpha_n$ is in ω_1, we have $F_{n,\alpha} \in \mathcal{D}$ for all $n \in \omega$. Thus the set $X - \bigcup_{n\in\omega} F_{n,\alpha}$ is countable, a contradiction. □

According to our convention, we shall interpret the number 2 as an ordinal, and identify it with the discrete space $\{0,1\}$ of all ordinals smaller than 2. For any set A, we denote by 2^A the family of all maps from A to $\{0,1\}$ equipped with the product topology. When A is countably infinite, the space 2^A is homeomorphic to the *Cantor ternary set*, thereafter referred to as the *Cantor set*.

A *point separating* family in a set A is a collection $\mathcal{E}$ of subsets of A such that given a pair of distinct points in A, there is a set $E \in \mathcal{E}$ which contains precisely one of them. The expression "$\mathcal{E}$ is a point separating family in A," is often abbreviated to "$\mathcal{E}$ separates points of A." The following existence results is due to Ulam [15] and Rothberger [13].

LEMMA 1.6. *In ω_1 there are an Ulam matrix $\{C_{n,\alpha}\}$ and a countable point separating family $\mathcal{E}$ such that each $C_{n,\alpha}$ is the intersection of some elements of $\mathcal{E}$.*

Proof. 1. (Ulam) Recall that an ordinal is identified with the set of all smaller ordinals, and order each $\xi \in \omega_1$ into a sequence $\xi = \{\xi_n : n \in \omega\}$. For $n \in \omega$ and $\alpha \in \omega_1$, let

$$C_{n,\alpha} = \{\xi \in \omega_1 : \xi_n = \alpha\} = \{\xi \in \omega_1 : \xi > \alpha \text{ and } \xi_n = \alpha\}.$$

If $\xi \in C_{n,\alpha} \cap C_{n,\beta}$, then $\alpha = \xi_n = \beta$, and we see that each row of the matrix $\{C_{n,\alpha}\}$ consists of disjoint sets. Moreover,

$$\bigcup_{n\in\omega} C_{n,\alpha} = \{\xi \in \omega_1 : \xi > \alpha\}.$$

Thus $\omega_1 - \bigcup_{n\in\omega} C_{n,\alpha} = \alpha + 1$ is countable, and $\{C_{n,\alpha}\}$ is an Ulam matrix in ω_1.

2. (Rothberger) Since the cardinality of 2^ω is at least ω_1, we can view ω_1 as a subspace of 2^ω, and denote by $\mathcal{U}$ the countable open base in ω_1. For $n \in \omega$ and $U \in \mathcal{U}$, let

$$E_{n,U} = \{\xi \in \omega_1 : \xi_n \in U\}.$$

If $\mathcal{U}_\alpha = \{U \in \mathcal{U} : \alpha \in U\}$, then $\bigcap \mathcal{U}_\alpha = \{\alpha\}$ and we have

$$\bigcap_{U\in\mathcal{U}_\alpha} E_{n,U} = \left\{\xi \in \omega_1 : \xi_n \in \bigcap_{U\in\mathcal{U}_\alpha} U\right\} = \{\xi \in \omega_1 : \xi_n = \alpha\} = C_{n,\alpha}.$$

As the family $\mathcal{E} = \{E_{n,U} : n \in \omega,\ U \in \mathcal{U}\}$ is countable, it suffices to show it separates points of ω_1. If $\alpha < \beta < \omega_1$, then $\alpha = \beta_n$ for an $n \in \omega$. As $\alpha_n < \alpha = \beta_n$, there is a $U \in \mathcal{U}$ such that $\beta_n \in U$ and $\alpha_n \notin U$. This means $E_{n,U}$ contains β but not α. □

COROLLARY 1.7. *A discrete space X with $|X| \leq \omega_1$ is small.*

Discrete spaces of appreciably larger cardinality than ω_1 are still small. Indeed, following the techniques of Ulam [5, Section 27], one can show that a discrete space X is small whenever the cardinality of X is smaller than the first *weakly inaccessible cardinal*. However, we shall not pursue these investigations: from the topological point of view, discrete spaces are rather boring. Instead we employ the full strength of Lemma 1.6, and show there is an uncountable small subspace of the Cantor set.

PROPOSITION 1.8. *The Cantor set contains a small subspace of cardinality ω_1.*

Proof. In ω_1 select an Ulam matrix $\{C_{n,\alpha}\}$ and a countable point separating family $\mathcal{E}$ as in Lemma 1.6. In view of Proposition 1.2,1, it suffices to find a small subspace of $2^{\mathcal{E}}$ whose cardinality is ω_1; for $2^{\mathcal{E}}$ and the Cantor set are homeomorphic. To this end, we use Marczewski's evaluation map $e : \omega_1 \to 2^{\mathcal{E}}$ introduced in [6]. A transparent description of Marczewski's map is obtained when each set $E \in \mathcal{E}$ is identified with its characteristic function $\xi \mapsto \langle E, \xi \rangle$. For $\alpha \in \omega_1$, define $e(\alpha) \in 2^{\mathcal{E}}$ by the formula

$$\big\langle e(\alpha), E \big\rangle = \langle E, \alpha \rangle = \begin{cases} 1 & \text{if } \alpha \in E, \\ 0 & \text{if } \alpha \in \omega_1 - E. \end{cases}$$

Since $\mathcal{E}$ separates points of ω_1, the map e is injective. Thus $X = e(\omega_1)$ has cardinality ω_1, and $\big\{e(C_{n,\alpha})\big\}$ is an Ulam matrix in X. Furthermore, $e(C_{n,\alpha}) = \bigcap_{k=1}^{\infty} e(E_k)$ where $E_1, E_2, \ldots$ are elements of $\mathcal{E}$. If $E \in \mathcal{E}$, then

$$e(E) = \big\{x \in X : x(E) = 1\big\}$$

is a closed subset of X. It follows that each $e(C_{n,\alpha})$ is a closed subset of X, and an application of Lemma 1.5 completes the argument. □

Corollary 1.9. *The Cantor set contains a set of cardinality ω_1 that has universal measure zero and is a λ-set.*

Corollary 1.9 is a direct consequence of Propositions 1.8 and 1.4. The reader should compare our relatively straightforward argument with that of [7, Theorem 5].

Remark 1.10. It is instructive to separate the four essential steps which constitute the proof of Proposition 1.8.

1. We construct a particular Ulam matrix $\{C_{n,\alpha}\}$ in ω_1.
2. By injecting ω_1 into 2^ω, we obtain a countable point separating family $\mathcal{E}$ in ω_1 so that each $C_{n,\alpha}$ is the intersection of some elements of $\mathcal{E}$.
3. Using Marczewski's evaluation map e, we map ω_1 bijectively onto an $X \subset 2^{\mathcal{E}}$, and note that the Ulam matrix $\{e(C_{n,\alpha})\}$ in X consists of closed subsets of X.
4. Observing that in no space an Ulam matrix consisting of closed sets and a discrimination can exist simultaneously, we conclude X is small.

2. Small fields

As the proof of Proposition 1.8 is nonconstructive, it tells us nothing about the topological structure of a small uncountable subspace of the Cantor set. To an extent, we shall remedy this situation by showing there is an uncountable small algebraic subfield $\mathbb{Q}^*$ of $\mathbb{R}$. Indeed, the algebraic structure of $\mathbb{Q}^*$ mollifies some topological pathologies exhibited by general small spaces. For instance, $\mathbb{Q}^*$ is a homogeneous subspace of $\mathbb{R}$.

It turns out that with no additional effort, we can replace $\mathbb{R}$ by any nondiscrete, separable, and completely metrizable field R. Below we list a few examples of such fields.

- The fields $\mathbb{R}$ and $\mathbb{C}$ of all real and complex numbers, respectively. In their usual topology, $\mathbb{R}$ and $\mathbb{C}$ are completely metrizable, and the countable fields $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-1})$ are dense in $\mathbb{R}$ and $\mathbb{C}$, respectively.

- The field $\mathbb{Q}_p$ of all *p-adic numbers* [1, Chapter 1, Section 3]. Given a prime number p and an integer x, let

$$\|x\|_p = \inf\{2^{-k} : p^k \text{ divides } x\}.$$

  It is easy to verify that $\|\cdot\|_p$ is a norm in the ring $\mathbb{Z}$ of all integers, called the *p-adic norm*. The completion $\mathbb{Z}_p$ of the normed space $(\mathbb{Z}, \|\cdot\|_p)$ is a compact ring with no zero divisors, and $\mathbb{Q}_p$ is the quotient field of $\mathbb{Z}_p$. The field $\mathbb{Q}_p$ has characteristic zero, is completely metrizable, and contains $\mathbb{Q}$ as a dense subfield.

- If f is a *meromorphic function* in a domain $\Omega \subset \mathbb{C}$, let

$$\|f\| = \sup\left\{\frac{|f(z)|}{1 + |f(z)|} : z \in \Omega \text{ and } f(z) \neq \infty\right\}.$$

  Clearly $\|\cdot\|$ is a norm on the field $\mathfrak{M}(\Omega)$ of all meromorphic functions in Ω, and the completion $\overline{\mathfrak{M}}(\Omega)$ of $(\mathfrak{M}(\Omega), \|\cdot\|)$ is a completely metrizable field of zero characteristic. The field of all rational functions in $\mathbb{C}$ with coefficients in $\mathbb{Q}(\sqrt{-1})$ is countable and dense in $\overline{\mathfrak{M}}(\Omega)$.

- Let $\mathfrak{L}(K)$ be the field of all *formal Laurent series* $x = \sum_{k=n}^{\infty} a_k t^k$ where n is an integer, $a_n, a_{n+1}, \dots$ belong to a *countable* field K of any characteristic, and $a_n \neq 0$ [1, Chapter 1, Section 4, Problems 6 and 7]. The norm ρ in $\mathfrak{L}(K)$, given by $\rho(x) = 2^{-n}$, defines a complete metric on $\mathfrak{L}(K)$. The countable field $K(t)$ of all rational functions in t with coefficients in K is dense in $\mathfrak{L}(K)$. Clearly, the characteristics of K and $\mathfrak{L}(K)$ are the same.

As R is separable, it contains a countable dense subfield; for a subfield of R generated by a countable set is countable. We fix such a subfield Q of R, and a complete metric d on R compatible with the topology of R.

The following lemma is due to Mycielski (see [8], and also [17, Theorem 6.5]). In the special case of $R = \mathbb{R}$ and $Q = \mathbb{Q}$, von Neumann [16] proved the lemma by a different method.

LEMMA 2.1. *There is a set $C \subset R$ that is algebraically independent over Q and homeomorphic to the Cantor set.*

Proof. Since Q is countable, the family of all nonzero polynomials over Q in any number of indeterminates can be ordered to a sequence $p_1, p_2, \ldots$. Suppose p_k is a polynomial in $s(k)$ indeterminates, and observe that the set

$$Z_k = \{(a_1, \ldots, a_{s(k)}) \in R^{s(k)} : p_k(a_1, \ldots, a_{s(k)}) = 0\}$$

is a closed and contains no nonempty open subset of $R^{s(k)}$.

Throughout this proof, the symbols $(i_1 \cdots i_n)$ and $(i_1 i_2 \cdots)$ denote, respectively, finite and infinite sequences of zeros and ones. The closure of a set $A \subset R$ is denoted by A^-.

Since R has no isolated points, for each nonempty open set $U \subset R$, we can find two disjoint nonempty open sets V and W with arbitrarily small diameters and such that $V^- \cup W^- \subset U$. Using this fact repeatedly, for all sequence $(i_1 \cdots i_n)$ we construct inductively nonempty open sets $U_{i_1 \cdots i_n}$ of diameters less than $1/n$ so that the following conditions are satisfied.

1. $U_{i_1 \cdots i_{n-1} 0} \cap U_{i_1 \cdots i_{n-1} 1} = \emptyset$ and $U^-_{i_1 \cdots i_{n-1} 0} \cup U^-_{i_1 \cdots i_{n-1} 1} \subset U_{i_1 \cdots i_{n-1}}$.
2. If $1 \leq k \leq n$ and $(i_1^1 \cdots i_n^1), \ldots, (i_1^{s(k)} \cdots i_n^{s(k)})$ are distinct, then
$$\left[U_{i_1^1 \cdots i_n^1} \times \cdots \times U_{i_1^{s(k)} \cdots i_n^{s(k)}}\right] \cap Z_k = \emptyset.$$

Assuming the sets $U_{i_1 \cdots i_{n-1}}$ have been already constructed, it is easy to produce nonempty open sets $U_{i_1 \cdots i_n}$ of diameters less than $1/n$ so that condition 1 is satisfied. Let $1 \leq k \leq n$, and let $\langle (i_1^1 \cdots i_n^1), \ldots, (i_1^{s(k)} \cdots i_n^{s(k)}) \rangle$ be distinct (note this is possible only when $s(k) \leq 2^n$). Since Z_k contains no nonempty open subset of $R^{s(k)}$, and since $\langle U_{i_1^1 \cdots i_n^1} \times \cdots \times U_{i_1^{s(k)} \cdots i_n^{s(k)}} \rangle$ is a nonempty open subset of $R^{s(k)}$, we can find a point

$$(a_1, \ldots, a_{s(k)}) \in \left[U_{i_1^1 \cdots i_n^1} \times \cdots \times U_{i_1^{s(k)} \cdots i_n^{s(k)}}\right] - Z_k\,.$$

Making the sets $U_{i_1^j \cdots i_n^j}$ smaller, we may assume that $\langle U_{i_1^1 \cdots i_n^1} \times \cdots \times U_{i_1^{s(k)} \cdots i_n^{s(k)}} \rangle$ is a neighborhood of $(a_1, \ldots, a_{s(k)})$ which is disjoint from the closed set Z_k. As such adjustments of $U_{i_1^j \cdots i_n^j}$ may be repeated finitely many times, condition 2 can also be satisfied.

Since (R,d) is complete, the set

$$C = \bigcap_{n=1}^{\infty} \bigcup_{(i_1\cdots i_n)} U^{-}_{i_1\cdots i_n} = \bigcup_{(i_1 i_2\cdots)} \bigcap_{n=1}^{\infty} U^{-}_{i_1\cdots i_n}$$

is homeomorphic to the Cantor set by condition 1. Let $a_1,\dots,a_r$ be distinct points of C, and let p_k be any polynomial with $s(k)=r$. If $n \geq k$ is an integer with $2^n \geq r$, then there are disjoint sets $U_{i_1^1\cdots i_n^1},\dots,U_{i_1^r\cdots i_n^r}$ containing, respectively, the points $a_1,\dots,a_r$. It follows from condition 2 that $p_k(a_1,\dots,a_r) \neq 0$. □

Lemma 2.1 and Proposition 1.8 suggest how to obtain an uncountable small subfield of R containing Q. We start a set $C \subset R$ which is algebraically independent over Q and homeomorphic to the Cantor set, and find a small uncountable subspace X of C. Then we show that the subfield $[X]$ of R generated by $Q \cup X$ is still small. To carry out this program, we must understand the structure of $[X]$.

As usual, we denote by $Q(t_1,\dots,t_n)$ the field of all rational functions in the indeterminates $t_1,\dots,t_n$ with coefficients in Q, and we let

$$Q'(t_1,\dots,t_n) = Q(t_1,\dots,t_n) - \bigcup_{i=1}^{n} Q(t_1,\dots,t_{i-1},t_{i+1},\dots,t_n)\,.$$

Thus $Q'(t_1,\dots,t_n)$ is the collection of all rational functions from $Q(t_1,\dots,t_n)$ that depend an all indeterminates $t_1,\dots,t_n$.

Recall d is a complete a metric in R compatible with the topology of R. Let $C \subset R$ be a fixed algebraically independent set over Q that is homeomorphic to the Cantor set. The set C is compact, and the natural order in the Cantor set defines an order $\preceq$ in C, which is closed in C^2 when viewed as a relation on C. For $X \subset C$ and $m,k = 1,2,\dots$, denote by $X_{m,k}$ the set of all $(x_1,\dots,x_m) \in X^m$ such that $x_1 \preceq \cdots \preceq x_m$ and $d(x_i,x_j) > 1/k$ for all $i,j = 1,\dots,m$ with $i \neq j$. Note that each $X_{m,k}$ is a relatively closed subsets of X^m; in particular, each $C_{m,k}$ is compact.

Since C is algebraically independent, every $r \in Q'(t_1,\dots,t_m)$ defines a continuous map $(a_1,\dots,a_m) \mapsto r(a_1,\dots,a_m)$ from $\bigcup_{k=1}^{\infty} C_{m,k}$ to R. This map and its various restrictions are still denoted by r.

Lemma 2.2. *For any $X \subset C$, we have*

$$[X] = Q \cup \bigcup_{m,k=1}^{\infty} \bigcup_{r \in Q'(t_1,\dots,t_m)} r(X_{m,k}).$$

Proof. Denote by K the right side of the desired equality. Since each map corresponding to an $r \in Q'(t_1,\dots,t_m)$ performs only the field operations in R, we have $K \subset [X]$. On the other hand, K is closed with respect to the field operations in R; for the set $\bigcup_{m=1}^{\infty} Q'(t_1,\dots,t_m)$ of all rational functions in any number of indeterminates with coefficients in Q is a field. If $r \in Q'(t_1)$ is given by $r(t_1) = t_1$, then $X = X_{1,k} = r(X_{1,k})$ for $k = 1, 2, \dots$. Thus $Q \cup X \subset K$ and the lemma is established. □

Lemma 2.3. *Let $A = \{a_1,\dots,a_m\}$ and $B = \{b_1,\dots,b_n\}$ be subsets of R each consisting of distinct elements, and let $A \cup B$ be algebraically independent over Q. If $r(a_1,\dots,a_m) = s(b_1,\dots,b_n)$ for some $r \in Q'(t_1,\dots,t_m)$ and $s \in Q'(t_1,\dots,t_n)$, then $A = B$.*

Proof. Assume $m \le n$, and renumerate the elements of A and B so that for a nonnegative integer $k \le m$ we have $a_i = b_i$ for $i = 1,\dots,k$, while the sets $\{a_{k+1},\dots,a_m\}$ and $\{b_{k+1},\dots,b_n\}$ are disjoint. The formula

$$h(t_1,\dots,t_n) = r(t_1,\dots,t_k,a_{k+1},\dots,a_m)$$

defines a rational function h in the indeterminates $t_1,\dots,t_n$ with coefficients in the transcendental extension $Q(a_{k+1},\dots,a_m)$ of Q. As $b_1,\dots,b_n$ are algebraically independent over $Q(a_{k+1},\dots,a_m)$ and

$$h(b_1,\dots,b_n) = r(a_1,\dots,a_m) = s(b_1,\dots,b_n)\,,$$

we see that $h = s$. Since s depends on the indeterminate t_n, we conclude $k = n$ and the lemma follows. □

Lemma 2.4. *Let $X \subset C$, let m and k be positive integers, and let r belong to $Q'(t_1,\dots,t_m)$. Then the map $r : X_{m,k} \to [X]$ is a homeomorphism and the set $r(X_{m,k})$ is relatively closed in $[X]$.*

Proof. It follows from Lemma 2.3 that the continuous map $r : C_{m,k} \to R$ is injective, and as $C_{m,k}$ is compact it is a homeomorphism. Thus it suffices to show that $r(X_{m,k})$ is a relatively closed subset of $[X]$. To this end, let z be a point of the relative closure of $r(X_{m,k})$ in $[X]$. As $r(C_{m,k})$ is a compact set containing $r(X_{m,k})$, we see that $z \in r(C_{m,k}) \cap [X]$. Hence $z = r(x)$ for an $x = (a_1, \dots, a_m)$ in $C_{m,k}$. Note $z \notin Q$ because $a_1, \dots, a_m$ are distinct points of C. Consequently, $z = s(y)$ for an $s \in Q'(t_1, \dots, t_n)$ and $y = (b_1, \dots, b_n)$ in $X_{n,p}$. Since $a_1 \preceq \cdots \preceq a_m$ and $b_1 \preceq \cdots \preceq b_n$, another application of Lemma 2.3 shows $x = y$. In particular, $m = n$ and x belongs to $C_{m,k} \cap X_{m,p} \subset X_{m,k}$. We conclude $z \in r(X_{m,k})$. □

THEOREM 2.5. *There is a small subfield Q^* of R which has cardinality ω_1 and contains Q.*

Proof. In view of Propositions 1.8 and 1.2,1, the set C contains a small subspace X of cardinality ω_1. Lemma 2.4 together with Proposition 1.2,1 and 2 implies that each $r(X_{m,k})$ is a closed small subspace of $[X]$. From Lemma 2.2 we obtain

$$[X] = \left(\bigcup_{x \in Q} \{x\} \right) \cup \bigcup_{m,k=1}^{\infty} \bigcup_{r \in Q'(t_1,\dots,t_m)} r(X_{m,k}) \, ,$$

and hence $[X]$ is small by Proposition 1.2,3. A straight forward calculation reveals the cardinality of $[X]$ is ω_1. □

3. Set-theoretic remarks

It follows from Theorem 2.5 that, without any set-theoretic assumptions, there is an uncountable small subfield $\mathbb{Q}^*$ of $\mathbb{R}$. In view of Propositions 1.4 and 1.3, the field $\mathbb{Q}^*$ has universal measure zero and is perfectly meager. All three concepts, small, universal measure zero, and perfectly meager, are based on similar ideas akin to that of σ-saturated ideal [5, Section 27]. Thus a natural question arises: do small subspaces of $\mathbb{R}$ coincide with the sets that have universal measure zero, or are perfectly meager, or both? While we are unable to answer this question within the usual axioms of set theory, i.e., within the Zermelo-Fraenkel axioms together with the axiom of choice (abbreviated as ZFC), we show the negative answer is consistent with ZFC. This result has been obtained previously in [4].

We say a set $X \subset \mathbb{R}$ is *concentrated on a set* $C \subset \mathbb{R}$ whenever $X - G$ is countable for each open set $G \subset \mathbb{R}$ containing C. A set $X \subset \mathbb{R}$ concentrated on a *countable* set $C \subset \mathbb{R}$ is called *concentrated*. Intuitively, each concentrated set is close to a countable set, and hence "small." In view of this, the next lemma is somewhat surprising.

LEMMA 3.1. *Let X be an uncountable subset of $\mathbb{R}$ that is concentrated on a countable set $C \subset X$. Then X is not small.*

Proof. The family $\mathcal{D} = \{F \in \mathcal{F}(X) : C \cap F = \emptyset\}$ is saturated in X; for C is countable. Let $F \in \mathcal{D}$, and let H be a closed subset of $\mathbb{R}$ for which $X \cap H = F$. As $C \subset X$ and $C \cap F = \emptyset$, the set C is contained in the open set $\mathbb{R} - H$. Thus $F = X - (\mathbb{R} - H)$ is a countable set. Since X is uncountable, $\mathcal{D}$ is a discrimination in X. □

LEMMA 3.2. *Each concentrated set $X \subset \mathbb{R}$ has universal measure zero.*

Proof. Let X be concentrated on a countable set $C \subset \mathbb{R}$, and let μ be a diffused finite Borel measure in $\mathbb{R}$. As μ is regular [3, Corollary 6.8], there is a G_δ set G with $C \subset G$ and $\mu(G) = 0$. Since $D = X - G$ is countable and μ is diffused, $\mu(D) = 0$. Observing $X \subset G \cup D$ completes the argument. □

The *continuum hypothesis* (abbreviated as CH) asserts the cardinality of $\mathbb{R}$ is ω_1. Both CH and $\neg$CH (the negation of CH) are consistent with ZFC. This means if ZFC is consistent, then so are ZFC + CH and ZFC + $\neg$CH (see [5]). We say $\{x_\alpha : \alpha < \kappa\}$ is an *enumeration* of a set X if κ is an ordinal and $\alpha \mapsto x_\alpha$ is a bijection between κ and X.

PROPOSITION 3.3. *Assuming CH, there is an uncountable set $L \subset \mathbb{R}$ that meets each meager subset of $\mathbb{R}$ in a countable set. In particular, L is concentrated on each countable dense subset of $\mathbb{R}$, and hence it has universal measure zero.*

Proof. In view of CH, we can enumerate all closed subsets of $\mathbb{R}$ whose interior is empty as $\{F_\alpha : \alpha < \omega_1\}$. Select $x_0 \in \mathbb{R} - F_0$, and proceeding by transfinite induction, for each ordinal $\alpha < \omega_1$ with $\alpha \geq 1$, let

$$C_\alpha = \{x_\beta : \beta < \alpha\} \cup \bigcup_{\beta<\alpha} F_\beta$$

and select an $x_\alpha \in \mathbb{R} - C_\alpha$. This is possible by the Baire category theorem [14, Chapter 7, Theorem 16], because C_α is meager. The set $L = \{x_\alpha : \alpha < \omega_1\}$ is uncountable, and $L \cap F_\alpha$ is a subset of the countable set $\{x_\beta : \beta \leq \alpha\}$. As each meager subset of $\mathbb{R}$ is covered by countably many sets F_α, the first claim is established.

If $G \subset \mathbb{R}$ is open and dense, then $\mathbb{R} - G$ is a closed set whose interior is empty. Thus $L - G = L \cap (\mathbb{R} - G)$ is countable by the previous claim. □

The set L of Proposition 3.3 is called a *Luzin set* [7, Section 2].

COROLLARY 3.4. *Assuming CH, there is a set $X \subset \mathbb{R}$ which has universal measure zero but it is not small.*

Proof. Use Proposition 3.3 to select a Luzin set L, and observe that $L \cup \mathbb{Q}$ is still a Luzin set. The corollary follows from and Lemmas 3.1 and 3.2. □

Rothberger [12] proved the next proposition. We omit its proof, which is complicated and rather technical.

PROPOSITION 3.5. *Assuming CH, there is an uncountable λ-set $X \subset \mathbb{R}$ concentrated on $\mathbb{Q}$.*

COROLLARY 3.6. *Assuming CH, there is a set $X \subset \mathbb{R}$ which is perfectly meager but not small.*

Proof. The proof is similar to that of Corollary 3.4. Use Proposition 3.5 to find a λ-set $X \subset \mathbb{R}$ concentrated on $\mathbb{Q}$, and let $Y = X \cup \mathbb{Q}$. By Proposition 1.3, the set X is perfectly meager, and so is Y. An application of Lemma 3.1 completes the argument. □

We turn our attention to the subsets of $\mathbb{R}$ that have universal measure zero and are perfectly meager simultaneously. Our tool for studying these sets will be so called Q-sets, which are "smaller" than λ-sets. A set $X \subset \mathbb{R}$ is called a *Q-set* if every subset of X is a relative G_δ set, and hence also a relative F_σ set.

PROPOSITION 3.7. *Every Q-set $X \subset \mathbb{R}$ with $|X| \leq \omega_1$ is small.*

Proof. Let $|X| = \omega_1$. In view of Lemma 1.6, there is an Ulam matrix $\{C_{n,\alpha}\}$ in X. Since each $C_{n,\alpha}$ is a relative F_σ set, there are sets $F^k_{n,\alpha} \in \mathcal{F}(X)$ such that $C_{n,\alpha} = \bigcup_{k=1}^\infty F^k_{n,\alpha}$ for each $n \in \omega$ and each $\alpha \in \omega_1$. Choose a bijection $f : \omega \times \omega \to \omega$, and denote $F^k_{n,\alpha}$ by $H_{f(n,k),\alpha}$. Then $\{H_{n,\alpha}\}$ is another Ulam matrix in X, and X is small by Lemma 1.5. □

Without proof we present the following consistency result established by Fleissner and Miller [2].

PROPOSITION 3.8. *It is consistent with ZFC that there is a Q-set $S \subset \mathbb{R}$ of cardinality ω_1 which is concentrated on $\mathbb{Q}$.*

THEOREM 3.9. *Let S be the set from Proposition 3.8.*

1. *S is small, hence it has universal measure zero and is perfectly meager.*
2. *$S \cup \mathbb{Q}$ has universal measure zero, is perfectly meager, but it is not small.*

Proof. The first claim follows from Propositions 3.7, 1.3, and 1.4. As $\mathbb{Q}$ is countable, the set $X = S \cup \mathbb{Q}$ still has universal measure zero and is perfectly meager. However, X is not small according to Lemma 3.1. □

Theorem 3.9 displays well the unintuitive property of small spaces: a small subspace of $\mathbb{R}$ may cease to be small when a countable subset of $\mathbb{R}$ is added to it. This unpleasantness disappears when in the definition of discrimination we employ *Borel sets* instead of closed sets.

We say a space X is *Borel-small* if there is no discrimination in X consisting of Borel sets. It is easy to show Borel-small spaces satisfy claims 1 and 2 of Proposition 1.2. Following the proof of Proposition 1.2,3, we see that a space X which is the union of its Borel-small subspaces $X_1, X_2, \dots$ is Borel-small whenever each X_n is a Borel subset of X. Since each countable subset of any space is a Borel set that is Borel-small, any Borel-small subspace of $\mathbb{R}$ remains Borel-small when we add to it a countable subset of $\mathbb{R}$.

PROPOSITION 3.10. *Each small space is a Borel-small. It is consistent with ZFC there is a Borel-small space $X \subset \mathbb{R}$ that is not a λ-set; in particular, X is not small.*

Proof. If $\mathcal{B}$ is a discrimination among the Borel subsets of a space X, then $\mathcal{D} = \mathcal{B} \cap \mathcal{F}(X)$ is a discrimination among the closed sets of X.

If $S \subset \mathbb{R}$ is the uncountable Q-set from Proposition 3.8, then $S \cup \mathbb{Q}$ is a Borel-small space. Suppose $S \cup \mathbb{Q}$ is a λ-set, and find a G_δ set $G \subset \mathbb{R}$ with $\mathbb{Q} = (S \cup \mathbb{Q}) \cap G$. It follows $\mathbb{Q} \subset G$ and $S \cap G \subset \mathbb{Q}$. The set $S - G$ is countable, since S is concentrated on $\mathbb{Q}$. Thus $S = (S - G) \cup (S \cap G)$ is countable, a contradiction. □

Our last proposition shows that Borel-small spaces, which may be "larger" than small spaces, are still pretty "small."

PROPOSITION 3.11. *Let X be a Borel-small subspace of $\mathbb{R}$. Then X has universal measure zero and it is perfectly meager.*

Proof. The argument employed in the prove of Proposition 1.4 reveals immediately X has universal measure zero.

As each subspace of a Borel-small space is Borel-small, we may assume X is a subset of a perfect set $P \subset \mathbb{R}$, and show X is meager in P. Let $\mathcal{C}$ be the family of all Borel subsets of P which are meager in P, and let $\mathcal{D} = \{C \cap X : C \in \mathcal{C}\}$. Suppose $\{D_\alpha : \alpha < \omega_1\}$ is an enumeration of a disjoint family of relative Borel subsets of X not in $\mathcal{D}$, and for each ordinal $\alpha < \omega_1$, find a Borel set $C_\alpha \subset P$ with $D_\alpha = C_\alpha \cap X$. The sets $B_\alpha = C_\alpha - \bigcup_{\beta<\alpha} C_\beta$ are disjoint Borel subsets of P. Since $D_\alpha \subset B_\alpha$ for each $\alpha < \omega_1$, no B_α is meager in P. As every Borel subset of P has the property of Baire [9, Theorem 4.3], each B_α contains a set $G_\alpha - N_\alpha$ where G_α is a nonempty relatively open subset of P, and N_α is meager in P. The set P is separable, and so there are distinct sets G_α and G_β with $G_\alpha \cap G_\beta \neq \emptyset$. The Baire category theorem [14, Chapter 7, Theorem 16] implies

$$B_\alpha \cap B_\beta \supset (G_\alpha \cap G_\beta) - (N_\alpha \cup N_\beta) \neq \emptyset\,,$$

a contradiction. Thus $\mathcal{D}$ is saturated in X. On the other hand, $\mathcal{D}$ contains all singletons of X, and it is not a discrimination in X. Hence $X \in \mathcal{D}$, and the proposition is proved. □

The question whether Borel-small subspaces of $\mathbb{R}$ coincide with those sets which have universal measure zero, or are perfectly meager, or both, is completely open.

References

[1] Z.E. Borevich and I.R. Shafarevich, *Number theory*, Academic Press, New York, 1966.
[2] W.G. Fleissner and A.W. Miller, *On Q–sets*, Proc. Amer. Math. Soc. **78** (1980), 280–284.
[3] R.J. Gardner and W.F. Pfeffer, *Borel measures*, Handbook of Set-Theoretic Topology (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 961–1043.
[4] R.J. Gardner and W.F. Pfeffer, *A note on small and other singular sets*, Atti Sem. Mat. Fis. Univ. Modena **39** (1991), 263–270.
[5] T. Jech., *Set theory*, Academic Press, New York, 1978.
[6] E. Szpilrajn (Marczewski), *The characteristic function of a sequence of sets and some of its applications*, Fund. Math. **31** (1938), 207–223.
[7] A.W. Miller, *Special subsets of the reals*, Handbook of Set-Theoretic Topology (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 201–233.
[8] J. Mycielski, *Independent sets in topological algebras*, Fund. Math. **55** (1964), 139–147.
[9] J.C. Oxtoby, *Measure and category*, Springer-Verlag, New York, 1971.
[10] W.F. Pfeffer, *Integrals and measures*, Marcel Dekker, New York, 1977.
[11] W.F. Pfeffer and K. Prikry, *Small spaces*, Proc. London Math. Soc. **58** (1989), no. 3, 418–438.
[12] F. Rothberger, *Sur un ensemble de preière catégorie qui est dépourvu de la propriété* λ, Fund. Math. **32** (1933), 294–300.
[13] F. Rothberger, *On families of real functions with a denumerable base*, Ann. Math. **45** (1944), 397–407.
[14] H.L. Royden, *Real analysis*, Macmillan, New York, 1968.
[15] S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math. **16** (1931), 140–150.
[16] J. von Neumann, *Ein System algebraisch unabhängiger Zahlen*, Math. Ann. **99** (1928), 134–141.
[17] S. Wagon, *The Banach-Tarski paradox*, Cambridge Univ. Press, New York, 1985.

Received September 30, 1998.