

## State Spaces of Orthomodular Structures

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SUMMARY. - *We present several known and one new description of orthomodular structures (orthomodular lattices, orthomodular posets and orthoalgebras). Originally, orthomodular structures were viewed as pasted families of Boolean algebras. Here we introduce semipasted families of Boolean algebras as an alternative description which is not as detailed, but substantially simpler. Semipasted families of Boolean algebras correspond to orthomodular structures in such a way that states and evaluation functionals are preserved. As semipasted families of Boolean algebras are quite general, they allow an easy construction of orthomodular structures with given state space properties. Based on this technique, we give a simplified proof of Shultz's Theorem on characterization of spaces of finitely additive states on orthomodular lattices. We also put some other results into the new context. We give a detailed exposition of the construction techniques as a tool for further applications, especially for finding counterexamples to questions about states on orthomodular structures.*

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## 1. Introduction

In a classical system, the observable events form a Boolean algebra. The states are described by a mapping which assigns to each event its probability. So the states may be identified with probability measures.

The logic of quantum mechanics is more general — it is non-distributive. For its system of events, several corresponding algebraical structures were suggested, e.g., orthomodular lattices, orthomodular posets, etc. Their study was initiated in [7] and further motivated in [18]. Combinatorial techniques (using hypergraphs) were introduced in this field by Greechie [9] and generalized by Dichtl [22]. Since that, they are an essential tool in the theory of orthomodular structures (see [16, 40]). Further constructions extended this technique — particular pastings used e.g., in [15, 20, 37], pasting of orthomodular posets [35], hypergraph representation of state spaces [34], and recently the technique of regulators [21]. At present, almost every researcher in the field of orthomodular structures has to use the combinatorial techniques at least to find examples or to demonstrate his notions and ideas. In this paper, we try to collect some of them in a unified and new formulation.

The paper is organized as follows: First we introduce a pasted family of Boolean algebras as a basic structure. We define its pasting and formulate conditions under which it is an orthoalgebra, orthomodular poset or orthomodular lattice. We use hypergraphs to describe pasted families of Boolean algebras, resp. the corresponding orthomodular structures, and we show that they allow to determine the elements of the structure, the states and the evaluation functionals. Then semipasted families of Boolean algebras are introduced as a new structure. They are again represented by hypergraphs. We derive a new type of correspondence — a functional isomorphism. It allows to determine the states and evaluation functionals, but not the elements of the structure. Its advantage is that each hypergraph corresponds to a semipasted family of Boolean algebras and each such family represents — up to a functional isomorphism — an orthomodular lattice. This makes its use in examples very easy. The effectiveness of this technique is then used in several simplified or new proofs of properties of the states spaces of orthomodular lat-

tices. Possibilities of further development are outlined at the end of the paper.

## 2. Classical logics

Before a generalization to quantum logics (described by orthomodular structures), we demonstrate the basic notions on classical logics (represented by Boolean algebras).

In the classical probability theory, the events of a system form a Boolean algebra,  $A$ . If  $A$  is finite, it is isomorphic to  $2^n$  for some  $n \in \mathbb{N}$  and it is completely determined by its atoms which are defined as follows:

**DEFINITION 2.1.** Let  $L$  be a poset with a least element,  $\mathbf{0}$ . For  $a, b \in L$ ,  $a \leq b$ , we define the *interval*  $[a, b]_L = \{c \in L : a \leq c \leq b\}$ . An *atom* in  $L$  is an element  $a \in L \setminus \{\mathbf{0}\}$  such that  $[\mathbf{0}, a]_L = \{\mathbf{0}, a\}$ . We denote by  $\mathcal{A}(L)$  the set of all atoms of  $L$ . A poset is called *chain-finite* iff each its chain (=linearly ordered subset) is finite.

Throughout this paper, intervals without indices are reserved for intervals of real numbers; all other intervals are indexed by the respective poset. We always consider an interval  $[a, b]_L$  with the partial ordering inherited from  $L$ . We denote the bounds of posets by  $\mathbf{0}$ ,  $\mathbf{1}$  (eventually with indices, e.g.,  $\mathbf{0}_L$ ,  $\mathbf{1}_L$ ), while the symbols  $0$ ,  $1$  are reserved for real numbers or constant functionals.

The system of events, described by a Boolean algebra,  $A$ , may be in different *states*. Each pair of an event,  $a \in A$ , and a state,  $s$ , is assigned a value,  $s(a) \in [0, 1]$ , called the *probability* of event  $a$  at state  $s$ . Thus states may be understood as mappings  $s: A \rightarrow [0, 1]$ . Here we assume only finite additivity of states:

**DEFINITION 2.2.** Let  $A$  be a Boolean algebra. A *state* on  $A$  is a mapping  $s: A \rightarrow [0, 1]$  such that

$$\text{(SBA1)} \quad s(\mathbf{1}) = 1,$$

$$\text{(SBA2)} \quad a, b \in A, \quad a \wedge b = \mathbf{0} \implies s(a \vee b) = s(a) + s(b).$$

A state is called *two-valued* if it attains only the values  $\mathbf{0}$  and  $\mathbf{1}$ .

We shall define states also on other structures — orthomodular lattices, hypergraphs, etc. We always denote by  $\mathcal{S}(L)$  the set of all states on  $L$  — the *state space* of  $L$ . We assume  $\mathcal{S}(L) \subseteq [0, 1]^L$  with the product (=weak) topology. It is always compact and convex. As such, it is a convex hull of the set of its extreme points which are called *pure states*. This means that a state  $s$  is pure iff it cannot be expressed as a nontrivial convex combination of other states. In Boolean algebras, pure states coincide with two-valued states [46]. In more general structures these two notions are different.

### 3. Motivating examples

In this section, we present several physical experiments which demonstrate some quantum phenomena and which are described by simple orthomodular structures. We will refer to them for demonstration of different descriptions and features of orthomodular structures.

EXAMPLE 3.1. Assume that we observe a fire-fly closed in a box divided by transparent walls into four quadrants (Fig. 1). The fire-fly

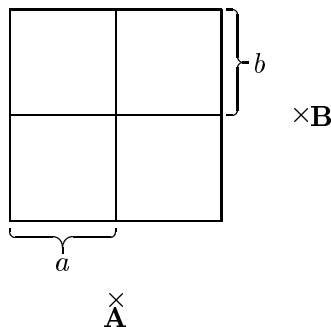


Figure 1: Experiment from Ex. 3.1

can move between the quadrants. Assume that the fire-fly keeps the light burning all the time. An observer at point **A** can distinguish whether the fire-fly is in the left or in the right half of the box. Similarly, an observer at point **B** can distinguish the upper and lower half. In the classical case, we may place two observers to points **A**,

**B** and distinguish four states corresponding to the presence of the fire-fly in particular quadrants.

In quantum systems, a simultaneous observation is often impossible. Measurements are destructive (they change the state of the system irreversibly), e.g., a single photon can be observed only once. (The same situation, characterized by irreversible changes of the state during measurements, is typical also in many other fields — sociology, psychology, artificial intelligence etc.) In our example, this phenomenon may be modelled by having only one observer, placed either in **A** or in **B**. So we may choose one of two possible observations, but we cannot perform both at the same time (and at the same state). For the observer placed in **A**, the observable events form a Boolean algebra  $A = \{\mathbf{0}_A, a, a'^A, \mathbf{1}_A\}$ , where  $a$  (resp.  $a'^A$ ) represents the event “the fire-fly is observed in the left (resp. right) half”, and  $\mathbf{0}_A$  (resp.  $\mathbf{1}_A$ ) represents the impossible (resp. sure) event. (We index the logical operations — including negation — and relations by the corresponding structure. We omit these indexes if this does not lead to confusion.) For the observer in **B**, the observable events form a Boolean algebra  $B = \{\mathbf{0}_B, b, b'^B, \mathbf{1}_B\}$ , where  $b$  is the event “the fire-fly is observed in the upper half”.

We have no tool to observe the conjunction of  $a$  and  $b$  and other events which are supposed to exist in the classical probability theory. Our system is described by two Boolean algebras,  $A$  and  $B$ . Their intersection is nonempty, because their bounds (impossible and sure events) are the same:  $\mathbf{0}_A = \mathbf{0}_B$ ,  $\mathbf{1}_A = \mathbf{1}_B$ . (From now on, we omit the indices when they are unimportant.)

All observable events from a “logic”  $L = \{\mathbf{0}, a, a'^A, b, b'^B, \mathbf{1}\}$  which inherits the ordering and negation of  $A$  and  $B$ .

Knowing the internal structure, we can consider four internal states of the system. They are described by the results of the observation performed at the states, so we can represent them as mappings from  $L$  to the set of truth values,  $\{0, 1\}$ . Each of these states corresponds to one row in the following table:

$s(a)$	$s(b)$
0	0
0	1
1	0
1	1

All the remaining values follow from the rules

$$(S0) \quad s(\mathbf{0}) = 0$$

and

$$(S1) \quad s(x') = 1 - s(x).$$

(Later on, we shall see how the states are recognized from the event structure, without a knowledge of the internal organization of the system.) The four states described so far are the pure states. Taking into account that the observations — influenced also by other circumstances — give only probabilistic results, we must admit also *mixtures of states*, i.e., their convex combinations. All states  $s$  on  $L$  satisfy (S0) and (S1) and

$$s(a) = p, \quad s(b) = q,$$

where  $p, q \in [0, 1]$  can be chosen arbitrarily.

EXAMPLE 3.2. We take the same system as in Ex. 3.1 with the only difference that the fire-fly can also put out the light. This situation corresponds to a new event,  $d$ , with the meaning “the fire-fly is not observed from  $\mathbf{A}$ ”. The events observable from position  $\mathbf{A}$  form a Boolean algebra  $A$ , isomorphic to  $2^3$ , having atoms  $\mathcal{A}(A) = \{a, d, (a \vee_A d)^{A'}\}$ . Similarly, the events observable from position  $\mathbf{B}$  form a Boolean algebra  $B$  with atoms  $\mathcal{A}(B) = \{b, d, (b \vee_B d)^{B'}\}$ . All observable events are  $L = \{\mathbf{0}, a, b, d, a \vee_A d, b \vee_B d, (a \vee_A d)^{A'}, (b \vee_B d)^{B'}, a'^A, b'^B, d'^A, \mathbf{1}\}$  (notice that  $d'^A = d'^B$ ). The pure states are given by the following table:

$s(a)$	$s(b)$	$s(d)$
0	0	0
0	1	0
1	0	0
1	1	0
0	0	1

All states  $s$  on  $L$  are uniquely determined by the values

$$s(a) = p, \quad s(b) = q, \quad s(d) = r,$$

where  $r \in [0, 1]$  is arbitrary and  $p, q \in [0, 1 - r]$ .

EXAMPLE 3.3. Consider a fire-fly in a box divided into three parts (Fig. 2). The fire-fly may put out the light. In contrast to the

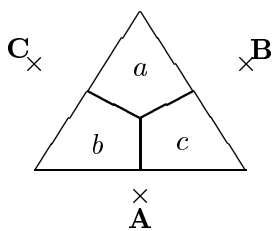


Figure 2: Experiment from Ex. 3.3

preceding examples, the internal walls are not transparent.

The events observable from position **A** form a Boolean algebra  $A$  with atoms  $\mathcal{A}(A) = \{b, c, d\}$ , where  $b$  (resp.  $c$ ) means “the fire-fly is observed in the left (resp. right) part”, and  $d$  means “the fire-fly is not observed”. (Notice that  $d$  corresponds to two possible internal situations: the fire-fly is either in the upper part or its light is not burning.) Similarly, the events observable from position **B**, resp. **C**, form a Boolean algebra  $B$ , resp.  $C$ , with  $\mathcal{A}(B) = \{a, c, e\}$ ,  $\mathcal{A}(C) = \{a, b, f\}$ . (The event  $a$  means “the fire-fly is observed in the upper part”, the meaning of the  $e, f$  is obvious.) The collection of all observable events is  $\{\mathbf{0}, a, b, c, d, e, f, a', b', c', d', e', f', \mathbf{1}\}$ . The pure states are given by the following table:

$s(a)$	$s(b)$	$s(c)$
1	0	0
0	1	0
0	0	1
0	0	0
1/2	1/2	1/2

Notice that the last pure state attains values different from 0, 1 and it cannot be interpreted as an internal state in classical terms. Nevertheless, it is not a convex combination of the preceding states.

All states  $s$  on  $L$  are uniquely determined by the values

$$s(a) = p, \quad s(b) = q, \quad s(c) = r,$$

where  $p, q, r \in [0, 1]$  are subject to the inequalities

$$p + q \leq 1, \quad p + r \leq 1, \quad q + r \leq 1.$$

The examples presented in this section will be used for demonstration of the new notions introduced in the sequel. In fact, they are crucial examples of the very basic orthomodular structures. Besides this, they show that it is natural to view a quantum experiment as a union of classical experiments, and the corresponding quantum logic as a union of classical logics (Boolean algebras). This approach will be developed in the following sections.

#### 4. Pasted families of Boolean algebras

In this section, we introduce pasted families of Boolean algebras as structures for description of quantum systems. This approach was typical for the early studies of orthomodular lattices [22, 9, 43] and it is close to the original motivation and interpretation of quantum logic. We return to it because it forms a natural link between hypergraphs (as a tool) and orthomodular structures (as the aim of our study). Moreover, this approach becomes particularly useful in the study of semipasted families of Boolean algebras which is built up the same way in Section 12.

DEFINITION 4.1. A *pasted family of Boolean algebras* (abbr. *PF*) is a family  $\mathcal{F}$  of Boolean algebras such that, for each  $A, B \in \mathcal{F}$ ,  $A \neq B$ ,

$$(PF1) \quad A \not\subseteq B,$$

$$(PF2) \quad A \cap B \text{ is a Boolean subalgebra of } A \text{ and of } B \text{ on which the operations of } A, B \text{ coincide,}$$

$$(PF3) \quad \forall a \in A \cap B \exists C \in \mathcal{F} : [\mathbf{0}, a]_A \cup [\mathbf{0}, a']_B \subseteq C.$$



REMARK 4.2. 1. Although we mean that *all* operations defined on Boolean algebras  $A, B$  coincide on  $A \cap B$ , this is equivalent to the condition that the orderings of  $A, B$  coincide.

2. The intersection  $A \cap B$  always contains the bounds  $\mathbf{0}, \mathbf{1}$ . These bounds, as well as orthocomplements (negations) are the same in all elements of  $\mathcal{F}$ , so there is no need to index them by the respective Boolean algebra.

3. The condition (PF3) is symmetric: There is also some  $D \in \mathcal{F}$  containing  $[\mathbf{0}, a']_A \cup [\mathbf{0}, a]_B$ .

4. Notice that elements of  $\mathcal{F}$  are Boolean algebras. We often refer to elements of  $\bigcup \mathcal{F}$  which are elements of the Boolean algebras in question (=events of the system).

For a PF  $\mathcal{F}$ , we use the notation  $\mathcal{A}(\mathcal{F}) = \bigcup_{B \in \mathcal{F}} \mathcal{A}(B)$  and we call the elements of  $\mathcal{A}(\mathcal{F})$  *atoms* of  $\mathcal{F}$  (they are atoms of the Boolean algebras in  $\mathcal{F}$ ).

DEFINITION 4.3. Two pasted families of Boolean algebras,  $\mathcal{F}$  and  $\mathcal{G}$ , are *isomorphic* iff there is a one-to-one mapping  $i: \bigcup \mathcal{F} \rightarrow \bigcup \mathcal{G}$  such that, for each  $B \in \mathcal{F}$ ,  $i|_B$  is a (Boolean) isomorphism of  $B$  and  $i(B)$ , and  $\mathcal{G} = \{i(B) : B \in \mathcal{F}\}$ .

DEFINITION 4.4. Let  $\mathcal{F}$  be a pasted family of Boolean algebras and let  $a, b \in \bigcup \mathcal{F}$ . We define the *distance*  $\mathbf{d}_{\mathcal{F}}(a, b)$  in  $\mathcal{F}$  as the minimal  $n$  for which there exists a sequence  $(B_1, \dots, B_n)$  in  $\mathcal{F}$  such that  $a \in B_1$ ,  $b \in B_n$  and  $B_i \cap B_{i+1} \not\supseteq \{\mathbf{0}, \mathbf{1}\}$  for  $i = 1, \dots, n - 1$ . We define  $\mathbf{d}_{\mathcal{F}}(a, b) = \infty$  if no such sequence exists, and we put  $\mathbf{d}_{\mathcal{F}}(a, a) = 0$  for all  $a \in \bigcup \mathcal{F}$ .

The definition of a state on a pasted family of Boolean algebras is a canonical extension of a state on a Boolean algebra.

DEFINITION 4.5. Let  $\mathcal{F}$  be a pasted family of Boolean algebras. A *state* on  $\mathcal{F}$  is a mapping  $s: \bigcup \mathcal{F} \rightarrow [0, 1]$  such that, for each  $B \in \mathcal{F}$ ,  $s|_B$  is a state on  $B$ .

EXAMPLE 4.6. In Exs. 3.1, 3.2,  $\{A, B\}$  is a pasted family of Boolean algebras. In Ex. 3.3,  $\{A, B, C\}$  is a pasted family of Boolean algebras. The states of the systems (as described in these examples) correspond to the states on the respective pasted families of Boolean algebras.

In order to avoid some problems with infinite subsets of PFs, we introduce the following notion:

DEFINITION 4.7. A pasted family of Boolean algebras  $\mathcal{F}$  is *chain-finite* iff there is no infinite set  $M \subseteq \bigcup \mathcal{F}$  such that each finite subset of  $M$  is contained in a Boolean algebra from  $\mathcal{F}$ .

In particular, all elements of a chain-finite PF are finite Boolean algebras.

## 5. Hypergraphs

Since [9, 43], hypergraphs are used as a powerful tool for description and graphical representation of orthomodular structures. In this section, we summarize the basic notions and the relationship of hypergraphs to pasted families of Boolean algebras.

DEFINITION 5.1. A *hypergraph* is a couple  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V}$  is a nonempty set and  $\mathcal{E}$  is a covering of  $\mathcal{V}$  by nonempty subsets of  $\mathcal{V}$  (i.e.,  $\bigcup \mathcal{E} = \mathcal{V}$ ). The elements of  $\mathcal{V}$ , resp.  $\mathcal{E}$ , are called *vertices*, resp. *edges* of  $\mathcal{H}$ .

DEFINITION 5.2. Two hypergraphs  $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ ,  $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$  are *isomorphic* iff there is a one-to-one mapping  $i: \mathcal{V}_1 \rightarrow \mathcal{V}_2$  such that  $\mathcal{E}_2 = \{i(E) : E \in \mathcal{E}_1\}$ .

DEFINITION 5.3. Let  $u, v$  be two vertices of a hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ . We define their *distance*  $\mathbf{d}_{\mathcal{H}}(u, v)$  in  $\mathcal{H}$  as the minimal  $n$  for which there exists a sequence  $(E_1, \dots, E_n)$  in  $\mathcal{E}$  such that  $u \in E_1$ ,  $v \in E_n$  and  $E_i \cap E_{i+1} \neq \emptyset$  for  $i = 1, \dots, n-1$ . We define  $\mathbf{d}_{\mathcal{H}}(u, v) = \infty$  if no such sequence exists, and we put  $\mathbf{d}_{\mathcal{H}}(v, v) = 0$  for all  $v \in \mathcal{V}$ .

DEFINITION 5.4. Let  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  be a hypergraph. A *state* on  $\mathcal{H}$  is a mapping  $s: \mathcal{V} \rightarrow [0, 1]$  such that, for each  $E \in \mathcal{E}$ ,

$$\sum_{v \in E} s(v) = 1.$$

This notion of a state was used without explicit formulation in [9, 43], and studied in detail in [10]. Now we are prepared to formulate the correspondence between pasted families of Boolean algebras and hypergraphs:

DEFINITION 5.5. Let  $\mathcal{F}$  be a chain-finite pasted family of Boolean algebras. The couple  $(\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V} = \mathcal{A}(\mathcal{F})$ ,  $\mathcal{E} = \{\mathcal{A}(B) : B \in \mathcal{F}\}$ , is a hypergraph called the *Greechie diagram* of  $\mathcal{F}$ .

In figures, we denote vertices of hypergraphs (Greechie diagrams) by small circles and edges by smooth curves.

EXAMPLE 5.6. The Greechie diagrams of the PFs from Ex. 4.6 are drawn in Fig. 3. (The vertices, resp. edges, are labeled by the corresponding atoms, resp. Boolean algebras.)

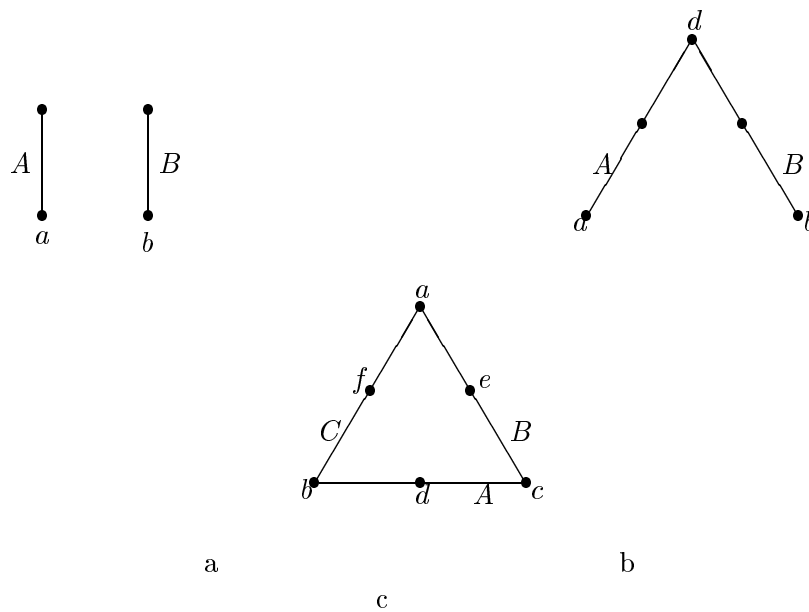


Figure 3: Greechie diagrams from Exs. 3.1, 3.2, 3.3 (cf. Exs. 4.6, 5.6)

PROPOSITION 5.7. *Two chain-finite pasted families of Boolean algebras are isomorphic iff their Greechie diagrams are isomorphic.*

The proof is routine.

The distance of atoms in a pasted family of Boolean algebras (Def. 4.4) is the distance in its Greechie diagram. States on a chain-

finite PF and on its Greechie diagram are in a natural one-to-one correspondence:

PROPOSITION 5.8. *Let  $\mathcal{F}$  be a chain-finite pasted family of Boolean algebras and let  $\mathcal{H}$  be its Greechie diagram. Then the restriction mapping  $h: \mathcal{S}(\mathcal{F}) \rightarrow \mathcal{S}(\mathcal{H})$  defined by  $h(s) = s|_{\mathcal{A}(\mathcal{F})}$  is an affine homeomorphism.*

*Proof.* Let  $s \in \mathcal{S}(\mathcal{F})$ . Each edge  $E$  of  $\mathcal{H}$  consists of the atoms of a (finite) Boolean algebra from  $\mathcal{F}$ , so  $\sum_{v \in E} s(v) = 1$  and  $s|_{\mathcal{A}(\mathcal{F})} \in \mathcal{S}(\mathcal{H})$ . Each element of  $\mathcal{F}$  can be expressed as a join of finitely many orthogonal atoms. This implies that two states on  $\mathcal{F}$  which coincide on all atoms must be identical. The mapping  $h$  is therefore injective and both  $h$  and  $h^{-1}$  are continuous. Obviously,  $h$  preserves affine combinations. It remains to prove that  $h$  is surjective.

Suppose that  $t \in \mathcal{S}(\mathcal{H})$ ; we shall find a state  $s \in \mathcal{S}(\mathcal{F})$  such that  $t = s|_{\mathcal{A}(\mathcal{F})}$ . Obviously, the only candidate is determined by the equation

$$s(b) = \sum_{a \in \mathcal{A}([0, b]_B)} t(a),$$

where  $B$  is any Boolean algebra from  $\mathcal{F}$  containing  $b$ . We only need to prove that the latter formula is consistent, i.e., independent of the choice of  $B$ . Assume that  $b \in B \cap C$  for some  $B, C \in \mathcal{F}$ . According to (PF3), there is a  $D \in \mathcal{F}$  containing  $[0, b]_C \cup [0, b'^B]_B$  and we may choose  $D$  such that the atoms of  $D$  are  $\mathcal{A}(D) = \mathcal{A}([0, b]_C) \cup \mathcal{A}([0, b'^B]_B)$ . From

$$\sum_{a \in \mathcal{A}(D)} t(a) = 1,$$

we obtain

$$\sum_{a \in \mathcal{A}([0, b]_C)} t(a) = 1 - \sum_{a \in \mathcal{A}([0, b'^B]_B)} t(a) = \sum_{a \in \mathcal{A}([0, b]_B)} t(a).$$

□

The condition of chain-finiteness is necessary in Props. 5.7 and 5.8:

EXAMPLE 5.9. Let  $A = 2^N$  (the power set of the set of natural numbers) and let  $B$  be the Boolean subalgebra of  $A$  consisting of all

finite subsets of  $N$  and their complements. The Boolean algebras  $A, B$  have the same sets of atoms. The restriction mapping  $h : \mathcal{S}(A) \rightarrow \mathcal{S}(B)$  defined by  $h(s) = s|_B$  is not injective, hence not a homeomorphism. We may view  $A$  and  $B$  also as singleton PFs,  $\{A\}$  and  $\{B\}$ . They have the same “Greechie diagram”; the state space of this diagram is homeomorphic to  $\mathcal{S}(A)$ , but not to  $\mathcal{S}(B)$ .

The “Greechie diagrams” of PFs which are not chain-finite do not allow to reconstruct the original structure of a PF and its state space. This is why we define Greechie diagrams only for chain-finite PFs. The corresponding notion in terms of hypergraphs is the following:

**DEFINITION 5.10.** A hypergraph is *chain-finite* iff it does not contain an infinite set  $V$  of vertices such that each finite subset of  $V$  is contained in an edge.

**EXAMPLE 5.11.** A hypergraph with finite edges which is not chain-finite: We take  $\mathcal{V} = \{a_n, b_n, c_n : n \in N\}$ ,  $\mathcal{E} = \{\{a_1, \dots, a_n, b_n, c_n\} : n \in N\}$ . Each finite subset of the set  $V = \{a_n : n \in N\}$  is contained in an edge.

The Greechie diagrams of chain-finite PFs are chain-finite hypergraphs.

**PROBLEM 5.12:** Which chain-finite hypergraphs are Greechie diagrams of pasted families of Boolean algebras?

**EXAMPLE 5.13.** The hypergraph in Fig. 4a is not a Greechie diagram of a PF because of condition (PF1). In a pasting of Boolean algebras  $A, B$ , we would have  $a = (c \vee_A d)^{A'} = (c \vee_B d)^{B'} = b$ , but  $a, b$  are denoted as distinct atoms. Fig. 4b shows a hypergraph which is a Greechie diagram of a PF.

**EXAMPLE 5.14.** For the same argument as in Ex. 5.13, the hypergraph in Fig. 5a is not a Greechie diagram of a PF. However, also the hypergraph in Fig. 5b is not a Greechie diagram of a PF. Indeed, we obtain  $a \vee_A b = (c \vee_A d)^{A'} = (c \vee_B d)^{B'} = e \vee_B f$ , hence  $a \vee_A b, e \vee_B f$  are the same elements in  $A$ , but  $a \vee_A b = (e \vee_B f)^{C'}$ , so they are complementary in  $C$  — a contradiction with (PF2).

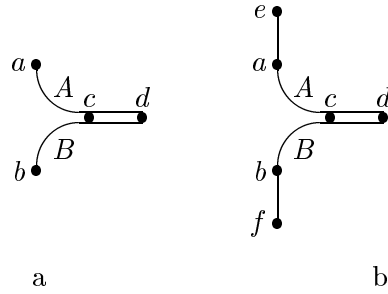


Figure 4: Hypergraphs from Ex. 5.13

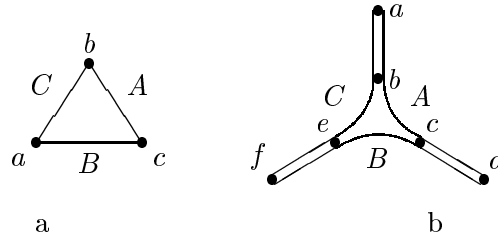


Figure 5: Hypergraphs from Ex. 5.14

EXAMPLE 5.15. The hypergraph in Fig. 6a is not a Greechie diagram of a PF — it violates (PF3). The element

$$i = a \vee_A b = (c \vee_A d)'^A = (c \vee_B d)'^B = e \vee_B f = (g \vee_C h)'^C \in A \cap C,$$

but there is no  $D \in \mathcal{F}$  containing  $[0, i]_A \cup [0, i]^C_C$ . The edge corresponding to  $D$  is added in the hypergraph in Fig. 6b which is a Greechie diagram of a PF.

### 6. Orthoalgebras

In this section we introduce orthoalgebras, algebraical structures used in the description of quantum mechanical systems. Their relation to pasted families of Boolean algebras will be clarified in the next section. Here we summarize basic notions and facts about orthoalgebras.

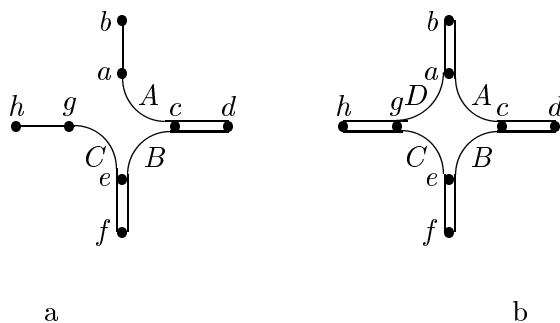


Figure 6: Hypergraphs from Ex. 5.15

DEFINITION 6.1. [8]: An *orthoalgebra* (OA) is a quadruple  $(L, \oplus_L, \mathbf{0}_L, \mathbf{1}_L)$ , where  $L$  is a set,  $\mathbf{0}_L, \mathbf{1}_L \in L$  and  $\oplus_L$  is a partial binary operation on  $L$  satisfying the following properties:

- (OA1)  $\forall a, b \in L : a \oplus_L b = b \oplus_L a,$
- (OA2)  $\forall a, b, c \in L : a \oplus_L (b \oplus_L c) = (a \oplus_L b) \oplus_L c,$
- (OA3)  $\forall a \in L \exists! d \in L : a \oplus_L d = \mathbf{1}_L,$
- (OA4)  $\forall a \in L : a \oplus_L a$  is defined iff  $a = \mathbf{0}_L.$

The operation  $\oplus_L$  is called the *orthosum*. As it is a partial operation, (OA1) and (OA2) should be read: If one side of the equality exists, then the other exists, too, and both sides are equal.

REMARK 6.2. Let  $(L, \oplus_L, \mathbf{0}_L, \mathbf{1}_L)$  be an orthoalgebra. For  $a, b \in L$ , we define  $a \leq_L b$  iff there is an element  $c \in L$  such that  $b = a \oplus_L c$ . Then  $\leq_L$  is a partial order inducing partial lattice operations  $\wedge_L, \vee_L$  on  $L$ . When we use them in expressions, we automatically assume their existence. We use the same convention for the orthosum. We define a unary operation  ${}^{\perp}L : L \rightarrow L$  assigning to each  $a \in L$  the unique element  $d$  satisfying (OA3). This is an involutive antiisomorphism of  $L$  such that  $a \wedge_L a^{\perp}L = \mathbf{0}_L$  for all  $a \in L$ . It is called an *ortho-complementation*. These operations equip  $L$  with the structure of an *orthoposet* (see [2, 16]), but not all orthoposets are orthoalgebras.

DEFINITION 6.3. Two orthoalgebras  $(K, \oplus_K, \mathbf{0}_K, \mathbf{1}_K)$ ,  $(L, \oplus_L, \mathbf{0}_L, \mathbf{1}_L)$  are called *isomorphic* iff there is a surjective mapping  $i: K \rightarrow L$  such that, for all  $a, b \in K$ ,  $i(a) \oplus_L i(b)$  exists iff  $a \oplus_K b$  exists, and if this is the case,  $i(a) \oplus_L i(b) = i(a \oplus_K b)$ .

Sometimes we speak of an orthoalgebra  $L$  instead of  $(L, \oplus_L, \mathbf{0}_L, \mathbf{1}_L)$  and we omit the indices of  $\oplus, \leq, \wedge, \vee, ', \mathbf{0}, \mathbf{1}$  when there is no risk of confusion. Two elements  $a, b$  of an orthoalgebra  $L$  are called *orthogonal* iff  $a \oplus b$  is defined (in symbols  $a \perp b$ ). This occurs iff  $a \leq b'$ .

EXAMPLE 6.4. Every Boolean algebra with the orthosum defined by  $a \oplus b = a \vee b$  whenever  $a \wedge b = \mathbf{0}$  is an orthoalgebra. This fact may be interpreted as follows: Each classical system (logic) is a special case of a quantum system (logic). Conversely, an orthoalgebra is a Boolean algebra iff it is a distributive lattice. Whenever we speak of a Boolean algebra as an orthoalgebra, we consider it this way.

The crucial example for applications in quantum physics is the following.

EXAMPLE 6.5. Let  $H$  be a (real or complex) Hilbert space. In the lattice  $L$  of closed subspaces of  $H$  we define the orthosum of two closed subspaces  $a, b$  as  $a \oplus b$  whenever  $a, b$  are orthogonal (with respect to the inner product). We obtain an orthoalgebra called a *Hilbert lattice*. Hilbert lattices are non-distributive in general. They play an essential role in the description of quantum mechanical systems [7, 18, 46]. However, an algebraic characterization of Hilbert lattices is a problem which still is not satisfactorily solved. In fact, the study of orthomodular structures was initiated in [7] and inspired by this problem — the original aim was to find an algebraic counterpart to structures successfully applied in quantum mechanics.

EXAMPLE 6.6. Let  $X$  be a nonempty set and let  $L$  be a *class of subsets* of  $X$ , i.e.,  $L \subseteq 2^X$  such that

$$\text{(CL1)} \quad \emptyset \in L,$$

$$\text{(CL2)} \quad a \in L \implies X \setminus a \in L,$$

$$\text{(CL3)} \quad (a, b \in L, a \cap b = \emptyset) \implies a \cup b \in L.$$



(The common definition of a *field of subsets* is generalized here in the sense that we require  $L$  to be closed only with respect to *disjoint* unions.) For each pair of disjoint sets  $a, b \in L$  we define  $a \oplus_L b = a \cup b$ . We take  $\mathbf{0}_L = \emptyset$ ,  $\mathbf{1}_L = X$ . Then  $(L, \oplus_L, \mathbf{0}_L, \mathbf{1}_L)$  is an orthoalgebra (see [12, 40]).

EXAMPLE 6.7. Let us make the latter example even more specific: We take  $X = \{1, 2, 3, 4\}$ ,  $L = \{\emptyset, a, a', b, b', X\}$ , where  $a = \{1, 2\}$ ,  $b = \{1, 3\}$  and  $a', b'$  are their set-theoretical complements in  $X$ . Then  $L$  is a class of subsets of  $X$ . It is a nondistributive modular lattice called *MO2*. We have already encountered it in Ex. 4.6; its Hasse diagram is in Fig. 7a.

DEFINITION 6.8. [40] A *concrete logic* is an orthoalgebra which is isomorphic to some class of subsets of a set (cf. Ex. 6.6).

The latter definition makes the term “concrete logic” independent of isomorphisms and of a particular set representation.

DEFINITION 6.9. A subset  $B$  of an orthoalgebra  $L$  is called a *Boolean subalgebra* iff

$$(BSA1) \quad \mathbf{0}_L, \mathbf{1}_L \in B,$$

$$(BSA2) \quad a \in B \implies a'^L \in B,$$

$$(BSA3) \quad (B, \oplus_B, \mathbf{0}_L, \mathbf{1}_L), \text{ where } \oplus_B \text{ is the restriction of } \oplus_L \text{ to } B, \text{ is a Boolean algebra.}$$

Two elements  $a, b$  in  $L$  are called *compatible*, in symbols  $a \leftrightarrow b$ , iff they are contained in a Boolean subalgebra of  $L$ .

In the quantum logical interpretation, compatible events belong to some classical subsystem; as such, they are simultaneously observable and all Boolean expressions on them are well defined in the orthoalgebra.

REMARK 6.10. Suppose that  $a, b$  are orthogonal elements of an orthoalgebra  $L$ . According to Def. 8.1, they are contained in at least one Boolean subalgebra of  $L$ . Suppose that  $a, b$  are contained in two Boolean subalgebras,  $A, B$ , of  $L$ . As the operations on  $A, B$  are induced by the ordering of  $L$ , they coincide on their intersection, and

$a \vee_A b = a \vee_B b$ . Any Boolean expression in  $a, b$  gives the same result in any Boolean subalgebra containing  $a, b$ . The collection of all Boolean subalgebras of  $L$  containing  $a, b$  has a minimal element — the Boolean subalgebra *generated* by  $a, b$ .

DEFINITION 6.11. A *block* in an orthoalgebra is a maximal Boolean subalgebra.

REMARK 6.12. A standard use of Zorn's Lemma implies that each Boolean subalgebra belongs to a maximal Boolean subalgebra.

This ensures the existence of blocks in any orthoalgebra. We shall prove more in the next section.

A characteristic feature of Boolean algebras is that they are uniquely complemented (see [2, 16]): For each element  $a$ , there is a unique element  $b$ , namely  $b = a'$ , such that  $a \wedge b = \mathbf{0}$ ,  $a \vee b = \mathbf{1}$  (i.e.,  $b$  is the lattice-theoretical *complement* of  $a$ ). In general orthoalgebras, an element  $a$  may have more than one complement ( $a'$  being always one of them). We call  $a'$  an *orthocomplement* to distinguish it from other lattice-theoretical complements. In Boolean algebras, the notions of complement and orthocomplement coincide.

Due to the preceding facts, the ordering of a Boolean algebra induces the orthosum uniquely and it is itself sufficient to determine the whole structure of a Boolean algebra.

PROPOSITION 6.13. For an element  $a$  of an orthoalgebra  $L$ ,  $a'^L$  is the only lattice-theoretical complement of  $a$  which is orthogonal to  $a$ .

*Proof.* If  $b \perp_L a$  and  $b$  is a complement of  $a$ , then there is a Boolean subalgebra  $A$  containing  $a, b$ , and  $a \oplus_L b = a \oplus_A b = a \vee_A b = \mathbf{1}$ , so  $b = a'^L$ .  $\square$

EXAMPLE 6.14. In the orthoalgebra from Ex. 6.7,  $a$  has complements  $a', b, b'$ , but  $a \not\perp b$ ,  $a \not\perp b'$ .

PROPOSITION 6.15. Let  $a, b, c$  be elements of an orthoalgebra  $L$  such that  $a \leq b \leq c$ . Then they are all contained in a Boolean subalgebra of  $L$ .

*Proof.* Take the subsets  $D = \{\mathbf{0}_L, a, (a \oplus_L b'^L)^L, b\}$ ,  $E = \{\mathbf{0}_L, c'^L, (c'^L \oplus_L b)^L, b'^L\}$ . For all  $d \in D$ ,  $e \in E$ , we have  $d \leq_L b$ ,  $e \leq_L b'^L$ , hence  $b \leq_L e'^L$  and  $d \leq_L e'^L$ . This means that  $d, e$  are orthogonal in  $L$  and  $d \oplus_L e$  is defined. The set  $\{d \oplus_L e : d \in D, e \in E\}$  is closed with respect to orthocomplements in  $L$ ; with the ordering inherited from  $L$ , it is a Boolean subalgebra of  $L$  containing  $a, b, c$ .  $\square$

**PROPOSITION 6.16.** *Let  $(L, \oplus_L, \mathbf{0}_L, \mathbf{1}_L)$  be an orthoalgebra and let  $e \in L \setminus \{\mathbf{0}_L\}$ . We take the interval  $K = [\mathbf{0}, e]_L$  and we define  $\mathbf{0}_K = \mathbf{0}_L$ ,  $\mathbf{1}_K = e$ , and*

$$a \oplus_K b = a \oplus_L b \text{ whenever } a \perp_L b \text{ and } a, b, a \oplus_L b \in K.$$

*Then  $(K, \oplus_K, \mathbf{0}_K, \mathbf{1}_K)$  is an orthoalgebra.*

*Proof.* (OA1), (OA4): Trivial.

(OA2): If  $(a \oplus_L b) \oplus_L c$  exists and belongs to  $K$ , then  $a \oplus_L b \in K$  and

$$(a \oplus_K b) \oplus_K c = (a \oplus_L b) \oplus_L c = a \oplus_L (b \oplus_L c) = a \oplus_K (b \oplus_K c)$$

(OA3): For each  $a \in K$ , the element  $a'^K = (a \oplus_L e'^L)^L$  is the unique element satisfying  $a \oplus_K a'^K = e = \mathbf{1}_K$  because  $a \oplus_L a'^K \oplus_L e'^L = \mathbf{1}_L$ .  $\square$

**REMARK 6.17.** Orthogonality in an interval need not coincide with orthogonality in the whole OA as we shall see in Rem. 8.6.

**REMARK 6.18.** Prop. 6.16 can be generalized to intervals with non-zero lower bounds, but we do not need this here.

## 7. Pasting

In this section we shall associate with a pasted family of Boolean algebras a single algebraic structure — its pasting — which appears to be an orthoalgebra. We show that all orthoalgebras can be obtained this way.

**DEFINITION 7.1.** Let  $\mathcal{F}$  be a pasted family of Boolean algebras. On  $L = \bigcup \mathcal{F}$ , we define the partial operation  $\oplus_L$  as the union of all  $\oplus_A$ ,  $A \in \mathcal{F}$ , i.e.,  $a \oplus_L b = c$  iff there is an  $A \in \mathcal{F}$  such that  $a \oplus_A b = c$ . The quadruple  $(L, \oplus_L, \mathbf{0}, \mathbf{1})$  is called the *pasting* of  $\mathcal{F}$ .

The consistency of the latter definition follows from (PF2).

EXAMPLE 7.2. The pastings of PFs from Ex. 4.6 have the Hasse diagrams in Fig. 7 (in Fig. 7c, the elements  $a, a'$  are marked twice in order to reduce the number of crossings).

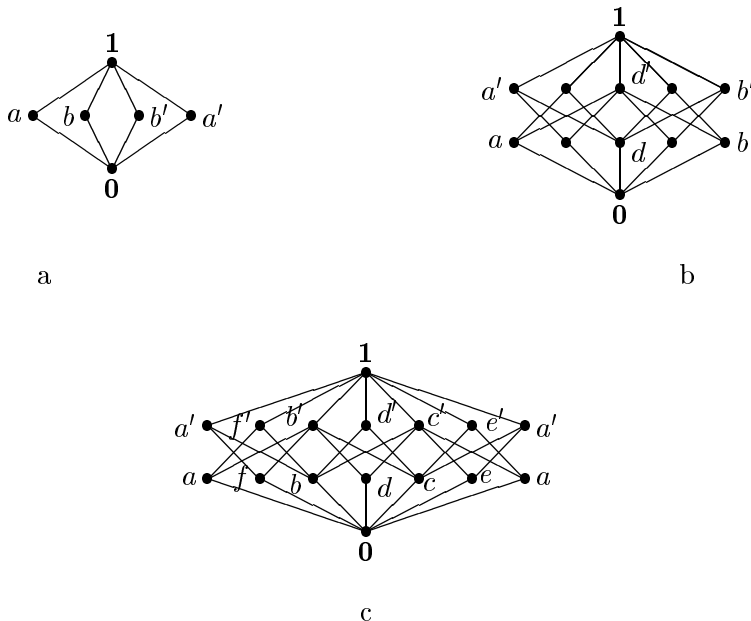


Figure 7: Hasse diagrams of the pastings of the pasted families of Boolean algebras from Exs. 3.1, 3.2, 3.3 (cf. Exs. 4.6, 7.2)

The following proposition states that orthoalgebras are exactly pastings of PFs. Although this fact is not very difficult to prove, it seems to be new. It is not mentioned in [16] because the study of orthoalgebras is relatively new. For partial results, see [16, §4, Prop. 13], [22], [41].

PROPOSITION 7.3. *The pasting of a pasted family of Boolean algebras is an orthoalgebra. Conversely, every orthoalgebra is a pasting of a pasted family of Boolean algebras, namely of the family of its blocks.*

*Proof.* Let  $L$  be the pasting of a pasted family of Boolean algebras  $\mathcal{F}$ . Conditions (OA1), (OA3), (OA4) are easily verified for  $L$ . Condition

(OA2) follows from (PF3): If  $(a \oplus_L b) \oplus_L c$  is defined, then there are  $A, B \in \mathcal{F}$  such that  $a \oplus_A b = a \oplus_L b$  and  $(a \oplus_L b) \oplus_B c = (a \oplus_L b) \oplus_L c$ . Applying (PF3) to  $d = a \oplus_L b \in A \cap B$ , we find a  $C \in \mathcal{F}$  containing  $[0, d]_A \cup [0, d']_B$ . Thus  $a, b, c \in C$  and all calculations can be made in  $C$ :

$$(a \oplus_L b) \oplus_L c = (a \oplus_C b) \oplus_C c = a \oplus_C (b \oplus_C c) = a \oplus_L (b \oplus_L c).$$

Conversely, let  $L$  be an orthoalgebra and  $\mathcal{F}$  the family of its blocks. First we prove that  $\mathcal{F}$  is a pasted family of Boolean algebras.

(PF1): Trivial.

(PF2): The orthosums on blocks  $A, B$  are only restrictions of the orthosum of  $L$ , so they coincide on  $A \cap B$ , which is therefore a Boolean subalgebra of  $A$  and of  $B$ .

(PF3): Let  $A, B$  be blocks of  $L$  and let  $a \in A \cap B$ . The intervals  $[0, a]_A, [0, a']_B$  are Boolean algebras. For all  $b \in [0, a]_A, c \in [0, a']_B$ , we have  $b \perp_L c$ , so  $b \oplus_L c$  exists. All elements  $b \oplus_L c$ , where  $b \in [0, a]_A$  and  $c \in [0, a']_B$ , form a Boolean subalgebra of  $L$  (isomorphic to the product of Boolean algebras  $[0, a]_A$  and  $[0, a']_B$ ). It is contained in a block. In particular, this block contains  $[0, a]_A \cup [0, a']_B$ .

We proved that  $\mathcal{F}$  is a pasted family of Boolean algebras, so it has a pasting. It remains to prove that this pasting coincides with  $L$ . Each element  $a$  of  $L$  belongs to the Boolean subalgebra  $\{0, a, a', 1\}$  of  $L$ . Thus  $L = \bigcup \mathcal{F}$ . For  $a, b \in L$ , the orthosum  $a \oplus_L b$  exists iff there is a block  $A$  such that  $a \oplus_A b$  exists. In this case  $a \oplus_L b = a \oplus_A b$ , so the orthosum of  $L$  is the orthosum of the pasting of  $\mathcal{F}$ .  $\square$

As far as we know, the latter proposition for OAs is new. It implies that the pasting technique for Boolean algebras is sufficiently general to give all OAs. Props. 7.3 and 7.3 show that orthoalgebras are exactly pastings of PFs. The notion of block is quite natural in OAs. The attempts to define blocks in more general structures were not very successful.

## 8. Orthomodular posets and lattices

In this section, we introduce special types of orthoalgebras — orthomodular posets and orthomodular lattices. We formulate conditions under which a pasting of a pasted family of Boolean algebras gives rise to an orthomodular poset, resp. an orthomodular lattice.

DEFINITION 8.1. An orthoalgebra  $L$  is

- an *orthomodular poset* (OMP) iff each orthogonal pair has a join in  $L$ ,
- an *orthomodular lattice* (OML) iff  $L$  is a lattice.

REMARK 8.2. We use here nonstandard definitions of OMPs and OMLs based on orthoalgebras. According to the standard definition, an orthomodular poset  $L$  is a poset with bounds  $\mathbf{0}$ ,  $\mathbf{1}$ , equipped with a unary operation  $' : L \rightarrow L$  (*orthocomplementation*) such that, for all  $a, b \in L$ ,

$$\text{(OMP1)} \quad a'' = a,$$

$$\text{(OMP2)} \quad a \leq b \implies b' \leq a',$$

$$\text{(OMP3)} \quad (b \leq a, b \leq a') \implies b = \mathbf{0},$$

$$\text{(OMP4)} \quad a \leq b' \implies a \vee b \text{ exists},$$

$$\text{(OMP5)} \quad a \leq d \implies d = a \vee (a' \wedge d).$$

Using the convention of Rem. 6.2, the condition (OMP3) may be written as  $a \wedge a' = \mathbf{0}$ . (OMP5) is called the *orthomodular law* (see [2, 16] for the discussion of its numerous equivalent formulations). An orthomodular lattice is an OMP which is a lattice. Our definition is based on different basic operations, but it describes categorically the same structures. We have already shown how the ordering and orthocomplementation are derived from the orthosum. Conversely, in an OMP  $L$ , the corresponding orthosum can be defined as  $a \oplus_L b = a \vee_L b$  whenever  $a \leq_L b'^L$ .

The inclusions between the classes of orthoalgebras studied here are displayed in Fig. 8. All these inclusions are proper (see the following examples). Boolean algebras and Hilbert lattices (Ex. 6.5), as well as *MO2* (Exs. 3.1, 6.7) and the pasting from Ex. 3.2, are OMLs.

EXAMPLE 8.3. Exs. 3.3, 4.6, 7.2 show an orthoalgebra which is not an OMP because orthogonal atoms  $a, b$  do not have a join — they have incomparable upper bounds  $c', f'$ .

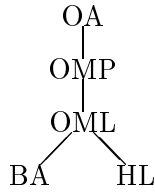


Figure 8: Inclusions between classes of orthoalgebras (BA=Boolean algebras, HL=Hilbert lattices)

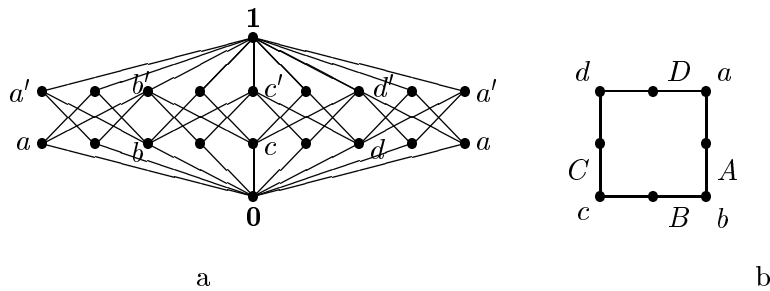


Figure 9: Hasse and Greechie diagrams of an OMP which is not an OML (Ex. 8.4)

EXAMPLE 8.4. The Hasse diagram in Fig. 9a (where  $a, a'$  are doubled to reduce the number of crossings) represents (the smallest) OMP which is not an OML, because (nonorthogonal) atoms  $b, d$  have incomparable upper bounds  $a', c'$ . Its Greechie diagram is in Fig. 9b.

Usually these structures, not PFs, are defined as primary objects.

PROPOSITION 8.5. *If  $L$  is an OMP, resp. an OML, then so is any nontrivial interval in  $L$ .*

REMARK 8.6. In an orthoalgebra which is not an orthomodular poset, orthogonality in an interval need not coincide with orthogonality in the whole OA. In the OA  $L$  from Exs. 3.3, 4.6,  $a \perp_L b$  and  $a, b \in [0, c'^L]_L$ , but  $a, b$  are not orthogonal in  $[0, c'^L]_L$ .

EXAMPLE 8.7. In OMPs and OMLs, if  $a \oplus b$  exists, then  $a \vee b$  exists, too, and  $a \oplus b = a \vee b$ . In OAs, the existence of  $a \oplus b$  does not imply the existence of  $a \vee b$ , see Exs. 3.3, 4.6, 7.2.

The results in the sequel will be formulated as strong as possible, i.e., positive results for the most general class of orthoalgebras, negative results (counterexamples) for OMLs. When we want to talk about any of the three classes (OAs, OMPs, OMLs), we speak of *orthomodular structures*.

Prop. 7.3 was proved for chain-finite OMLs in [22] and for OMPs in [41]. Now we shall characterize the PFs which give OMPs as their pastings. We need the following notion:

**DEFINITION 8.8.** Let  $\mathcal{F}$  be a pasted family of Boolean algebras. A sequence  $((A_0, a_0), (A_1, a_1), \dots, (A_{n-1}, a_{n-1}))$ , where  $A_i \in \mathcal{F}$ ,  $a_i \in A_i \cap A_{i+1} \setminus \{\mathbf{0}\}$  (indices mod  $n$ ), is called an *n-cycle* in  $\mathcal{F}$  iff  $[\mathbf{0}, a_i]_{A_i} = [\mathbf{0}, a_i]_{A_{i+1}}$  and  $a_{i-1} \perp_{A_i} a_i$  for  $i = 0, \dots, n-1$ . If, moreover,  $A_0, \dots, A_{n-1}$  are mutually distinct and  $a_i$  is an atom of both  $A_i$  and  $A_{i+1}$  for all  $i = 0, \dots, n-1$ , then  $((A_0, a_0), (A_1, a_1), \dots, (A_{n-1}, a_{n-1}))$  is called an *n-loop* in  $\mathcal{F}$ .

**EXAMPLE 8.9.** In Ex. 3.3, resp. Ex. 4.6, (see Fig. 3c),  $((A, c), (B, a), (C, b))$  is a 3-loop. In Ex. 8.4 (see Fig. 9b),  $((A, b), (B, c), (C, d), (D, a))$  is a 4-loop. The hypergraph in Fig. 6b is a Greechie diagram of an OML in which  $((A, c \vee d), (B, e \vee f), (C, g \vee h), (D, a \vee b))$  is a 4-cycle, but not a 4-loop.

**PROPOSITION 8.10.** *The pasting  $L$  of a pasted family of Boolean algebras  $\mathcal{F}$  is an orthomodular poset iff for each 3-cycle  $((A_0, a_0), (A_1, a_1), (A_2, a_2))$  in  $\mathcal{F}$  there is a  $D \in \mathcal{F}$  containing  $a_0, a_1, a_2$ .*

*Proof.* For the first implication, let us assume that  $L$  is an OMP and  $((A_0, a_0), (A_1, a_1), (A_2, a_2))$  is a 3-cycle in  $\mathcal{F}$ . Then  $a_0, a_1, a_2$  are mutually orthogonal in  $L$ , there exists the join  $d = a_0 \vee_L a_1 = a_0 \vee_{A_1} a_1$  and  $d \leq_L a'_2$ . Thus there is a  $D \in \mathcal{F}$  such that  $d, a_2 \in D$  and  $d \leq_D a'_2$ . According to (PF3),  $D$  can be chosen such that  $[\mathbf{0}, d]_D = [\mathbf{0}, d]_{A_1}$ , so  $D$  contains  $a_0, a_1, a_2$ .

For the reverse implication, let  $b, c \in L$ ,  $b \perp_L c$ . There is a Boolean algebra  $A \in \mathcal{F}$  such that  $b \perp_A c$ . We shall prove that  $b \vee_A c$  is also the join  $b \vee_L c$  computed in  $L$ . Suppose that  $d \in L$  and  $b \leq_L d$ ,  $c \leq_L d$ . There are Boolean algebras  $B, C \in \mathcal{F}$  satisfying  $b \leq_B d$ ,  $c \leq_C d$ . Moreover, according to (PF3) we may choose  $B, C$  such that  $[\mathbf{0}, b]_B = [\mathbf{0}, b]_A$ ,  $[\mathbf{0}, c]_C = [\mathbf{0}, c]_A$ , and  $[\mathbf{0}, d^L]_B = [\mathbf{0}, d^L]_C$ .



The triple  $((A, b), (B, d^L), (C, c))$  is a 3-cycle unless some of  $b, c, d^L$  is zero (this case is trivial). According to our assumption, there is a Boolean algebra  $D \in \mathcal{F}$  containing  $b, c, d^L$ . As  $b \vee_A c$  belongs to the Boolean subalgebra of  $L$  generated by  $b, c$ , it belongs also to  $D$ . We obtain  $b \vee_A c = b \vee_D c \leq_D d$ , so  $b \vee_A c = b \vee_L c$ .  $\square$

EXAMPLE 8.11. A typical example of a PF containing a 3-loop and giving an OML as its pasting is presented by the Greechie diagram in Fig. 10.

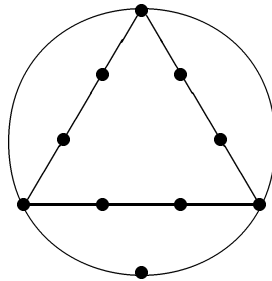


Figure 10: Greechie diagram of an OML containing a 3-loop (Ex. 8.11)

There are conditions (see [22, 16, 35]) ensuring that a pasting of a PF is an OML, but they are not far from a reformulation of the condition that it is a lattice.

PROPOSITION 8.12. [22] *Let  $\mathcal{F}$  be a chain-finite pasted family of Boolean algebras. Its pasting  $L$  is an orthomodular lattice iff for each 4-cycle  $((A_0, a_0), (A_1, a_1), (A_2, a_2), (A_3, a_3))$  in  $\mathcal{F}$  there is a 4-cycle  $((B_0, b), (B_1, b'), (B_2, b), (B_3, b'))$  in  $\mathcal{F}$  such that  $a_0 \leq b, a_2 \leq b, a_1 \leq b', a_3 \leq b'$ .*

*Proof.* (See also [22, 16] or [35].) Suppose that  $\mathcal{F}$  satisfies the condition of the latter proposition. According to Prop. 7.3,  $L$  is a chain-finite orthoalgebra. It remains to prove that  $L$  is a lattice. Let  $a_0, a_2 \in L$ . Let  $U$  be the set of all upper bounds of  $a_0, a_2$ . As

$L$  is chain-finite,  $U$  has a minimal element. Suppose that  $c_1, c_3$  are minimal elements of  $U$ . Then there are  $A_0, A_1, A_2, A_3 \in \mathcal{F}$  such that  $a_0 \leq_{A_0} c_3$ ,  $a_0 \leq_{A_1} c_1$ ,  $a_2 \leq_{A_2} c_1$ ,  $a_2 \leq_{A_3} c_3$ . Then  $((A_0, a_0), (A_1, c'_1), (A_2, a_2), (A_3, c'_3))$  is a 4-cycle in  $\mathcal{F}$  (except for the trivial cases  $\mathbf{0} \in \{a_0, c'_1, a_2, c'_3\}$ ). According to the assumption (with  $a_1 = c'_1, a_3 = c'_3$ ), we find an element  $b \in U$  satisfying  $b \leq c_1, b \leq c_3$ . Because of minimality of  $c_1, c_3$ , we obtain  $c_1 = c_3 = b = a_0 \vee_L a_2$ .  $\square$

EXAMPLE 8.13. A typical example of a PF containing a 4-loop and giving an OML as its pasting is presented by the Greechie diagram in Fig. 11.

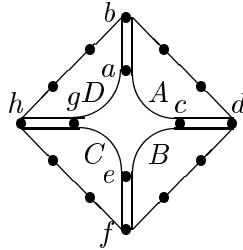


Figure 11: Greechie diagram of an OML containing a 4-loop (Ex. 8.13)

Sufficient conditions for a PF to give an OMP or an OML as its pasting were given long before in Greechie’s Loop Lemma [9]:

PROPOSITION 8.14. *Let  $\mathcal{F}$  be a pasted family of Boolean algebras such that the intersection of each pair  $A, B \in \mathcal{F}$  is of the form  $\{\mathbf{0}, \mathbf{1}\}$  or  $\{\mathbf{0}, a, a', \mathbf{1}\}$ , where  $a$  is an atom of both  $A$  and  $B$ . Then the pasting of  $\mathcal{F}$  is*

- on orthomodular poset iff  $\mathcal{F}$  has no 3-loops,
- on orthomodular lattice iff  $\mathcal{F}$  has no 3-loops and 4-loops.

*Proof.* The first part follows easily from Prop. 8.10, the second part from Prop. 8.12. See also [9] or [43] for a direct proof.  $\square$

EXAMPLE 8.15. Exs. 8.11, 8.13 show PFs to which Prop. 8.14 is not applicable. Nevertheless, their pastings are OMLs.

## 9. State spaces of orthoalgebras

Here we define states on orthoalgebras and we present basic facts about state spaces.

DEFINITION 9.1. Let  $L$  be an orthoalgebra. A *state* on  $L$  is a mapping  $s : L \rightarrow [0, 1]$  such that

$$\text{(SOA1)} \quad s(\mathbf{1}) = 1,$$

$$\text{(SOA2)} \quad a, b \in L, a \perp_L b \implies s(a \oplus_L b) = s(a) + s(b).$$

REMARK 9.2. If  $L$  is an OMP, (SOA2) attains the standard form  $a \perp_L b \implies s(a \vee_L b) = s(a) + s(b)$  (assumption  $a \wedge_L b = \mathbf{0}$  is too weak here).

States on orthoalgebras (Def. 9.1) correspond to states on pasted families of Boolean algebras (Def. 4.5):

PROPOSITION 9.3. *Let  $\mathcal{F}$  be a pasted family of Boolean algebras. A function  $s : \bigcup \mathcal{F} \rightarrow [0, 1]$  is a state on  $\mathcal{F}$  iff it is a state on the pasting of  $\mathcal{F}$ .*

The proof is straightforward. For chain-finite orthoalgebras, Prop. 5.8 extends the latter proposition to a correspondence between states on an orthoalgebra and on its Greechie diagram:

PROPOSITION 9.4. *Let  $L$  be a chain-finite orthoalgebra and  $\mathcal{H}$  its Greechie diagram. The restriction mapping  $h : \mathcal{S}(L) \rightarrow \mathcal{S}(\mathcal{H})$  defined by  $h(s) = s|_{\mathcal{A}(L)}$  is an affine homeomorphism.*

Exs. 3.1, 3.2, 3.3 may serve as examples of state spaces of orthoalgebras. Further examples follow.

EXAMPLE 9.5. [14] The hypergraph in Fig. 12 is the Greechie diagram of an orthoalgebra. The description of its states is obtained from Ex. 3.3 with an additional condition  $s(d) + s(e) + s(f) = 1$ . With the notation of Ex. 3.3, we obtain the restriction  $p + q + r = 1$ . Each state on any block has a unique extension to the whole orthoalgebra. The state space is a triangle. Its analogy to the state space of the Boolean algebra  $2^3$  will be formulated in Section 13.

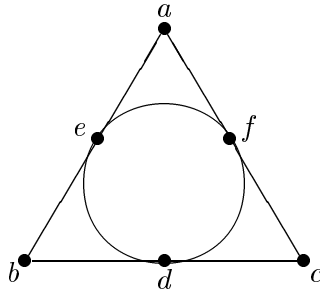


Figure 12: Greechie diagram of the orthoalgebra from Ex. 9.5

EXAMPLE 9.6. [14] One can easily verify that the hypergraph in Fig. 13 admits only one state (evaluating each vertex to  $1/3$ ). It is the Greechie diagram of an orthoalgebra admitting exactly one state. This state is *faithful*, i.e., it attains nonzero values on all nonzero elements.

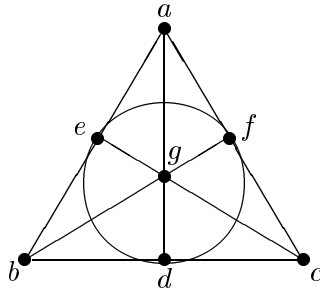


Figure 13: Greechie diagram of an orthoalgebra admitting exactly one (faithful) state (Ex. 9.6)

EXAMPLE 9.7. In the hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  in Fig. 14,  $\mathcal{V}$  allows disjoint coverings by 9 or 10 edges. This implies, for each  $s \in \mathcal{S}(\mathcal{H})$ , the equalities  $9 = \sum_{v \in \mathcal{V}} s(v) = 10$ , a contradiction. The hypergraph  $\mathcal{H}$  does not admit any state. According to Prop. 8.14,  $\mathcal{H}$  is the

Greechie diagram of an OML without states (Prop. 9.4). (This is an unpublished example by R. Mayet simplifying the example by R. Greechie [9].)

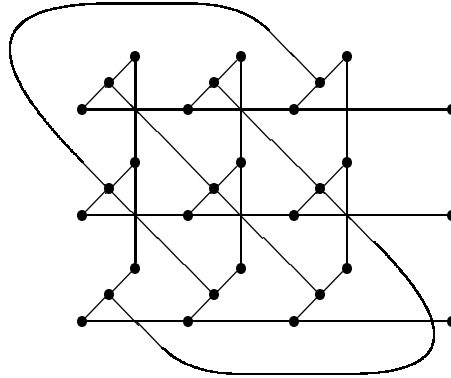


Figure 14: Greechie diagram of an OML admitting no states (Ex. 9.7)

Preceding examples have very few states. Despite their peculiarity, they became the base of many important results about state spaces of orthomodular structures, e.g., [27, 33, 37, 43].

**PROPOSITION 9.8.** *The state space of an orthoalgebra  $L$  is convex and compact in the weak topology.*

*Proof.* Convexity is straightforward. The state space of  $L$  is a subset of  $[0, 1]^L$  which is a compact set (Tikhonov Theorem). It is a closed subset because it is determined by a collection of equalities of the form (SOA2) and summation of states is a continuous operation in  $R^L$ .  $\square$

Props. 9.3, 9.4 allow to extend Prop. 9.8 also to pasted families of Boolean algebras and hypergraphs. The reverse implication will be proved in Section 15.

## 10. Constructions with orthomodular structures

In this section, we introduce the basic construction techniques with orthomodular structures — direct products, horizontal sums, and identification of atoms. For more advanced techniques, we refer to [14, 15, 16, 35]. The proofs missing in this section can be found, e.g., in [2, 16, 40].

**DEFINITION 10.1.** Let  $\mathcal{F}$  be a family of orthoalgebras. We take the Cartesian product  $L = \prod_{K \in \mathcal{F}} K$  and we endow it with the orthosum  $\oplus_L$  and orthocomplementation  $'^L$  defined pointwise, i.e., for all  $a, b \in L$ ,  $a = (a_K)_{K \in \mathcal{F}}$ ,  $b = (b_K)_{K \in \mathcal{F}}$ ,  $c = (c_K)_{K \in \mathcal{F}}$ , we define

$$a \oplus_L b = c \iff \forall K \in \mathcal{F} : a_K \oplus_K b_K = c_K.$$

We define  $\mathbf{0}_L$ , resp.  $\mathbf{1}_L$ , as the element of  $\prod_{K \in \mathcal{F}} K$  which has all coordinates equal to  $\mathbf{0}_K$ , resp.  $\mathbf{1}_K$ . Then  $(L, \oplus_L, \mathbf{0}_L, \mathbf{1}_L)$  is an orthoalgebra called the *product* of the family  $\mathcal{F}$ . The product of orthomodular posets (resp. orthomodular lattices) is an orthomodular poset (resp. an orthomodular lattice).

**EXAMPLE 10.2.** The pasting in Ex. 3.2 (see also Ex. 4.6) is the product of  $MO2$  and  $2^1$  (the elements  $(\mathbf{0}, \mathbf{1}), (\mathbf{1}, \mathbf{0})$  correspond to  $d, d'$ , resp.). Fig. 4b (resp. Fig. 6b) is the Greechie diagram of the product  $MO2 \times 2^2$  (resp.  $MO2 \times MO2$ ).

**EXAMPLE 10.3.** Take the OML  $L$  from Ex. 9.7, and form the product of  $L$  and the Boolean algebra  $2^1$ . Each state on the product vanishes on the elements  $(a, \mathbf{0})$  and attains 1 at all  $(a, \mathbf{1})$ ,  $a \in L$ . The product admits exactly one state; this state is two-valued. (This example is based on the idea of P. Pták [39].)

**DEFINITION 10.4.** Let  $\mathcal{F}$  be a family of orthoalgebras. We make a family  $\mathcal{G}$  of copies of orthoalgebras from  $\mathcal{F}$  which are disjoint except that they have the same least element,  $\mathbf{0}$ , and the same greatest element,  $\mathbf{1}$ . Thus, for each  $K, M \in \mathcal{G}$ ,  $K \neq M$ , we have  $K \cap M = \{\mathbf{0}, \mathbf{1}\}$ . We take the union  $L = \bigcup \mathcal{G}$  and we endow it with the orthosum  $\oplus_L$  defined by

$$a \oplus_L b = c \iff \exists K \in \mathcal{G} : (a, b, c \in K, a \oplus_K b = c).$$

Then  $(L, \oplus_L, \mathbf{0}, \mathbf{1})$  is an orthoalgebra called the *horizontal sum* of the family  $\mathcal{F}$ . The horizontal sum of orthomodular posets (resp. orthomodular lattices) is an orthomodular poset (resp. an orthomodular lattice).

The analogy between Defs. 10.4 and 7.1 allows us to consider the horizontal sum as a pasting of orthoalgebras. (It is called  *$\mathbf{0-1}$ -pasting* in [40].) The common generalization is given in [35] for OMPs and OMLs and in [14] for OAs.

EXAMPLE 10.5. *MO2* (Ex. 3.1) is the horizontal sum of two Boolean algebras  $2^2$ .

Some modifications of the following technique were used several times, but it was explicitly formulated only in [21]. We present it here for orthomodular lattices, using the definition from Rem. 8.2.

DEFINITION 10.6. Let  $a, b$  be elements of an orthoalgebra  $L$ . We define their *distance*  $\mathbf{d}_L(a, b)$  in  $L$  as the minimal  $n$  for which there exists a sequence  $(B_1, \dots, B_n)$  of blocks in  $L$  such that  $a \in B_1, b \in B_n$  and  $B_i \cap B_{i+1} \not\supseteq \{\mathbf{0}, \mathbf{1}\}$  for  $i = 1, \dots, n - 1$ . We define  $\mathbf{d}_L(a, b) = \infty$  if no such sequence exists, and we put  $\mathbf{d}_L(a, a) = 0$  for all  $a \in L$ .

According to Prop. 7.3, the latter definition is meaningful and the distance in a pasted family of Boolean algebras (Def. 4.4) coincides with the distance in its pasting.

THEOREM 10.7. *Let  $L$  be an orthomodular lattice. Let  $M$  be a set of atoms of  $L$  with mutual distance (in  $L$ ) at least 5. We define an equivalence relation  $\approx$  on  $L$  such that  $a \approx b$  iff*

- $a = b$  or
- $a, b \in M$  or
- $a^{!L}, b^{!L} \in M$ .

For each  $a \in L$ , we denote  $[a] = \{b \in L : b \approx a\}$ . We take the set  $K = \{[a] : a \in L\}$ , and we endow it with the unary operation  ${}^{!K}$  and a relation  $\leq_K$  defined by

$$[a]^{!K} = [a^{!L}],$$

$$[a] \leq_K [b] \iff \exists a_1 \in [a] \exists b_1 \in [b] : a_1 \leq_L b_1.$$

Then  $(K, \leq_K, {}^1K)$  is an orthomodular lattice. Moreover, the atoms (resp. blocks) of  $K$  are images of the atoms (resp. isomorphic blocks) of  $L$  under the quotient mapping. In this case, we say that  $K$  originated by identification of atoms of  $M$  in  $L$  and we denote this fact by  $K = L/M$ .

*Proof.* Apparently the relation  $\approx$  is an equivalence. The classes of equivalence (=elements of  $K$ ) are  $M, M'^K$  and  $\{a\}$ ,  $a \in L \setminus (M \cup M'^K)$ .

Notice that  $[\mathbf{0}_L] \leq_K [a]$  for all  $[a] \in K$ , and  $[a] \leq_K [\mathbf{0}_L]$  iff  $a = \mathbf{0}_L$ . Also  $[a] \leq_K [\mathbf{1}_L] = \{\mathbf{1}_L\}$  for all  $[a] \in K$ , and  $[\mathbf{1}_L] \leq_K [a]$  iff  $a = \mathbf{1}_L$ . No class of equivalence  $\approx$  contains two different orthogonal elements. Thus  $a_1 \perp_L a_2$  for  $a_1, a_2 \in [a] \in K$  implies  $a_1 = a_2 = \mathbf{0}_L$ ,  $[a] = [\mathbf{0}_L] = \{\mathbf{0}_L\}$ .

Consistency of  ${}^1K$ , (OMP1), (OMP2): The equivalence  $\approx$  preserves the orthocomplementation in the sense that  $a'^L \approx b'^L \iff a \approx b$ . Thus the operation  ${}^1K$  is well-defined, involutive and antitone. In particular,  $M'^K = \{a'^L : a \in M\}$ .

We have to prove that  $\leq_K$  is an ordering. Reflexivity is trivial.

Antisymmetry of  $\leq_K$ : Suppose that  $[a] \leq_K [b]$  and  $[b] \leq_K [a]$ . There are  $a_1, a_2 \in [a]$ ,  $b_1, b_2 \in [b]$  such that  $a_1 \leq_L b_1$ ,  $b_2 \leq_L a_2$ . If  $a_1 = a_2$  and  $b_1 = b_2$ , then  $a_1 = b_1$  and  $[a] = [b]$ . Assume, without any loss of generality, that  $a_1 \neq a_2$ , so  $[a]$  is not a singleton,  $[a] \in \{M, M'^K\}$ . Our assumption ensures that  $[b]$  is different from  $[\mathbf{0}_L], [\mathbf{1}_L]$ . If  $[a] = M$ , then  $b_2 \in [a_2]_L = \{a_2\}$  and  $b_2 = a_2$ . If  $[a] = M'^K$ , then  $b_1 \in [a_1]_L = \{a_1\}$  and  $b_1 = a_1$ . In all cases, we obtain  $[a] = [b]$ .

Transitivity of  $\leq_K$ : Assume that  $[a] \leq_K [b]$  and  $[b] \leq_K [c]$ , i.e., there are  $a_1 \in [a]$ ,  $c_2 \in [c]$  and  $b_1, b_2 \in [b]$  such that  $a_1 \leq_L b_1$  and  $b_2 \leq_L c_2$ . If  $b_1 = b_2$ , then  $a_1 \leq_L c_2$ . If  $[b] = M$ , then  $[a] = [b]$  or  $[a] = [\mathbf{0}_L]$ . If  $[b] = M'^K$ , then  $[c] = [b]$  or  $[c] = [\mathbf{1}_L]$ . In all three cases,  $[a] \leq_K [c]$ .

We proved that  $\leq_K$  is a partial ordering. The minimal, resp. maximal, element of  $K$  is  $\mathbf{0}_K = [\mathbf{0}_L]$ , resp.  $\mathbf{1}_K = [\mathbf{1}_L]$ . Atoms of  $K$  are exactly the images of atoms of  $L$ . In particular,  $M$  is an atom of  $K$ .



(OMP3): Suppose that there are  $[a] \in K$ ,  $[b] \in K$  such that  $[b] \leq_K [a]$  and  $[b] \leq_K [a]^{'K}$ . Then  $[b] \leq_K [a] \leq_K [b]^{'K}$ , so  $[b]$  contains orthogonal elements which means that  $[b] = \mathbf{0}_K$ .

Existence of joins: Let  $[a], [b] \in K$ . We avoid the trivial cases assuming that  $[a] \neq \mathbf{0}_K \neq [b]$ ,  $[a] \neq [b]$ , and  $[a], [b]$  have an upper bound different from  $\mathbf{1}_K$ . As  $M \vee_K M^{'K} = \mathbf{1}_K$ , at least one of the classes  $[a], [b]$  is a singleton. Notice that two elements of  $L$  having a join less than  $\mathbf{1}_L$  have distance at most 2. As the mutual distance of elements of  $M$  is greater than 4, there are unique  $a_1 \in [a]$ ,  $b_1 \in [b]$  such that  $a_1 \vee_L b_1 < \mathbf{1}_L$ . We have  $[a_1 \vee_L b_1] = [a] \vee_K [b]$ .

Preservation of blocks: As the quotient mapping preserves the ordering and orthocomplementation, each Boolean subalgebra of  $L$  is mapped onto an isomorphic Boolean subalgebra of  $K$ . We need to prove that also each Boolean subalgebra of  $K$  is an image of an isomorphic Boolean subalgebra of  $L$ . Let  $B$  be a Boolean subalgebra of  $K$ . If  $M \notin B$ , then each element of  $B$  has a unique preimage and  $A = \{a \in L : [a] \in B\}$  is a Boolean subalgebra of  $L$  isomorphic to  $B$ . If  $M \in B$ , we have to distinguish two cases. If  $B = \{\mathbf{0}_K, \mathbf{1}_K, M, M^{'K}\}$ , then we take any  $a \in M$  and  $B$  is the image of the Boolean subalgebra  $A = \{\mathbf{0}_L, \mathbf{1}_L, a, a^{'L}\}$  of  $L$ . In the remaining case, there is a  $[b] \in B \setminus \{\mathbf{0}_K, \mathbf{1}_K, M, M^{'K}\}$ . It has a unique preimage  $b \in L \setminus (M \cup M^{'K} \cup \{\mathbf{0}_L, \mathbf{1}_L\})$ . As  $M$  is an atom in  $K$ , the compatibility of  $M$  and  $[b]$  implies that one of the relations  $M \leq_K [b]$ ,  $M \leq_K [b]^{'K}$  holds. Thus there exists an atom  $a \in M$  such that one of the relations  $a \leq_L b$ ,  $a \leq_L b^{'L}$  is satisfied. The distance of the elements of  $M$  ensures that  $a \in L$  with these properties is unique and independent of the choice of  $[b] \in B \setminus \{\mathbf{0}_K, \mathbf{1}_K, M, M^{'K}\}$ . Thus  $a$  is compatible (in  $L$ ) to all  $b \in L$  such that  $[b] \in B \setminus \{M, M^{'K}\}$ . We see that  $B$  is the image of the Boolean subalgebra  $A = \{a, a^{'L}\} \cup \{b : [b] \in B \setminus \{M, M^{'K}\}\}$  of  $L$ . Looking at maximal Boolean subalgebras, we see that blocks of  $K$  are exactly images of blocks of  $L$ .

(OMP5): Follows easily from preservation of blocks.  $\square$

REMARK 10.8. The technique of Th. 10.7 is applicable also to orthomodular posets and orthoalgebras. In order to obtain an OMP, resp. OA, it is sufficient to assume the mutual distance of atoms in  $M$  at least 4, resp. 3.

REMARK 10.9. Th. 10.7 can be applied subsequently to more sets of atoms  $M_1, M_2, \dots$ , resulting in an OML  $L/M_1/M_2/\dots$ . The only problem is that the distance of the atoms of  $M_2$  in  $L/M_1$  may be smaller than in  $L$ , etc. Therefore it is necessary to choose these sets in such a way that the assumption on minimal distance is not violated during the procedure.

## 11. Evaluation functionals

Until now, we worked with event structures (PFs, OAs, OMPs, OMLs, SFs) and state spaces. For a structure  $L$ , states can be considered as elements of its dual,  $L^* = R^L$ , more exactly,  $[0, 1]^L$ . There is a natural embedding  $\mathbf{e}$  of  $L$  into its second dual,  $L^{**} = R^{L^*}$ , more exactly,  $[0, 1]^{\mathcal{S}(L)}$ , defined by

$$\mathbf{e}(a)(s) = s(a) \text{ for all } a \in L, s \in \mathcal{S}(L).$$

The functional  $\mathbf{e}(a): \mathcal{S}(L) \rightarrow [0, 1]$  is called the *evaluation functional* associated with  $a$ . We also extend the mapping  $\mathbf{e} : L \rightarrow \mathbf{e}(L)$  to subsets of  $L$ ; in particular, we use the notation  $\mathbf{e}(L) = \{\mathbf{e}(a) : a \in L\}$ .

REMARK 11.1. The elements of  $\mathbf{e}(L)$  are continuous affine functionals on  $\mathcal{S}(L)$ . The set  $\mathbf{e}(L)$  is partially ordered by the usual order of real-valued functionals. There is a greatest and a least evaluation functional, namely  $\mathbf{e}(1)$  and  $\mathbf{e}(0)$ . (These are the constant functions 1 and 0 on  $\mathcal{S}(L)$ .) For each evaluation functional  $\mathbf{e}(a)$ , its complementary functional  $1 - \mathbf{e}(a)$  is the evaluation functional associated with  $a'$ . This allows to define an “orthocomplementation” on  $\mathbf{e}(L)$  by  $\mathbf{e}(a)' = \mathbf{e}(a')$ .

The structure of  $\mathbf{e}(L)$  reflects in some sense the structure of  $L$ . They coincide in the following — very important — case:

DEFINITION 11.2. We say that an orthoalgebra  $L$  admits an *order-determining* set of states iff

$$(OD) \quad \forall a, b \in L : (a \leq_L b \iff \forall s \in \mathcal{S}(L) : s(a) \leq s(b)).$$

EXAMPLE 11.3. The orthoalgebras with Greechie diagrams in Fig. 3 admit order-determining sets of states.

EXAMPLE 11.4. In the OML of Ex. 9.6, there are only four evaluation functionals: the constants 0, 1/3, 2/3, 1. It does not admit an order-determining set of states.

EXAMPLE 11.5. In the OML of Ex. 10.3, the constants  $0 = \mathbf{e}(\mathbf{0})$ ,  $1 = \mathbf{e}(\mathbf{1})$  are the only evaluation functionals. It does not admit an order-determining set of states.

Prop. 9.3 states that a pasted family of Boolean algebras and its pasting have the same states. Now we add another analogy:

PROPOSITION 11.6. *A chain-finite pasted family of Boolean algebras and its pasting have the same sets of evaluation functionals.*

PROPOSITION 11.7. *An orthoalgebra  $L$  admits an order-determining set of states iff*

$$\forall a, b \in L : (a \leq_L b \iff \mathbf{e}(a) \leq \mathbf{e}(b)).$$

*If this is the case,  $\mathbf{e}(L)$  with the ordering and orthocomplementation from Rem. 11.1 is an orthoalgebra isomorphic to  $L$  under the isomorphism  $\mathbf{e}$ .*

The proof is elementary.

If we do not assume that  $L$  admits an order-determining set of states, not much is known about the poset of evaluation functionals.

PROBLEM 11.8: What is the structure of  $\mathbf{e}(L)$  (=the collection of evaluation functionals) for an orthomodular structure  $L$ ?

## 12. Semipasted families of Boolean algebras

Pasted families of Boolean algebras are the basic combinatorial tool for constructions of orthomodular structures. Although they simplify the work substantially, they are still very complex in some cases. Here we introduce a new tool — semipasted families of Boolean algebras. They give us much more freedom in constructions of orthomodular structures with given state spaces properties.

DEFINITION 12.1. *A semipasted family of Boolean algebras (SF) is a family  $\mathcal{F}$  of Boolean algebras such that, for each  $A, B \in \mathcal{F}$ ,  $A \cap B$  is an ideal in  $A$  and in  $B$  on which the orderings of  $A$  and  $B$  coincide.*

As an alternative, semipasted families of Boolean algebras may be viewed as *simplicial complexes* [19].

REMARK 12.2. Different Boolean algebras in a SF have the same lower bound,  $\mathbf{0}$ , but — in contrast to PFs — different upper bounds.

We define *atoms* of a SF  $\mathcal{F}$  just as for PFs, and we use the notation  $\mathcal{A}(\mathcal{F}) = \bigcup_{B \in \mathcal{F}} \mathcal{A}(B)$ . The *isomorphisms* of SFs and *states* on SFs are defined just as in PFs — see Def. 4.3, 4.5. Also the definitions of a *chain-finite* SF and of a *distance* in SFs are direct analogies of Def. 4.7 and Def. 4.4.

DEFINITION 12.3. The *Greechie diagram* of a chain-finite semipasted family of Boolean algebras  $\mathcal{F}$  is the hypergraph  $(\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V} = \mathcal{A}(\mathcal{F})$ ,  $\mathcal{E} = \{\mathcal{A}(B) : B \in \mathcal{F}\}$ .

The notions introduced in this section correspond to the same notions for hypergraphs. In particular, a SF is chain-finite iff its Greechie diagram is chain-finite. Two chain-finite SFs are isomorphic iff their Greechie diagrams are isomorphic.

PROPOSITION 12.4. *Let  $\mathcal{F}$  be a chain-finite semipasted family of Boolean algebras and  $\mathcal{H}$  its Greechie diagram. Then the restriction mapping  $h: \mathcal{S}(\mathcal{F}) \rightarrow \mathcal{S}(\mathcal{H})$  defined by  $h(s) = s|_{\mathcal{A}(\mathcal{F})}$  is an affine homeomorphism.*

*Proof.* We proceed analogously to the proof of Prop. 5.8. We do not need (PF3): If  $A, B \in \mathcal{F}$  and  $a \in A \cap B$ , then the intervals  $[\mathbf{0}, a]_A$ ,  $[\mathbf{0}, a]_B$  contain the same atoms. This simplifies the proof substantially.  $\square$

A chain-finite hypergraph may be viewed as a Greechie diagram in two ways:

1. as a Greechie diagram of a pasted family of Boolean algebras and also of the corresponding orthoalgebra,
2. as a Greechie diagram of a semipasted family of Boolean algebras.

In both cases the state space remains the same. This can be easily demonstrated on a hypergraph  $\mathcal{H}$  with two edges:

1. If  $\mathcal{H}$  is considered as the Greechie diagram of a pasted family of Boolean algebras,  $\{A, B\}$ , then  $A \cap B = I \cup I'$ , where  $I$  is an ideal and  $I' = \{a' : a \in I\}$  is its dual filter.
2. If  $\mathcal{H}$  is considered as the Greechie diagram of a semipasted family of Boolean algebras,  $\{A, B\}$ , then  $A \cap B = I$ , where  $I$  is an ideal (see Exs. 12.5, 12.6).

The restrictions for the state space are the same, because the value of a state  $s$  on  $a' \in I'$  is uniquely determined by the value on  $a \in I$ ;  $s(a') = 1 - s(a)$ .

It is possible to form pastings of SFs analogously to PFs. The resulting structure is a poset with a least element, but in general with many maximal elements.

EXAMPLE 12.5. The hypergraph in Fig. 3b can be understood as the Greechie diagram of a semipasted family of Boolean algebras. The Hasse diagram of its pasting (as a SF) is in Fig. 15 (cf. Fig. 7b).

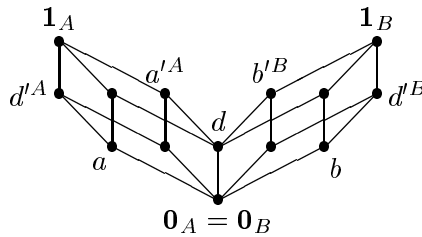


Figure 15: Hasse diagram of a pasting of a semipasted family of Boolean algebras corresponding to the Greechie diagram in Fig. 3b (Ex. 12.5)

What we gain by the use of SFs is more freedom in their construction — some hypergraphs are not Greechie diagrams of PFs, but they are still Greechie diagrams of SFs:

EXAMPLE 12.6. The hypergraph in Fig. 4a is the Greechie diagram of a SF. The Hasse diagram of its pasting is in Fig. 16b.

EXAMPLE 12.7. The hypergraph  $\mathcal{H}$  in Fig. 5a is the Greechie diagram of a semipasted family of Boolean algebras  $\mathcal{F} = \{A, B, C\}$ ,

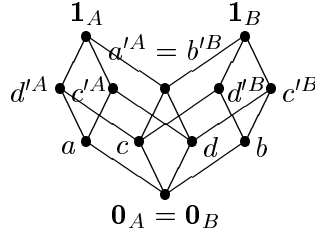


Figure 16: Hasse diagram of a pasting of a semipasted family of Boolean algebras corresponding to the Greechie diagram in Fig. 4a (Ex. 12.6)

where  $A, B, C$  are isomorphic to  $2^2$  and each two of them intersect in  $\mathbf{0}$  and in an atom. The pasting of  $\mathcal{F}$  is drawn in Fig. 17. There is only one state on  $\mathcal{H}$ ; it evaluates all vertices to  $1/2$ . The corresponding state on  $\mathcal{F}$  is faithful.

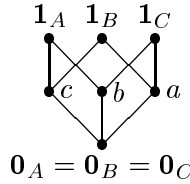


Figure 17: Hasse diagram of the pasting of a semipasted family of Boolean algebras corresponding to the Greechie diagram in Fig. 5a (Ex. 12.7)

The following proposition will play an important role in the sequel.

**PROPOSITION 12.8.** *Every chain-finite hypergraph is a Greechie diagram of some semipasted family of Boolean algebras.*

*Proof.* The chain-finite hypergraph  $(\mathcal{V}, \mathcal{E})$  is the Greechie diagram of the SF  $\{2^E : E \in \mathcal{E}\}$ .  $\square$

Pastings of SFs allow us to define the *relative inverse* (an extension of the orthocomplementation, having the relative upper bound as a second argument). Each pair of elements of a pasting of a SF

has a meet. Every two elements having an upper bound are compatible. Thus the pastings of SFs form a special class of *relative inverse posets* (see [17]), but not much is known about characterizations of this class.

PROBLEM 12.9: Characterize (algebraically) the class of posets which can be obtained as pastings of semipasted families of Boolean algebras.

### 13. Functional embedding and functional isomorphism

Now we shall formulate the crucial notion of this paper — the correspondence between state spaces and sets of evaluation functionals called the functional embedding, resp. functional isomorphism. It can be formulated in a more general context:

DEFINITION 13.1. Let  $F_1$  (resp.  $F_2$ ) be a set of functionals on a subset  $S_1$  (resp.  $S_2$ ) of a topological linear space  $V_1$  (resp.  $V_2$ ). We call a mapping  $g : F_1 \rightarrow F_2$  a *functional embedding* iff it is injective and there is an affine homeomorphism  $h : S_1 \rightarrow S_2$  such that

$$[f_2 = g(f_1), s_2 = h(s_1)] \implies f_2(s_2) = f_1(s_1)$$

for all  $f_1 \in F_1, s_1 \in S_1$ . If, moreover,  $g$  is surjective, it is called a *functional isomorphism* and  $F_1, F_2$  are called *functionally isomorphic*.

Functional embedding is a correspondence of sets of functionals which assumes that their domains are affinely homeomorphic. We shall apply this notion to the sets of all evaluation functionals of different structures — OMLs, OMPs, OAs, PFs and SFs. Whenever  $K, L$  are two of these structures and  $\mathbf{e}(K), \mathbf{e}(L)$  are functionally isomorphic, we say also that  $K, L$  are functionally isomorphic. We sometimes extend a functional embedding  $g : \mathbf{e}(K) \rightarrow \mathbf{e}(L)$  to subsets of  $\mathbf{e}(K)$ .

The importance of the functional isomorphism follows from the fact that it preserves many properties of state spaces, but it allows to represent some complex structures by much simpler ones which

are functionally isomorphic. In the following sections, we shall develop an efficient tool for applications of this idea. The affine homeomorphism between state spaces will be often a simple restriction mapping.

EXAMPLE 13.2. The orthoalgebra  $L$  from Ex. 9.5 is functionally isomorphic to the Boolean algebra  $2^3$ . Its evaluation functionals are  $\mathbf{e}(L) = \{0, \mathbf{e}(b), \mathbf{e}(c), \mathbf{e}(d), \mathbf{e}(b'), \mathbf{e}(c'), \mathbf{e}(d'), 1\}$ .

Following the conditions of Prop. 8.14, it is much more difficult to find an example analogous to Ex. 13.2 among OMPs of even OMLs. The simplest known non-Boolean OML with this property is the following:

EXAMPLE 13.3. [24] Define a hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ , where

$$\mathcal{V} = \{a_i : i = 0, \dots, 65\},$$

$$\mathcal{E} = \{\{a_{2i}, a_{2i+1}, a_{2i+2}\} : i = 0, \dots, 32\}$$

$$\cup \{\{a_{2i-7}, a_{2i}, a_{2i+13}\} : i = 0, \dots, 32\}$$

(indices mod 66). Each state  $s$  on  $\mathcal{H}$  is uniquely determined by the 66 values  $s(a_i) \in [0, 1]$ , and these values have to satisfy 66 equations, one for each edge in  $\mathcal{E}$ . (These equations are not independent.) It was verified by a computer that the solutions  $s : \mathcal{V} \rightarrow [0, 1]$  are exactly those functions which satisfy:

$$s(a_{i+3k}) = s(a_i), \quad i = 0, 1, 2, \quad k = 0, \dots, 21,$$

$$s(a_0) + s(a_1) + s(a_2) = 1.$$

The hypergraph  $\mathcal{H}$  satisfies the conditions of Prop. 8.14, hence it is the Greechie diagram of an OML which is functionally isomorphic to  $2^3$ .

EXAMPLE 13.4. Let  $\mathcal{V}$  be a 4-element set and  $\mathcal{E}$  the collection of all 3-element subsets of  $\mathcal{V}$ . Each state on the hypergraph  $(\mathcal{V}, \mathcal{E})$  (see Fig. 18) attains  $1/3$  at each vertex. Analogously to Ex. 5.13,  $(\mathcal{V}, \mathcal{E})$  is not a Greechie diagram of a PF and of an OA. Nevertheless, it is the Greechie diagram of a SF (Prop. 12.8), say  $\mathcal{F}$ , which is functionally isomorphic to the OA from Ex. 9.6. The Greechie diagram of  $\mathcal{F}$



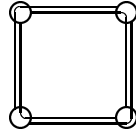


Figure 18: Greechie diagram of a semipasted family of Boolean algebras admitting exactly one (faithful) state (Ex. 13.4)

is simpler than that from Fig. 13. It is desirable to find an OML functionally isomorphic to  $\mathcal{F}$ , too. We shall do it in the sequel using a much more general tool. (In this particular case, an OML with these properties was constructed directly in [28] using Prop. 8.14 and the idea of Ex. 13.3; it has 44 atoms. Alternative solution may be found in [47].)

Functional embeddings of different structures will play a crucial role. Props. 9.3 and 11.6 have the following easy consequence:

PROPOSITION 13.5. *Every chain-finite pasted family of Boolean algebras is functionally isomorphic to its pasting.*

There are functional isomorphisms between structures with non-identical state spaces:

PROPOSITION 13.6. *Every chain-finite pasted family of Boolean algebras is functionally isomorphic to a semipasted family of Boolean algebras.*

*Proof.* It suffices to take the SF with the same Greechie diagram (which exists according to Prop. 12.8) and apply Props. 5.8 and 12.4.  $\square$

Conversely, starting from a SF, we would like to find a functionally isomorphic PF. This makes difficulties, because not all Greechie diagrams of SFs are Greechie diagrams of PFs. We shall solve this problem in the next section (Th. 14.1).

REMARK 13.7. In [34], functional isomorphism  $g: \mathbf{e}(\mathcal{A}(K)) \rightarrow \mathbf{e}(\mathcal{A}(L))$  (only evaluation functionals corresponding to atoms are considered)

was introduced under the notion of *state isomorphism*. It is a stronger condition — for chain-finite orthoalgebras  $K, L$ , the functional isomorphism  $g: \mathbf{e}(\mathcal{A}(K)) \rightarrow \mathbf{e}(\mathcal{A}(L))$  allows an extension to a functional isomorphism  $g^*: \mathbf{e}(K) \rightarrow \mathbf{e}(L)$ , but the reverse correspondence need not exist. The advantage of the approach of [34] is that it preserves more state space properties (e.g., faithfulness of states). It is applicable to a rather general class of chain-finite hypergraphs (as the representing Greechie diagrams), but not to all. A more serious disadvantage is that it does not allow an extension to structures with infinite chains. Functional isomorphism of semipasted families of Boolean algebras overcomes this difficulty.

#### 14. Orthomodular lattices functionally isomorphic to semipasted families of Boolean algebras

As a principal tool, we shall construct OMLs functionally isomorphic to chain-finite SFs. This extremely simplifies the construction of OMLs with those properties of the state space which are preserved by a functional isomorphism. Instead of constructing the Greechie diagram of an OML according to Prop. 8.12 or 8.14, it suffices to find a Greechie diagram of a SF (which is an arbitrary chain-finite hypergraph) and use the following theorem:

**THEOREM 14.1.** *Let  $\mathcal{F}$  be a chain-finite semipasted family of Boolean algebras. Then there is an orthomodular lattice  $L$  which is functionally isomorphic to  $\mathcal{F}$ .*

The proof will be divided into several lemmas.

**LEMMA 14.2.** *Let  $\mathcal{F}$  contain only one Boolean algebra,  $\mathcal{F} = \{2^n\}$ , where  $n \geq 3$ . Then there is a finite orthomodular lattice  $K_n$  and a functional isomorphism  $g_n: \mathbf{e}(\mathcal{F}) \rightarrow \mathbf{e}(K_n)$ . Moreover,  $K_n$  contains a set of atoms  $G_n \subseteq \mathcal{A}(K_n)$  with mutual distance at least 3 and such that  $g_n(\mathbf{e}(\mathcal{A}(\mathcal{F}))) = \mathbf{e}(G_n)$ .*

*Proof.* The proof will be made by induction in  $n$ . For the inductive step, we need an additional condition:

**(IND)** There are atoms  $b_1, \dots, b_n, c_1, \dots, c_n \in \mathcal{A}(K_n)$  with mutual distance at least 2 and such that  $\mathbf{e}(b_i) = \mathbf{e}(c_i)$ ,  $i = 1, \dots, n$ , and  $\mathbf{e}(\{b_1, \dots, b_n\}) = g_n(\mathbf{e}(\mathcal{A}(\mathcal{F})))$ .

For  $n = 3$ , we may take for  $K_3$  the OML from Ex. 13.3 and choose  $G_3 = \{a_0, a_{16}, a_{32}\}$ ,  $(b_1, b_2, b_3) = (a_0, a_4, a_8)$ ,  $(c_1, c_2, c_3) = (a_{12}, a_{16}, a_{20})$ .

Assume now that we have an OML  $K_n$  with  $G_n \subseteq \mathcal{A}(K_n)$  satisfying the conditions of Lemma 14.2 and  $b_1, \dots, b_n, c_1, \dots, c_n$  satisfying (IND). Let  $(\mathcal{V}_n, \mathcal{E}_n)$  be the Greechie diagram of  $K_n$ .

We take the product  $K_n \times 2^1$ . Its Greechie diagram,  $(\mathcal{U}_n, \mathcal{D}_n)$ , is obtained by adding a new vertex,  $d \notin \mathcal{V}_n$ , and putting  $\mathcal{U}_n = \mathcal{V}_n \cup \{d\}$ ,  $\mathcal{D}_n = \{E \cup \{d\} : E \in \mathcal{E}_n\}$ . There is a functional isomorphism  $f_n: \mathbf{e}(2^{n+1}) \rightarrow \mathbf{e}(K_n \times 2^1)$  such that

$$\begin{aligned} f_n(\mathbf{e}(\mathcal{A}(2^{n+1}))) &= \mathbf{e}(G_n) \cup \{\mathbf{e}(d)\} = \mathbf{e}(\{d, b_1, \dots, b_n\}) \\ &= \mathbf{e}(\{d, c_1, \dots, c_n\}). \end{aligned}$$

The mutual distance of  $b_1, \dots, b_n, c_1, \dots, c_n$  in  $(\mathcal{U}_n, \mathcal{D}_n)$  becomes 2.

We choose a number  $p \in \mathbb{N}$  which is a common multiple of  $2, \dots, n$  satisfying  $p \geq 2(n+1)^2 - 1$ . We take  $p+1$  disjoint copies  $(\mathcal{U}_n^k, \mathcal{D}_n^k)$ ,  $k = 0, \dots, p$ , of the hypergraph  $(\mathcal{U}_n, \mathcal{D}_n)$ . The atoms corresponding to  $b_i, c_i, d$  in the  $k$ -th copy are denoted by  $b_i^k, c_i^k, d^k$ ,  $i = 1, \dots, n$ . We identify  $c_i^k$  with  $b_i^{k+i}$  for all  $i = 1, \dots, n$  and  $k = 0, \dots, p$  (upper indices mod  $p+1$ ). We construct a new hypergraph,  $(\mathcal{V}_{n+1}, \mathcal{E}_{n+1})$ , where

$$\begin{aligned} \mathcal{V}_{n+1} &= \bigcup_{k=0}^p \mathcal{U}_n^k, \\ \mathcal{E}_{n+1} &= \bigcup_{k=0}^p \mathcal{D}_n^k. \end{aligned}$$

For each  $j, k = 0, \dots, p$ ,  $j \neq k$ , the intersection  $\mathcal{U}_n^j \cap \mathcal{U}_n^k$  consists of at most one vertex. This ensures that the hypergraph  $(\mathcal{V}_{n+1}, \mathcal{E}_{n+1})$  satisfies the conditions of Prop. 8.12 and it is the Greechie diagram of an OML,  $K_{n+1}$ . We shall prove that it satisfies the conditions of Lemma 14.2.

For each  $i = 1, \dots, n$ , we obtain

$$\mathbf{e}(b_i^1) = \mathbf{e}(c_i^1) = \mathbf{e}(b_i^{1+i}) = \mathbf{e}(c_i^{1+i}) = \mathbf{e}(b_i^{1+2i}) = \dots$$

$$\dots = \mathbf{e}(c_i^{1+p-i}) = \mathbf{e}(b_i^0).$$

Due to the cyclic symmetry of our construction, we have  $\mathbf{e}(b_i^j) = \mathbf{e}(b_i^k)$  for all  $i = 1, \dots, n$  and  $j, k = 0, \dots, p$ . This implies  $\mathbf{e}(d^j) = \mathbf{e}(d^k)$  for all  $j, k = 0, \dots, p$ . Thus  $K_{n+1}$  is functionally isomorphic to a single copy  $K_n \times 2^1$  and also to  $2^{n+1}$ . For the atoms satisfying (IND) in  $K_{n+1}$ , we may choose, e.g.,

$$d^0; \quad b_i^{(n+1)i}, \quad i = 1, \dots, n; \quad d^{(n+1)^2}; \quad b_i^{(n+1)(n+1+i)}, \quad i = 1, \dots, n.$$

As they have mutual distance at least 3, the subset  $G_{n+1}$  required in Lemma 14.2 can be easily chosen from these atoms (the first half suffices). □

LEMMA 14.3. *Suppose that each Boolean algebra in  $\mathcal{F}$  has at least 3 atoms. Then  $\mathcal{F}$  is functionally isomorphic to an orthomodular lattice.*

*Proof.* For each  $A \in \mathcal{F}$ , we apply Lemma 14.2 to  $A$ , obtaining an OML,  $L_A$ . We take a copy  $(\mathcal{V}_A, \mathcal{E}_A)$  of the Greechie diagram of  $L_A$  such that the set of vertices  $G_A \subseteq \mathcal{V}_A$  (corresponding to  $G$  in Lemma 14.2) is identified with the atoms of  $A$ . (This induces also the identification with some vertices of the Greechie diagrams corresponding to other Boolean algebras from  $\mathcal{F}$  having common atoms with  $A$ .) After this identification, we form a new hypergraph  $(\mathcal{V}, \mathcal{E})$ , where

$$\mathcal{V} = \bigcup_{A \in \mathcal{F}} \mathcal{V}_A,$$

$$\mathcal{E} = \bigcup_{A \in \mathcal{F}} \mathcal{E}_A.$$

(Thus we replaced each edge  $\mathcal{A}(A)$  in the Greechie diagram of  $\mathcal{F}$  with  $(\mathcal{V}_A, \mathcal{E}_A)$ .) As the vertices of  $G_A$ ,  $A \in \mathcal{F}$ , have mutual distance at least 3, no nontrivial 4-cycles were created by this construction. Thus  $(\mathcal{V}, \mathcal{E})$  satisfies the condition of Prop. 8.12 and it is the Greechie diagram of a chain-finite OML,  $L$ . As  $L_A$  is functionally isomorphic to  $A$  and  $(\mathcal{V}_A, \mathcal{E}_A)$  restricts the values of states on  $G_A$  in the same way as  $A$ ,  $L$  is functionally isomorphic to  $\mathcal{F}$ . □

LEMMA 14.4. *There is a finite hypergraph  $\mathcal{H}_0$  having at least 3 vertices in each edge, admitting exactly one state and such that this state is two-valued.*

*Proof.* We may take the Greechie diagram of the OML from Ex. 10.3 or the hypergraph  $(\mathcal{V}_0, \mathcal{E}_0)$ , where  $\mathcal{V}_0 = \{a, b, c, d, e\}$ ,  $\mathcal{E}_0 = \{\mathcal{V}_0, \{a, b, c\}, \{a, d, e\}\}$ .  $\square$

*Proof of Th. 14.1.* We take a hypergraph  $(\mathcal{V}_0, \mathcal{E}_0)$  satisfying the conditions of Lemma 14.4. There is a vertex  $a \in \mathcal{V}_0$  evaluated to 1 by the only state on  $(\mathcal{V}_0, \mathcal{E}_0)$ . Let  $(\mathcal{V}_1, \mathcal{E}_1)$  be the Greechie diagram of  $\mathcal{F}$ . (We assume  $\mathcal{V}_0 \cap \mathcal{V}_1 = \emptyset$ .) We form a hypergraph  $(\mathcal{V}_2, \mathcal{E}_2)$ , where  $\mathcal{V}_2 = \mathcal{V}_0 \cup \mathcal{V}_1$ ,  $\mathcal{E}_2 = \mathcal{E}_0 \cup \mathcal{E}_1$ . It is the Greechie diagram of a SF functionally isomorphic to  $\mathcal{F}$ .

We shall modify  $(\mathcal{V}_2, \mathcal{E}_2)$  so that we shall preserve the functional isomorphism with  $\mathcal{F}$  and fulfil the assumptions of Lemma 14.3. To each edge  $E \in \mathcal{E}_2$  with less than 3 atoms, we shall add two more vertices on which all states vanish. Let  $\mathcal{D} = \{E \in \mathcal{E}_2 : \text{card } E < 3\}$ . We take a set  $\mathcal{U}_2 = \{b_E, c_E : E \in \mathcal{D}\}$  disjoint from  $\mathcal{V}_2$  (the elements  $b_E, c_E, E \in \mathcal{D}$ , are supposed to be mutually distinct). We form a hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ , where

$$\mathcal{V} = \mathcal{V}_2 \cup \mathcal{U}_2,$$

$$\mathcal{E} = (\mathcal{E}_2 \setminus \mathcal{D}) \cup \{E \cup \{b_E, c_E\} : E \in \mathcal{D}\} \cup \{\{a, b_E, c_E\} : E \in \mathcal{D}\}.$$

Each state on  $\mathcal{H}$  vanishes on  $\mathcal{U}_2$ . The hypergraph  $\mathcal{H}$  is the Greechie diagram of a SF which is functionally isomorphic to  $\mathcal{F}$  and satisfies the assumption of Lemma 14.3. This finishes the proof.  $\square$

According to Prop. 12.8, every chain-finite hypergraph is the Greechie diagram of a SF. Due to Th. 14.1, this SF is functionally isomorphic to an OML. This means that every chain-finite hypergraph represents the state space of an OML in the sense of functional isomorphism of the corresponding SF. This tool simplifies many tasks.

EXAMPLE 14.5. The hypergraph  $\mathcal{H}$  in Fig. 19a (vertex  $a$  is contained in a singleton edge) is the Greechie diagram of a semipasted family of Boolean algebras  $\mathcal{F} = \{A, B\}$ , where  $A = \{\mathbf{0}_A, a, a'^A, \mathbf{1}_A\}$ ,  $B = \{\mathbf{0}_B, \mathbf{1}_B\}$ ,  $\mathbf{0}_B = \mathbf{0}_A$ ,  $\mathbf{1}_B = a$ . The Hasse diagram of the pasting of  $\mathcal{F}$  is in Fig. 19b. Again,  $\mathcal{H}$  is not a Greechie diagram of a pasted

family of Boolean algebras — it violates (PF1), (PF2). There is only one state,  $s$ , on  $\mathcal{H}$ , because  $s(a) = 1$ . This state (and also the corresponding state on  $\mathcal{F}$ ) is two-valued. Together with Th. 14.1, this example guarantees the existence of an OML with exactly one (two-valued) state (cf. Ex. 10.3).

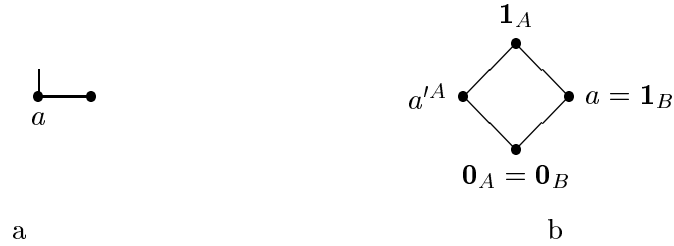


Figure 19: Greechie diagram of a semipasted family of Boolean algebras admitting exactly one (two-valued) state and Hasse diagram of its pasting (Ex. 14.5)

EXAMPLE 14.6. The hypergraph  $\mathcal{H}$  in Fig. 20a is the Greechie diagram of a SF which does not admit any state. The Hasse diagram of its pasting is in Fig. 20b. Together with Th. 14.1, this example guarantees the existence of an OML without any state (cf. Ex. 9.7).



Figure 20: Greechie diagram of a semipasted family of Boolean algebras admitting no state and Hasse diagram of its pasting (Ex. 14.6)

EXAMPLE 14.7. The hypergraph in Fig. 21a is the Greechie diagram of a SF. The Hasse diagram of its pasting is in Fig. 21b.



Figure 21: Greechie diagram of a semipasted family of Boolean algebras and Hasse diagram of its pasting (Ex. 14.7)

EXAMPLE 14.8. The SF from Ex. 12.7 has exactly one state and three (constant) evaluation functionals, 0,  $1/2$  and 1. According to Th. 14.1, there is an OML with these evaluation functionals. No direct construction of an OML with these properties seems to be described in literature. It is not easy to find it without the use of Th. 14.1 or at least some techniques from its proof.

It is important that the proof of Th. 14.1 is constructive — it allows to describe in detail the resulting OML, although it is very large and the construction is not optimal. (We tried to minimize the complexity of the proof rather than the complexity of the resulting OML.) It is somewhat surprising that we obtained the same characterization for all three classes of orthomodular structures in question. Up to functional isomorphism, there is no distinction between OMLs, OMPs and OAs. An explicit formulation follows (a weaker version for OMPs can be derived from [34]):

COROLLARY 14.9. (Pták's Principle) *Every orthoalgebra (in particular, every orthomodular poset) is functionally isomorphic to an orthomodular lattice.*

Exs. 9.5, 13.3 show that the OML functionally isomorphic to an OA may become much more complex. Our technique guarantees its existence. We have even a more efficient tool based on semipasted families of Boolean algebras, because their Greechie diagrams are arbitrary chain-finite hypergraphs. In combination with Props. 13.5 and 13.6, we have a correspondence with other structures studied in

this paper (with an exception of hypergraphs): Every SF, PF, OA, OMP or OML is functionally isomorphic to some SF, PF, OA, OMP and OML.

### 15. Characterization of state spaces — finitely additive case

In this section, we use Prop. 12.8 and Th. 14.1 to give a strengthening (and a simplified proof) of the Shultz's Theorem characterizing state spaces of OMLs (the main result of [43]):

**THEOREM 15.1.** *Let  $C$  be a compact convex subset of a locally convex topological linear space. Let  $f : C \rightarrow [0, 1]$  be a continuous affine functional. Then there is a chain-finite orthomodular lattice  $L$ , an affine homeomorphism  $h : C \rightarrow \mathcal{S}(L)$  and an atom  $a \in L$  such that  $f = \mathbf{e}(a) \circ h$ .*

We divide the proof into several lemmas. We start with additional tools.

**LEMMA 15.2.** *Let  $j, k \in \mathbb{N}$ ,  $j \leq k$ . There is a finite semipasted family of Boolean algebras  $\mathcal{F}_{j,k}$  with atoms  $x, y$  such that*

1. *each state  $s$  on  $\mathcal{F}_{j,k}$  is uniquely determined by its values on  $x, y$ ,*
2. *a state  $s$  on  $\mathcal{F}_{j,k}$  with values  $s(x) = p$ ,  $s(y) = q$  exists iff  $p, q \in [0, 1]$ ,*

$$q \in \left[ \frac{j-1}{k}p, \frac{j}{k}p \right].$$

*Proof.* (See Fig. 22 for the state space of  $\mathcal{F}_{j,k}$ .) We take for  $\mathcal{F}_{j,k}$  the SF with the Greechie diagram in Fig. 23. Each state  $s \in \mathcal{S}(\mathcal{F}_{j,k})$  satisfies

$$s(u_i) = 1 - s(u^\#), \quad i = 1, \dots, n,$$

hence

$$\begin{aligned} \mathbf{e}(u_i) &= \mathbf{e}(u_m) \text{ for all } i, m = 1, \dots, n, \\ s(x) &= k s(u_1), \\ s(y) &\leq j s(u_1), \end{aligned}$$



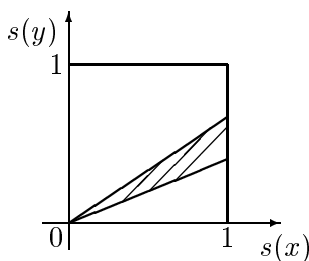


Figure 22: The state space from Lemma 15.2

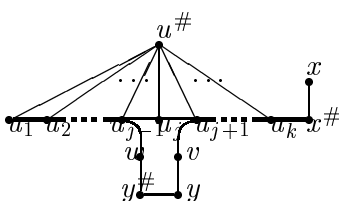


Figure 23: Greechie diagram of the SF  $\mathcal{F}_{j,k}$  from Lemma 15.2

$$s(y) \geq (j - 1) s(u_1).$$

□

LEMMA 15.3. *Let  $r$  be a positive real. There is a chain-finite semi-pasted family of Boolean algebras  $\mathcal{G}_r$  with atoms  $x, y$  such that*

1. *each state  $s$  on  $\mathcal{G}_r$  is uniquely determined by its values on  $x, y$ ,*
2. *a state  $s$  on  $\mathcal{G}_r$  with values  $s(x) = p, s(y) = q$  exists iff  $p, q \in [0, 1], q = rp$ .*

*Proof.* (See Fig. 24 for the state space of  $\mathcal{G}_r$ .) Without any loss of generality, we assume that  $r \leq 1$  (otherwise, we interchange the roles of  $x$  and  $y$ ). We take sequences of nonnegative integers  $(j_n)_{n \in \mathbb{N}}, (k_n)_{n \in \mathbb{N}}$  such that

$$\bigcap_{n \in \mathbb{N}} \left[ \frac{j_n - 1}{k_n}, \frac{j_n}{k_n} \right] = \{r\}.$$

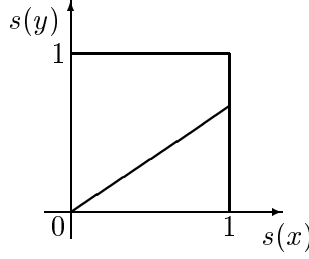


Figure 24: The state space from Lemma 15.3

We apply Lemma 15.2 to find a sequence  $(\mathcal{F}_{j_n, k_n})_{n \in \mathbb{N}}$  of SFs, where we identify the atoms  $x, y$  (with the meaning from Lemma 15.2), i.e.,  $\mathcal{A}(\mathcal{F}_{j_n, k_n}) \cap \mathcal{A}(\mathcal{F}_{j_m, k_m}) = \{x, y\}$  for  $m \neq n$ . The union  $\bigcup_{n \in \mathbb{N}} \mathcal{F}_{j_n, k_n}$  is a SF  $\mathcal{G}_r$  with the required properties.  $\square$

LEMMA 15.4. *Let  $\mathcal{G}$  be a chain-finite semipasted family of Boolean algebras. Let  $C$  be the set of all states  $s$  on  $\mathcal{G}$  satisfying the inequality*

$$(I) \quad \sum_{i \leq n} p_i s(u_i) \leq q,$$

where  $q, p_i \in \mathbb{R}$  and  $u_i \in \mathcal{A}(\mathcal{G})$ ,  $i = 1, \dots, n$ . Then there is a chain-finite semipasted family of Boolean algebras  $\mathcal{F}$  with an atom  $e$  such that

1.  $\mathcal{G} \subseteq \mathcal{F}$ ,
2. each state  $s \in C$  has a unique extension to a state on  $\mathcal{F}$ ,
3. each state  $s$  on  $\mathcal{F}$  satisfies (I),
4. each state  $s$  on  $\mathcal{F}$  satisfies the equality in (I) iff  $s(e) = 0$ .

In particular,  $\mathcal{S}(\mathcal{F})$  is affinely homeomorphic to  $C$ .

*Proof.* We assume, without any loss of generality, that  $p_1, \dots, p_n$  are nonzero. If some  $p_j$  is negative, we find a Boolean algebra  $A \in \mathcal{G}$  containing  $u_j$ . Let  $a_1, \dots, a_m$  be the atoms of  $[0, u_j]_A$ . Then

$$p_j s(u_j) = p_j \left( 1 - \sum_{k \leq m} s(a_k) \right)$$

and we may rewrite (I) into an equivalent form

$$\sum_{i \leq n, i \neq j} p_i s(u_i) + |p_j| \cdot \sum_{k \leq m} s(a_k) \leq q + |p_j|.$$

The latter inequality is again of the general form of (I) (with more atoms:  $x_i, i \in \{1, \dots, n\} \setminus \{j\}$ , and  $a_k, k \in \{1, \dots, m\}$ ), so we may suppose, without any loss of generality, that all coefficients  $p_i$  in (I) are positive.

If  $q < 0$  then  $C = \emptyset$ . In this case it suffices to take the union of  $\mathcal{G}$  with the SF from Ex. 9.7; the choice of  $e$  is irrelevant.

If  $q = 0$  then (I) reduces to the equations  $s(u_i) = 0, i = 1, \dots, n$ . We take  $n$  copies  $\mathcal{F}_i, i = 1, \dots, n$ , of the SF from Ex. 14.5 (or the SF with the Greechie diagram from Lemma 14.4) with atoms  $a_i \in \mathcal{A}(\mathcal{F}_i)$  corresponding to  $a$  from Ex. 14.5. We choose  $\mathcal{F}_i$  such that  $\mathcal{A}(\mathcal{G}) \cap \mathcal{A}(\mathcal{F}_i) = \{u_i\}$  and  $u_i \perp_{\mathcal{F}_i} a_i, i = 1, \dots, n$ . The union  $\mathcal{F} = \mathcal{G} \cup \bigcup_{i \leq n} \mathcal{F}_i$  satisfies the conditions of Lemma 15.4; we may choose any  $u_i$  for  $e$ .

Suppose finally that  $q > 0$ . We take  $n$  copies  $\mathcal{F}_i, i = 1, \dots, n$ , of the SF  $\mathcal{G}_r$  from Lemma 15.3 applied to  $r = p_i/q$ , with atoms  $x_i, y_i$  corresponding to  $x, y$  from Lemma 15.3. We choose  $\mathcal{F}_i$  such that  $x_i = u_i, \mathcal{A}(\mathcal{G}) \cap \mathcal{A}(\mathcal{F}_i) = \{u_i\}$ . We take one new atom,  $e$ , and form a Boolean algebra  $B$  with  $\mathcal{A}(B) = \{e, y_1, \dots, y_n\}$ . We take for  $\mathcal{F}$  the union  $\mathcal{G} \cup \{B\} \cup \bigcup_{i \leq n} \mathcal{F}_i$ . For each state  $s$  on  $\mathcal{F}$ , we obtain

$$\sum_{i \leq n} p_i s(u_i) = q \cdot \sum_{i \leq n} s(y_i) = q - q s(e) \leq q$$

so  $\mathcal{F}$  with  $e \in \mathcal{A}(\mathcal{F})$  have the desired properties. □

*Proof of Th. 15.1.* Without any loss of generality, we may suppose that  $C$  is a subset of  $[0, 1]^V$  for some set  $V$  (see [43] for details). The set  $C$  can be described as the set of all  $s \in [0, 1]^V$  satisfying certain family of inequalities of the form (I).

For each  $v \in V$ , we take a Boolean algebra  $A_v$  isomorphic to  $2^2$  and having  $v$  as one of its atoms. We assume that for each  $v_1, v_2 \in V$  the intersection  $A_{v_1} \cap A_{v_2}$  contains only the zero element. The semipasted family of Boolean algebras  $\{A_v : v \in V\}$  has a state space affinely homeomorphic to  $[0, 1]^V$ . For each inequality of the

type (I), determining the set  $C$ , we apply Lemma 15.4 and obtain a SF (containing  $\{A_v : v \in V\}$ ) which admits only the states satisfying this inequality. The union of all these SFs forms a SF,  $\mathcal{F}_0$ , allowing an affine homeomorphism  $h_0: C \rightarrow \mathcal{S}(\mathcal{F}_0)$ .

In order to get an evaluation functional corresponding to  $f$ , observe that  $f \circ h_0^{-1}$  is a continuous affine functional on  $\mathcal{S}(\mathcal{F}_0)$ , hence it is of the form

$$(f \circ h_0^{-1})(s) = \sum_{i \leq n} p_i s(v_i) - q$$

for some  $q, p_i \in R$ ,  $v_i \in V$  ( $i = 1, \dots, n$ ). We add to  $\mathcal{F}_0$  a Boolean algebra  $B$  isomorphic to  $2^2$  and we take for  $a$  one of its atoms. Then we apply twice Lemma 15.4 to  $\mathcal{F}_0 \cup \{B\}$  and the equality

$$s(a) = \sum_{i \leq n} p_i s(v_i) - q$$

which corresponds to two inequalities of the form (I). We obtain a semipasted family of Boolean algebras  $\mathcal{F}$ . The mapping  $g: \mathcal{S}(\mathcal{F}_0) \rightarrow \mathcal{S}(\mathcal{F})$  which maps a state on  $\mathcal{F}_0$  onto its extension to  $\mathcal{F}$  is one-to-one and it is an affine homeomorphism. Thus  $h = g \circ h_0: C \rightarrow \mathcal{S}(\mathcal{F})$  is an affine homeomorphism. For all  $s \in \mathcal{S}(\mathcal{F})$ , we have

$$(f \circ h^{-1})(s) = s(a) = \mathbf{e}(a)(s),$$

so  $f = \mathbf{e}(a) \circ h$ . Th. 14.1 finishes the proof.  $\square$

## 16. Countably additive states

Until now, we worked with structures closed with respect to finite operations, and with finitely additive states on them. The analogy with the classical measure and probability theory sometimes requires countable additivity of states. In this section we collect the necessary definitions.

**DEFINITION 16.1.** An OMP  $L$  is called  $\sigma$ -orthocomplete (also a  $\sigma$ -OMP) iff each sequence of mutually orthogonal elements of  $L$  has a join.

REMARK 16.2. Def. 16.1 uses joins instead of orthosums, so it is not applicable to orthoalgebras. There is not a unique way how to generalize this notion to an orthoalgebra. Greechie (personal communication) studied 5 different definitions of a  $\sigma$ -orthoalgebra and none of them became commonly accepted. Therefore in the study of  $\sigma$ -orthocomplete orthomodular structures we shall restrict our attention to OMPs (or OMLs).

DEFINITION 16.3. A state  $s$  on a  $\sigma$ -OMP  $L$  is called  $\sigma$ -additive if for each sequence  $(a_n)_{n \in \mathbb{N}}$  of mutually orthogonal elements of  $L$  it satisfies the equality

$$s\left(\bigvee_{n \in \mathbb{N}} a_n\right) = \sum_{n \in \mathbb{N}} s(a_n).$$

The set of all  $\sigma$ -additive states on an OMP  $L$  is denoted by  $\mathcal{S}_\sigma(L)$ . For each  $a \in L$ , we denote by  $\mathbf{e}_\sigma(a): \mathcal{S}_\sigma(L) \rightarrow [0, 1]$  the evaluation functional associated with  $a$  restricted to  $\mathcal{S}_\sigma(L)$ :

$$\mathbf{e}_\sigma(a) = \mathbf{e}(a)|_{\mathcal{S}_\sigma(L)}.$$

We extend  $\mathbf{e}_\sigma$  to subsets of  $L$ ; in particular,  $\mathbf{e}_\sigma(L) = \{\mathbf{e}_\sigma(a) : a \in L\}$ .

## 17. Functional embeddings of concrete logics

In OMSs, the study of properties which are preserved by a functional isomorphism may be simplified by choosing a simpler functionally isomorphic structure. Due to Prop. 11.7, this approach is not applicable when we require an OD set of states. Nevertheless, the notions introduced here can still be successfully applied to formulate some facts, and they bring a new point of view. In this section, we shall apply them to concrete logics (see Def. 6.8). The following proposition gives a characterization of concrete logics. Recall that a state  $s$  on a class  $L$  of subsets of  $X$  is *concentrated* in a point  $x \in X$  iff

$$\forall a \in L : s(a) = \begin{cases} 1 & \text{if } x \in a, \\ 0 & \text{if } x \notin a. \end{cases}$$

PROPOSITION 17.1. [12] *An OMP is a concrete logic iff it admits an order-determining set of two-valued states.*

*Proof.* 1. Let  $L$  be an OMP. Suppose that  $L$  admits an order-determining set of two-valued states and denote this set by  $X \subseteq \mathcal{S}(L)$ . We define a mapping  $i: L \rightarrow 2^X$  by  $i(a) = \{s \in X : s(a) = 1\} = \mathbf{e}(a)^{-1}(1) \cap X$ . A routine verification shows that  $i(L) = \{i(a) : a \in L\}$  is a class of subsets of  $X$  and  $i$  is an isomorphism between  $L$  and  $i(L)$ .

2. In every class of subsets, the set of all states concentrated in points is an order-determining set of states. The isomorphism of the OMPs preserves this property.  $\square$

REMARK 17.2. We formulated Prop. 17.1 for OMPs. It is valid also for OAs, but this does not bring anything new — an OA admitting an order-determining set of two-valued states is an OMP (see [14]).

Surprisingly, there exist nontrivial functional embeddings between Boolean  $\sigma$ -algebras and non-Boolean  $\sigma$ -orthocomplete concrete logics. This construction is based on the following measure-theoretic result due to Solovay:

THEOREM 17.3. [45] *Let  $U$  be the set of the first uncountable cardinality. Each  $\sigma$ -additive state on  $2^U$  is a countable convex combination of states concentrated in points of  $U$ .*

Now we are prepared to present our examples of “almost Boolean” concrete logics which are not only non-Boolean, but they allow to embed a given concrete logic (except for a cardinality limitation). It is a modification of [3, Th. 4] using our new terms.

THEOREM 17.4. *Let  $K$  be a  $\sigma$ -orthocomplete class of subsets of a countable set  $X$ . Then there is a  $\sigma$ -orthocomplete concrete logic  $L$  with the following properties:*

1.  $L$  contains a sub- $\sigma$ -OMP isomorphic to  $K$  (i.e., a subset of  $L$  which, with the operations inherited from  $L$ , forms a  $\sigma$ -OMP isomorphic to  $K$ ).
2.  $L$  is not a Boolean algebra.
3. There are Boolean  $\sigma$ -algebras  $A, B$  and functional embeddings  $f: \mathbf{e}_\sigma(A) \rightarrow \mathbf{e}_\sigma(L)$ ,  $g: \mathbf{e}_\sigma(L) \rightarrow \mathbf{e}_\sigma(B)$ .

*Proof.* We take a set  $U$  of the first uncountable cardinality. We shall construct  $L$  as a class of subsets of the set  $Y = U \times X$ . We define mappings  $T: 2^Y \rightarrow 2^U$ ,  $T^*: 2^Y \rightarrow 2^U$  by

$$u \in T(M) \iff \{x \in X : (u, x) \in M\} \notin K,$$

$$u \in T^*(M) \iff \{x \in X : (u, x) \in M\} \notin \{\emptyset, X\}.$$

We define  $L = \{a \subseteq Y : T(a) \text{ is countable}\}$ . Obviously,  $\emptyset \in L$  and  $L$  is closed under complements in  $Y$ . For a mutually disjoint sequence  $(a_n)_{n \in N}$  in  $L$ , we have

$$T\left(\bigcup_{n \in N} a_n\right) \subseteq \bigcup_{n \in N} T(a_n),$$

hence  $\bigcup_{n \in N} a_n \in L$ . We verified that  $L$  is a  $\sigma$ -orthocomplete class of subsets of  $Y$ .

We define Boolean  $\sigma$ -algebras  $A, B$  of subsets of  $Y$  by  $A = \{a \subseteq Y : T^*(a) \text{ is countable}\}$ ,  $B = 2^Y$ . We have inclusions  $A \subseteq L \subseteq B$ . The existence of the required functional embeddings will be proved by showing that each  $\sigma$ -additive state  $s$  on  $A$  has a unique extension to a  $\sigma$ -additive state on  $B$ . (Obviously, each  $\sigma$ -additive state on  $B$  is an extension of a  $\sigma$ -additive state on  $A$ .)

Let  $s$  be a  $\sigma$ -additive state on  $A$ . As  $\{V \times X : V \subseteq U\}$  is a Boolean sub- $\sigma$ -algebra of  $A$  isomorphic to  $2^U$ , there is a countable set  $V_0 \subseteq U$  such that  $s(V_0 \times X) = 1$  (Th. 17.3). As  $A$  contains all subsets of the countable set  $V_0 \times X$ ,  $s$  is a countable convex combination of states concentrated in points of  $V_0 \times X$ . These concentrated states allow extensions to ( $\sigma$ -additive) concentrated states on  $B$  and the required extension of  $s$  is obtained as their countable convex combination. This is the only extension of  $s$  to  $B$ . The extension mapping is an affine homeomorphism of the state spaces.

We obtain canonical functional embeddings: Each evaluation functional associated with an element of the smaller OMP is mapped to the evaluation functional associated with the same element in the larger OMP.  $\square$

The functional embeddings imply that the OML  $L$  from Th. 17.4 has many properties of Boolean  $\sigma$ -algebras, e.g., the Jauch-Piron property [4, 42], the Radon-Nikodým property [31], and the space of measures on  $L$  is a lattice [3].

## 18. Overview of applications

Once the construction of OMLs functionally isomorphic to any semi-pasted family of Boolean algebras is established, it simplifies many proofs and new investigations in orthomodular structures.

The technique of Th. 14.1 allowed to prove the existence of embeddings of orthomodular structures into orthomodular structures with given state spaces, centers and automorphism groups [14, 27, 30, 33]. The use of a functional isomorphism and Th. 14.1 allowed also to find examples of non-Boolean OMLs which possess the Radon-Nikodým property [13, 31] and which are *fully embeddable* (see [14, 38] for the exact definition and examples). Based on the ideas similar to Th. 15.1 and Lemmas 15.3, 15.4, a characterization of spaces of  $\sigma$ -additive states was found in [36]. Its strengthening to  $\sigma$ -orthocomplete OMPs still remains an open problem. Recently, the problem of existence of  $\sigma$ -additive signed measures not allowing Jordan-Hahn decomposition to  $\sigma$ -additive positive measures was solved by the use of Th. 14.1 and Lemma 15.3 in [44].

There are numerous other problems in which the use of functional isomorphism appeared to be useful. There is a limitation of this technique — due to Prop. 11.7, it does not bring new constructions of orthomodular structures admitting order-determining sets of states. In this field, only the new pasting technique using *regulators* [21] brought a progress. It lead to solutions of such old problems as the existence of a continuum of varieties of OMLs related to states (formulated in [20], solved in [21]) or the uniqueness problem for bounded observables (formulated in [11], solved in [29]).

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