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Willmore Canal Surfaces in Euclidean Space

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Summary. - *We study envelopes of 1-parameter families of spheres (including planes) in Euclidean space which are critical points of the Willmore functional (Willmore canal surfaces). We prove that Willmore canal surfaces are isothermic surfaces and hence conformally equivalent to surfaces of revolution, cones or cylinders. We provide explicit formulae for all solution surfaces. In the generic case the formulae involve Weierstrass's elliptic functions. There are two exceptional cases which can be integrated by using elementary functions only, namely the catenoid and the stereographic projection of the minimal Clifford torus in S^3. To obtain the solution surfaces we explicitly integrate the linear differential system defining the Willmore canal surfaces.*

1. Introduction

An immersed surface $f : S \to \mathbb{E}^3$ is a *Willmore surface* if it is a critical point of the functional $\int (H^2 - K)dA$ on immersed surfaces,

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where H and K are the mean and Gaussian curvatures of the immersion, and dA is the induced area element. This is the case precisely when f satisfies the corresponding Euler-Lagrange equation given by the fourth-order, nonlinear, elliptic PDE

$$\Delta H + 2(H^2 - K)H = 0. \qquad \text{(E-L)}$$

It is well-known that the the Möbius group G of conformal transformations is a symmetry group for (E-L) [2, 4, 13].

In this paper we provide explicit formulae in terms of elliptic functions for all Willmore surfaces with no umbilical points which are obtained as envelopes of a 1-parameter family of spheres, the *Willmore canal surfaces*. Examples include rotational Willmore surfaces.

This work has its origins in the observation of Bryant–Griffiths [5] that the profiles of rotational Willmore surfaces can be considered as solutions of a completely integrable Hamiltonian system. Their observation is based on the remarkable fact, first discovered in [5], that rotational Willmore surfaces can be obtained by revolving free elastic curves in the hyperbolic half plane $\mathbb{H}^2 \subset \mathbb{R}^2$ around its ideal boundary. Free elastic curves in $\mathbb{H}^2$ are critical points of the total square curvature functional $\alpha \mapsto \int \kappa_\alpha^2 dt$ on smooth curves α for compactly supported variations. See also the work of Langer and Singer [9], and Pinkall [12].

The key fact to carry out the integration process is that Willmore canal surfaces are isothermic (Theorem 4.1). This in turn implies that they are Möbius equivalent to surfaces of revolution, cones or cylinders by a classical result of Darboux [7, 14] [1] Accordingly, the result of Bryant–Griffiths on Willmore surfaces of revolution has a natural generalization to Willmore canal surfaces, which can be described in terms of elastic curves in two-dimensional space forms. To our knowledge, this was essentially known to Pinkall (see also Section 4).

The approach to the integration problem presented in the paper is direct and makes use of the method of moving frames as developed by

[1]We have been informed that this result was also indicated by K. Voss in a conference at Oberwolfach [15].

R. Bryant in [4]. Using the conformal invariance and the additional geometric constraint, we are reduced to solving a system of ODE's.

The paper is organized as follows. In Section 2 we recall some basic facts about Möbius geometry ([2, 4]). In Section 2 we develop the method of moving frames for a canal surface in Euclidean space: we construct adapted frames along the surface, introduce a set of differential invariants q_1, q_2, p_1, p_2, p_3 (the invariant functions) and write the structure equations of the surface. In Section 4 we characterize Willmore canal surfaces in terms of the invariant functions and show that the function p_2 must necessarily vanish (Theorem 4.1), that is they are isothermic surfaces ([11]). Willmore canal surfaces are then interpreted as solutions of a completely integrable Pfaffian differential system on $G \times \mathbb{R}^2$ and are divided into four types: *positive, negative, null and special type.* In Section 5 Willmore canal surfaces of positive type are parameterized (up to G-equivalence) by a constant $k \in \mathbb{R}$: when $k > 0$ we find surfaces of revolution, when $k = 0$ we obtain a cylinder with plane directrix curve and generating lines orthogonal to the plane of the curve, when $k < 0$ we get a cone with vertex in the origin and directrix curve on the sphere of radius $\sqrt{2}$ centered at the origin. Explicit formulae for the profiles and the directrix curves are found in terms of elliptic functions (Theorem 5.1). Willmore canal surfaces of negative type are dealt with in Section 6. Also in this case we find surfaces of revolution whose profiles are explicitly described by elliptic functions (Theorem 6.1). In Section 7 Willmore canal surfaces of null type are proved to be equivalent to a catenoid (Theorem 7.1), while Willmore canal surface of special type are equivalent to the stereographic projection of the minimal Clifford torus in S^3 (Theorem 7.2).

The basic reference on elliptic functions has been [10] and our notations are consistent with this reference.

2. Preliminaries

Let $\mathbb{R}^5$ have coordinates $X^0, \ldots, X^4$, and give $\mathbb{R}^5$ the scalar product of signature (4,1)

$$\langle X, Y \rangle = -(X^0Y^4 + X^4Y^0) + X^1Y^1 + X^2Y^2 + X^3Y^3 = g_{IJ}X^IY^J. \tag{1}$$

The *Möbius Group* G is defined to be the identity component of the semi-orthogonal group of (1). It consists of elements $A = (A^I_J) \in GL(5;\mathbb{R})$ such that $\det A = 1$; $\langle AX, AX\rangle = \langle X, X\rangle$; $A^0_J + A^4_J > 0$, $J = 0, 4$.

Let $\mathbb{L}^5$ denote $\mathbb{R}^5$ endowed with the Lorentz metric (1). A *Möbius frame* is a basis $(A_0, \dots, A_4)$ of $\mathbb{L}^5$ such that

$$\langle A_I, A_J\rangle = g_{IJ}, \quad 0 \le I, J \le 4; \quad A^0_J + A^4_J > 0, \quad J = 0, 4. \tag{2}$$

G acts simply transitively on the Möbius frames and, up to the choice of a reference frame, the manifold of all such frames may be identified with G. Let $(e_0, \dots, e_4)$ be the standard basis of $\mathbb{R}^5$, and for any $A \in G$ let $A_J = Ae_J$ denote the J-th column vector of A. Regarding the A_J's as $\mathbb{R}^5$-valued functions on G, there exist unique 1-forms $\{\omega^I_J\}_{0\le I,J\le 4}$, such that

$$dA_I = \omega^J_I A_J, \quad 0 \le I \le 4, \tag{3}$$

where ω^I_J are the components of the Maurer-Cartan form $\omega = A^{-1}dA$ of G. Differentiating (2) and (3), we get the *structure equations for the frame manifold*

$$\omega^K_I g_{KJ} + \omega^K_J g_{KI} = 0, \tag{4}$$

$$d\omega^I_J = -\omega^I_K \wedge \omega^K_J, \quad 0 \le I, J \le 4. \tag{5}$$

The *Möbius space* is the 3-quadric in the 4-dimensional real projective space $\mathbb{RP}^4$ defined by the homogeneous equation

$$-2X^0X^4 + (X^1)^2 + (X^2)^2 + (X^3)^2 = 0. \tag{6}$$

It is diffeomorphic to the 3-sphere $S^3 \subset \mathbb{R}^4$ under the mapping

$$\phi : {}^t(y^1, \dots, y^4) \in S^3 \mapsto \left[{}^t(\frac{1-y^4}{\sqrt{2}}, y^1, y^2, y^3, \frac{1+y^4}{\sqrt{2}})\right] \in \mathbb{RP}^4,$$

where $[X]$ denotes the point represented by the non-zero vector $X \in \mathbb{L}^5$. The south pole ${}^t(0,0,0,-1)$ of S^3 in $\mathbb{R}^4$ is identified with the *origin* $P_0 = [e_0]$ of S^3 in $\mathbb{RP}^4$, and the north pole ${}^t(0,0,0,1)$ is identified with the *point at infinity* $P_\infty = [e_4]$. The Möbius group acts transitively on S^3 by $B \cdot [X] = [BX]$, for all $B \in G, [X] \in S^3$. The isotropy subgroup of G at the origin is

$$G_0 = \left\{ B(r;A;p) = \begin{pmatrix} r^{-1} & {}^t pA & r\frac{{}^t pp}{2} \\ 0 & A & rp \\ 0 & 0 & r \end{pmatrix} : r > 0, A \in SO(3), p \in \mathbb{R}^3 \right\}, \tag{7}$$

and the mapping $\pi_{S^3} : B \in G \mapsto [B_0] \in S^3$ makes G into a G_0-principal fibration over S^3.

In our study we will compare surface theory in Euclidean space $\mathbb{E}^3$ with the conformally invariant surface theory based on the Möbius group. The transition from Euclidean geometry to Möbius geometry is realized by the faithful representation

$$\rho : (A,p) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ p & A & 0 \\ \frac{{}^t pp}{2} & {}^t pA & 1 \end{pmatrix}, \quad p = {}^t(p^1,p^2,p^3) \in \mathbb{E}^3, \quad A \in SO(3), \tag{8}$$

of the group of Euclidean motions $\mathbb{E}(3) = SO(3) \ltimes \mathbb{R}^3$ into G.

The space of oriented 2-spheres (excluding point spheres) and oriented planes in $\mathbb{E}^3$ is naturally identified with the Lorentz 4-sphere

$$\mathcal{Q} = \left\{ X \in \mathbb{L}^5 : -2X^0X^4 + (X^1)^2 + (X^2)^2 + (X^3)^2 = 1 \right\} \tag{9}$$

in Minkowski 5-space $\mathbb{L}^5$; coordinates in $\mathbb{L}^5$ are essentially Darboux's pentaspherical coordinates for such spheres [7]. Let $\sigma(p,r)$ denote the oriented sphere in $\mathbb{E}^3$ with center $p = (p^1,p^2,p^3)$ and signed radius $r \in \mathbb{R}, r \neq 0$, and $\pi(n,p_0)$ denote the oriented plane through p_0 orthogonal to $n = (n^1,n^2,n^3) \in S^2 \subset \mathbb{E}^3$. Then the *space of oriented spheres* Σ_0 is identified with the open and dense subset $\mathcal{Q}_0 = \left\{ X \in \mathcal{Q} : X^0 \neq 0 \right\}$ by

$$\sigma(p,r) \mapsto {}^t\left(\frac{1}{r}, \frac{p^1}{r}, \frac{p^2}{r}, \frac{p^3}{r}, \frac{{}^t pp - r^2}{2r}\right),$$

and the *space of oriented planes* Π with the 3-dimensional hypersurface $\mathcal{Q}_1 = \left\{ X \in \mathcal{Q} : X^0 = 0 \right\}$ by

$$\pi(n,p_0) \mapsto {}^t(0,n^1,n^2,n^3,h).$$

Conversely, the polar hyperplane to any $X \in \mathcal{Q}$ intersects S^3 in a surface $\mathcal{S}(X)$, whose stereographic projection yields an oriented sphere associated with X or, if $P_\infty \in \mathcal{S}(X)$, a plane associated with X. The relative positions of S_1, S_2 in $\Sigma = \Sigma_0 \cup \Pi$ are read from the corresponding vectors $X_1, X_2 \in \mathcal{Q}$ as follows:

1. If $0 \leq |\langle X_1, X_2 \rangle| < 1$, S_1 and S_2 intersect along a circle (possibly a line), and $\langle X_1, X_2 \rangle = 0$ if and only if S_1 and S_2 are orthogonal each other.

2. $|\langle X_1, X_2 \rangle| = 1$ if and only if S_1 and S_2 are tangent. When $\langle X_1, X_2 \rangle = 1$, the orientations agree and S_1, S_2 are said to be in *oriented contact*. When $\langle X_1, X_2 \rangle = -1$, the orientations are opposite.

3. $|\langle X_1, X_2 \rangle| > 1$ if and only if S_1 and S_2 are disjoint.

REMARK 2.1. $B = (B_0, \ldots, B_4) \in G$. *Then, the unit space-like vectors* B_1, B_2, B_3 *represent oriented spheres/planes* S_j, $j = 1, 2, 3$. S_i *and* S_j, $i \neq j$, *do intersect orthogonally. The points* $[B_0] \in S^3$ *and* $[B_4] \in S^3$ *are the intersection points of* S_i.

The Möbius group acts transitively on $\mathcal{Q}$ by $B \cdot X = BX$. If we take e_3 as an origin for $\mathcal{Q}$, the isotropy subgroup of G at e_3 is

$$\hat{G} = \left\{ B \in G : B_3^J = 0, \quad J \neq 3, \quad B_3^3 = 1 \right\},$$

and the mapping $\pi_{\mathcal{Q}} : B \in G \mapsto B_3 \in \mathcal{Q}$ makes $G \to G/\hat{G}$ into a $\hat{G}$-principal fibration. Note that, with the due identifications, $\mathcal{Q}$ is the symmetric space $O(4,1)/O(3,1)$ (de Sitter space-time).

3. Conformal Geometry of Canal Surfaces

Let S be a connected surface and let $f : S \to \mathbb{E}^3 \subset S^3$ be a smooth immersion (not necessarily one-to-one). A local *Möbius frame* along (S, f) is defined to be a smooth map $B : \mathcal{U} \subset S \to G$ defined on an open subset $\mathcal{U}$ of S such that $f(x) = [B_0(x)]$, for each $x \in \mathcal{U}$. That is, a Möbius frame is a local cross section of the induced principal G_0-bundle $f^*(G) \to S$.

For any Möbius frame field $B : \mathcal{U} \subset S \to G$ along f we let $\beta = B^{-1}dB = B^*(\omega)$. Any other Möbius frame field $\hat{B}$ on $\mathcal{U}$ is given by $\hat{B} = BX(r;A;p)$, where $X = X(r;A;p) : \mathcal{U} \to G_0$ is a smooth map, and $\hat{\beta} = X^{-1}\beta X + X^{-1}dX$.

Definition 3.1. *In the classical surface theory, a* sphere congruence *is a smooth mapping $\Gamma : S \to \mathcal{Q}$, where S is a 2-dimensional manifold. Let $f : S \to \mathbb{E}^3$ be an immersed surface of $\mathbb{E}^3$ and let $\Gamma : S \to \mathcal{Q}$ be a sphere congruence on S. (S, f) is said to be an* envelope *of Γ if the sphere/plane represented by $\Gamma(x)$ and the affine tangent plane $\pi(x, f) = f(x) + df(T_xS)$ to the surface are in oriented contact at $f(x)$, for each $x \in S$. A connected, orientable, immersed surface $f : S \to \mathbb{E}^3$ is said to be a* canal surface *if it is enveloped by a* rank 1 *mapping $\Gamma : S \to$*
mathcal1, i.e., if it is enveloped by a 1-parameter family of oriented spheres/planes in $\mathbb{E}^3$. Γ is said Euclidean *if $\Gamma(x)$ represents a proper sphere in $\mathbb{E}^3$, for each $x \in S$.*

Remark 3.2. *The rank 1 assumption is equivalent to the surface being umbilic free (cf. Proposition 3.4).*

From now on, (S, f) will denote a canal surface enveloped by Γ. Let us assume the existence of a global *Darboux orthonormal framing* along $f : S \to \mathbb{E}^3$, i.e., a lift of f to $\mathbb{E}(3)$

$$e = (f; e_1, e_2, e_3) : S \to \mathbb{E}(3),$$

where $(e_1(x), e_2(x))$ is a basis for $df_x(T_xS)$, and $e_3(x)$ is the unit normal vector at $f(x)$, for each $x \in S$. Let (θ^1, θ^2) be the dual coframe relative to (e_1, e_2). We have

$$\begin{aligned} df &= \theta^1 e_1 + \theta^2 e_2, \ de_1 = \theta_1^2 e_2 + \theta_1^3 e_3, \\ de_2 &= -\theta_1^2 e_1 + \theta_2^3 e_3, \ de_3 = -\theta_1^3 e_1 - \theta_2^3 e_2, \end{aligned} \tag{10}$$

where $\theta_1^2 = a\theta^1 + b\theta^2$, $\theta_1^3 = h_{11}\theta^1 + h_{12}\theta^2$, $\theta_2^3 = h_{12}\theta^1 + h_{22}\theta^2$, and $a, b, h_{11}, h_{12}, h_{22}$ are smooth functions. Note that it is not restrictive to work with Darboux frames e which are *adapted* to Γ, i.e., such that e_1 lies in the vertical distribution $\mathcal{V}_\Gamma \subset T(S)$ determined by Γ.

According to the faithful representation ρ of $\mathbb{E}(3) = SO(3) \ltimes \mathbb{R}^3$ into G defined in §1, the adapted Darboux framing e gives rise to a Möbius frame field $A(f; e_1, e_2, e_3) = (A_0, A_1, A_2, A_3, A_4)$ along f, where

- $A_0(x) = {}^t(1, f(x), \frac{1}{2}\|f(x)\|^2)$ is the positive null-vector representing $f(x)$ in S^3,
- $A_i(x), i = 1, 2, 3$, represent the oriented planes through $f(x)$ orthogonal to e_i,
- $A_4(x)$ is the constant vector e_4,

for each $x \in S$. In particular, $A_3(x)$ represents the affine tangent plane at $f(x)$ oriented by the unit normal e_3.

LEMMA 3.3. *Let (S, f) be a canal surface in $\mathbb{E}^3$. Then any adapted Darboux frame e along (S, f) is principal, i.e., h_{12} vanishes identically on S.*

Proof. Since the property is local, we may assume that Γ is Euclidean. Let e be an adapted Darboux frame and let $C(x; e)$ and $r(x; e)$ denote the center and the signed radius of the oriented sphere corresponding to $\Gamma(x; e)$, respectively. We then have

$$d\Gamma \wedge \theta^2 = 0, \quad dC \wedge \theta^2 = 0, \quad dr \wedge \theta^2 = 0. \tag{11}$$

The condition that $\Gamma(x; e)$ is in oriented contact with $A_3(x; e)$ at $f(x)$ yields

$$C = f + re_3. \tag{12}$$

Differentiating (12) and using (11), we find $(df + rde_3) \wedge \theta^2 = 0$. Then, by (10),

$$(1 - rh_{11})e_1 - rh_{12}e_2 = 0, \tag{13}$$

which implies $h_{12} = 0$. □

According to Lemma 3.3, h_{11} and h_{22} are the principal curvatures, $H = \frac{1}{2}(h_{11} + h_{22})$ is the mean curvature and $K = h_{11}h_{22}$ is the Gaussian curvature.

We are now in a position to prove

PROPOSITION 3.4. *Let (S, f) be a canal surface in $\mathbb{E}^3$. Then (S, f) has no umbilics.*

Proof. As above, Γ may be assumed to be Euclidean. Let e be an adapted Darboux frame along (S, f). By (13), $r = (h_{11})^{-1}$ and then, for each $x \in S$,

$$\Gamma(x; e) = h_{11}(x)A_0(x) + A_3(x). \tag{14}$$

Codazzi's equations imply

$$\begin{aligned} dh_{11} &= (h_{11} - h_{22})[(u - b)\theta^1 + a\theta^2], \\ dh_{22} &= (h_{11} - h_{22})[b\theta^1 + (v - a)\theta^2]. \end{aligned} \tag{15}$$

where u, v are functions on S defined by $dH = \frac{1}{2}(h_{11} - h_{22})(u\theta^1 + v\theta^2)$. Moreover,

$$\begin{aligned} dA_0 &= \theta^1 A_1 + \theta^2 A_2, \quad dA_1 = \theta^2_1 A_2 + h_{11}\theta^1 A_3 + \theta^1 A_4 \\ dA_2 &= -\theta^2_1 A_1 + h_{22}\theta^2 A_3 + \theta^2 A_4, \\ dA_3 &= -h_{11}\theta^1 A_1 - h_{22}\theta^2 A_2. \end{aligned} \tag{16}$$

Thus, exterior differentiation of (14) yields

$$d\Gamma = (h_{11} - h_{22})(bA_0 + A_2)\theta^2. \tag{17}$$

Since Γ is rank-one, $h_{11} - h_{22} \neq 0$ on S. □

On S, we shall consider the orientation compatible with the choice $h_{11} - h_{22} > 0$.

Following the construction in [4], since (S, f) is umbilic-free, we may adapt $A = A(f; e_1, e_2, e_3)$ further to a Möbius frame $B = (B_0, \ldots, B_4)$, where

$$\begin{aligned} B_0 &= \frac{1}{2}(h_{11} - h_{22})A_0, \quad B_1 = A_1 - 2uA_0, \\ B_2 &= A_2 + 2vA_0, \quad B_3 = A_3 + HA_0, \\ B_4 &= \frac{2}{h_{11} - h_{22}}\left\{e_4 + \left[\frac{1}{2}H^2 + u^2 + v^2\right]A_0 - uA_1 + vA_2 + HA_3\right\}. \end{aligned} \tag{18}$$

The associated Maurer-Cartan form $\beta = B^{-1}dB$ will be

$$\begin{pmatrix} -2q_2\beta_0^1 + 2q_1\beta_0^2 & p_1\beta_0^1 + p_2\beta_0^2 & -p_2\beta_0^1 + p_3\beta_0^2 & 0 & 0 \\ \beta_0^1 & 0 & -q_1\beta_0^1 - q_2\beta_0^2 & -\beta_0^1 & p_1\beta_0^1 + p_2\beta_0^2 \\ \beta_0^2 & q_1\beta_0^1 + q_2\beta_0^2 & 0 & \beta_0^2 & -p_2\beta_0^1 + p_3\beta_0^2 \\ 0 & \beta_0^1 & -\beta_0^2 & 0 & 0 \\ 0 & \beta_0^1 & \beta_0^2 & 0 & 2q_2\beta_0^1 - 2q_1\beta_0^2 \end{pmatrix}, \tag{19}$$

where

$$\beta_0^1 = \frac{1}{2}(h_{11} - h_{22})\theta^1, \quad \beta_0^2 = \frac{1}{2}(h_{11} - h_{22})\theta^2 \tag{20}$$

and p_1, p_2, p_3, q_1, q_2 are real-valued smooth functions[2]. We call p_1, p_2, p_3, q_1, q_2 the *invariant functions* and (β_0^1, β_0^2) the *normal coframing* of the immersion (with respect to the given normal frame field B). From the structure equations (5) we compute

$$d\beta_0^1 = -q_1\beta_0^1 \wedge \beta_0^2, \quad d\beta_0^2 = -q_2\beta_0^1 \wedge \beta_0^2, \tag{21}$$

$$\begin{aligned} dq_1 \wedge \beta_0^1 + dq_2 \wedge \beta_0^2 &= (1 + p_1 + p_3 + {q_1}^2 + {q_2}^2)\beta_0^1 \wedge \beta_0^2, \\ dq_2 \wedge \beta_0^1 - dq_1 \wedge \beta_0^2 &= -p_2\beta_0^1 \wedge \beta_0^2, \\ dp_1 \wedge \beta_0^1 + dp_2 \wedge \beta_0^2 &= [4q_2p_2 + q_1(3p_1 + p_3)]\beta_0^1 \wedge \beta_0^2, \\ dp_2 \wedge \beta_0^1 - dp_3 \wedge \beta_0^2 &= [4q_1p_2 - q_2(p_1 + 3p_3)]\beta_0^1 \wedge \beta_0^2. \end{aligned} \tag{22}$$

(21, 22) will be referred to as the *structure equations of the immersion.*

REMARK 3.5. *$B_3 : S \to \mathcal{Q}$ is the* central congruence*, also known as the* Gauss conformal map*, while $\hat{f} : x \in S \mapsto [B_4(x)] \in S^3$ is the* conformal transform *of $f : S \to S^3$ ([2],[4]). In terms of the normal frame (18), the sphere congruence $\Gamma : S \to \mathcal{Q}$, $\Gamma = h_{11}A_0 + A_3$ of (S, f) is given by $\Gamma = B_0 + B_3$.*

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[2] A Möbius frame whose Maurer-Cartan form is normalized as in (19) is called a *normal* frame field [4]. Actually, the totality of normal frame fields forms a $\mathbb{Z}_2$-principal bundle $\mathcal{F}(f, S)$ over S; if $B = (B_0, \ldots, B_4)$ is a normal frame, any other normal frame takes the form $(B_0, -B_1, -B_2, B_3, B_4)$. Two surfaces (S, f) and (S', f') are said to be *G-equivalent* if there exists $B \in G$ such that $B \cdot f'(S') = f(S)$. Observe that, up to G-equivalence, any umbilic-free surface admits a globally defined normal frame.

PROPOSITION 3.6. *Let (S, f) be a canal surface. Then the invariant function q_2 vanishes identically.*

Proof. By construction, $d\Gamma \wedge \theta^2 = 0$. By (19) and (20), we then get

$$d\Gamma \wedge \beta_0^2 = (\beta_0^0 \wedge \beta_0^2)B_0 = 0,$$

hence $q_2 = 0$. □

DEFINITION 3.7. *(cf. [3]) Let (S, f) be a canal surface enveloped by Γ. We say that (S, f) is* regular *if $\Gamma(S)$ is the image of a curve $X : S \to \mathcal{Q}$.*

REMARK 3.8. *Let I be an oriented 1-dimensional manifold and $X : I \to \mathcal{Q}$ a smooth immersion. Then X is a space-like curve if and only if the tangent line $T_t(X)$ of X at $t \in I$ is space-like as a line in $\mathbb{L}^5$. In this case, there exists a unique $\dot{X} : I \to \mathbb{L}^5$ such that $\dot{X}(t)$ is a positive vector of $T_t(X)$, $\langle \dot{X}(t), \dot{X}(t) \rangle = 1$, and $\langle X(t), \dot{X}(t) \rangle = 0$. Let $\mathcal{N}_X(t) = \{X(t), \dot{X}(t)\}^\perp \subset \mathbb{L}^5$ and define the circle bundle $\mu_X : \mathcal{M}(X) \to I$ by*

$$\mathcal{M}(X) = \left\{(t, [Y]) \in I \times S^3 : Y \in \mathcal{N}_X(t)\right\}, \quad \mu_X(t, [Y]) = t.$$

$\mathcal{M}(X)$ is canonically immersed in S^3 by

$$(t, [Y]) \in \mathcal{M}(X) \mapsto [Y] \in S^3.$$

It is easy to show the following.

LEMMA 3.9. *Let (S, f) be a regular canal surface and let $X : I \to \mathcal{Q}$ be any smooth 1-dimensional parameterization of $\Gamma(S)$. Then, (I, X) is a space-like curve and $f(S)$ is contained in $\mathcal{M}(X) \subset S^3$.*

4. Willmore Canal Surfaces

Let $S \hookrightarrow \mathbb{E}^3$ be a regular canal surface. According to the preceding discussion, we may assume the existence of a smooth immersion (not necessarily one-to-one) $f : \mathbb{R}^2 \to \mathbb{E}^3$ such that $S = f(\mathbb{R}^2)$ and of a globally defined normal framing $B : \mathbb{R}^2 \to G$ along $(\mathbb{R}^2, f)$ such that $q_2(x, y) = 0$, for all $(x, y) \in \mathbb{R}^2$. If, in addition, S is Willmore, then

$p_1(x,y) - p_3(x,y) = 0$ ([4, 11]). The latter is the Euler-Lagrange equation $\Delta H + 2(H^2 - K)H = 0$ expressed in terms of the invariant functions.

The following is a key result in our discussion.

THEOREM 4.1. *On a Willmore canal surface the invariant function p_2 vanishes identically. This amounts to saying that Willmore canal surfaces are isothermic surfaces.*

Proof. By (21)

$$d\beta_0^1 = -q_1\beta_0^1 \wedge \beta_0^2, \quad d\beta_0^2 = 0. \tag{23}$$

By (22),

$$dq_1 = p_2\beta_0^1 - (1 + 2p_1 + {q_1}^2)\beta_0^2. \tag{24}$$

By exterior differentiation of (24), we get

$$dp_2 \wedge \beta_0^1 - 2dp_1 \wedge \beta_0^2 = q_1 p_2 \beta_0^1 \wedge \beta_0^2. \tag{25}$$

From the 3rd and 4th of the equations (22) and (25) we obtain

$$\begin{aligned} dp_1 &= q_1(p_2\beta_0^1 + p_1\beta_0^2) + X\beta_0^2, \\ dp_2 &= 5q_1(p_1\beta_0^1 - p_2\beta_0^2) + X\beta_0^1, \end{aligned} \tag{26}$$

where $X : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Differentiation of (26) yields

$$dX = -p_2(3p_1 + 6{q_1}^2 + 1)\beta_0^1 - (5{p_2}^2 + 30{q_1}^2 p_1 - 10{p_1}^2 + 11q_1 X)\beta_0^2. \tag{27}$$

Differentiating (27) we obtain

$$p_2(5p_1q_1 + X)\beta_0^1 \wedge \beta_0^2 = 0. \tag{28}$$

If there is a point s_0 on the surface such that $p_2(s_0) \neq 0$, then $5p_1q_1 + X = 0$ on an open neighbourhood $\mathcal{U}$ of s_0. Exterior differentiation of (28) and the use of (24) yield

$$5{p_2}^2\beta_0^1 \wedge \beta_{0|\mathcal{U}}^2 = 0,$$

a contradiction. Hence $p_2 = 0$. The vanishing of p_2 is equivalent to the isothermic property [11, Proposition 1.3, p. 33] □

REMARK 4.2. *The vanishing of p_2 is connected to the conformal deformation problem of surfaces. Actually, the vanishing of p_2 is a necessary and sufficient condition to have conformal deformation [11]. The notion of conformal deformation of surfaces is related to the general deformation theory of submanifolds in homogeneous spaces [8].*

It is a classical result of Darboux [7] that isothermic canal surfaces are Möbius equivalent to surfaces of revolution, cones, or cylinders. We then have:

COROLLARY 4.3. *Willmore canal surfaces are Möbius equivalent to surfaces of revolution, cones, or cylinders.*

REMARK 4.4. *According as S be a cone, a cylinder, or a surface of revolution, let $\alpha_S \subset S^2$, $\alpha_S \subset \mathbb{R}^2$, $\alpha_S \subset \mathbb{H}^2$ denote the directrix curves of the cone, the cylinder, or the profile of the surface of revolution, respectively. Then the Euler–Lagrange equation (E-L) implies that α_S is a free elastic curve in the corresponding 2-dimensional space form.*

DEFINITION 4.5. *A local coordinate system (u, v) is said to be* adapted *to the Willmore canal surface if*

$$\beta_0^1 = g(v)du, \quad \beta_0^2 = dv, \tag{29}$$

where $g : \mathbb{R} \to \mathbb{R}^+$ is a positive smooth function.

LEMMA 4.6. *Adapted coordinate systems exist near any point of S.*

Proof. Since β_0^2 is a closed form, we may find for any $s_0 \in \mathbb{R}^2$ a local coordinate system $(z, v) = \Phi : \mathcal{U} \to \mathbb{R}^2$ defined in an open neighbourhood $\mathcal{U}$ of s_0 such that

$$\beta_0^1 = T dz, \quad \beta_0^2 = dv,$$

where $T : \Phi(\mathcal{U}) \to \mathbb{R}$ is a positive smooth function. From $d\beta_0^1 = -q_1\beta_0^1 \wedge dv$ we get $q_1 = \frac{\partial}{\partial v}(\log T)$. Since p_2 vanishes identically we then have $dq_1 \wedge dv = 0$. This implies $\frac{\partial^2}{\partial v \partial z}(\log T) = 0$ and hence

$$T(z, v) = e^{G(v)} e^{H(z)}.$$

The new coordinate system (u, v) defined by $du = e^{H(z)}dz$, v is an adapted one. $\square$

REMARK 4.7. *If $\Phi = (u, v) : \mathcal{U} \to \mathbb{R}^2$ and $\Phi' = (u', v') : \mathcal{U} \to \mathbb{R}^2$ are adapted local coordinates, then*

$$g' = \frac{1}{r} g, \quad u' = ru + a, \quad v' = v + b, \tag{30}$$

for r a positive constant, and a, b arbitrary constants. Therefore, using the simply connectedness of $\mathbb{R}^2$ and (30), it follows that there exist globally defined adapted coordinates $\Phi = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$.

From the structure equations we get

$$q_1 dv = d(\log g), \tag{31}$$

$$dq_1 = -(1 + 2p_1 + {q_1}^2) dv, \tag{32}$$

$$dp_1 = -4 p_1 q_1 dv. \tag{33}$$

This implies

$$p_1 = h g^{-4}, \tag{34}$$

$$\frac{d^2 g}{dv^2} + g + \frac{2h}{g^3} = 0, \tag{35}$$

where h is a constant. Equation (35) yields

$$\left(\frac{dg}{dv}\right)^2 = -g^2 + \frac{2h}{g^2} + k, \tag{36}$$

where k is a constant of integration. If g is not constant, (36) yields

$$g^2 = \frac{1}{2}\left(\sqrt{k^2 + 8h}\cos(2v + \gamma) + k\right),$$

where γ is a constant. Choosing on $\mathbb{R}^2$ coordinates $(u, v + \frac{1}{2}\gamma)$, we may assume that

$$g^2 = \frac{1}{2}\left(\sqrt{k^2 + 8h}\cos(2v) + k\right). \tag{37}$$

The constant k only depends on the surface and will be called the *reduced modulus* of the surface. We may give a classification of Willmore canal surfaces into four types:

$$\begin{cases} \textit{positivetype}: & g \text{ non constant and } h > 0 \\ \textit{negativetype}: & g \text{ non constant and } h < 0 \\ \textit{nulltype}: & g \text{ non constant and } h = 0 \\ \textit{specialtype}: & g \text{ constant} \end{cases}$$

With respect to adapted coordinates (u, v), the normal framing B of a Willmore canal surface satisfies the following ODE system[3]

$$
\begin{aligned}
dB_0 &= gB_1du + \left(\frac{2}{g}\frac{dg}{dv}B_0 + B_2\right)dv,\\
dB_1 &= \left(\frac{h}{g^3}B_0 + \frac{dg}{dv}B_2 + gB_3 + gB_4\right)du,\\
dB_2 &= -\frac{dg}{dv}B_1du + \left(\frac{h}{g^4}B_0 - B_3 + B_4\right)dv,\\
dB_3 &= -gB_1du + B_2dv, \quad dB_4 = \frac{h}{g^3}B_1du + \left(\frac{h}{g^4}B_2 - \frac{2}{g}\frac{dg}{dv}B_4\right)dv.
\end{aligned}
\tag{38}
$$

5. Willmore Canal Surfaces of Positive Type

Preparatory Material. Let $S = f(\mathbb{R}^2)$ be a Willmore canal surface of positive type. Replacing, if necessary, (u, v) with $(h^{-\frac{1}{4}}u, v)$, we may assume that $h = 1$ so that

$$g^2(v) = \frac{1}{2}\left(\sqrt{k^2+8}\cos(2v) + k\right), \quad k \in \mathbb{R}. \tag{39}$$

We put

$$e_1 = \frac{1}{2}\left(\frac{1}{2}\sqrt{k^2+8} - \frac{k}{6}\right) > e_2 = \frac{k}{6} > e_3 = -\frac{1}{2}\left(\frac{1}{2}\sqrt{k^2+8} + \frac{k}{6}\right) \tag{40}$$

and consider the Weierstras's elliptic function $\wp(z)$ satisfying

$$\left(\frac{d\wp}{dz}\right)^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3). \tag{41}$$

---

[3]Let $M = G \times \mathbb{R}^2$ and let q_1, p_1 be coordinates of $\mathbb{R}^2$. On M we introduce exterior differential 1-forms η^a, $a = 1, \ldots, 10$ by

$$
\begin{aligned}
&\eta^1 = \omega_0^3, && \eta^2 = \omega_0^1 - \omega_1^3, && \eta^3 = \omega_0^2 + \omega_2^3, && \eta^4 = \omega_3^0,\\
&\eta^5 = \omega_1^2 - q_1\omega_0^1, && \eta^6 = \omega_0^0 - 2q_1\omega_0^2, && \eta^7 = \omega_1^0 - p_1\omega_0^1, &&\\
&\eta^8 = \omega_2^0 - p_1\omega_0^2, && \eta^9 = dp_1 + 4p_1q_1\omega_0^2, && \eta^{10} = dq_1 + (1 + 2p_1 + q_1{}^2)\omega_0^2. &&
\end{aligned}
$$

Notice that the exterior differential system $\eta^1 = \eta^2 = \cdots = \eta^{10} = 0$, $\omega_0^1 \wedge \omega_0^2 \neq 0$ is completely integrable and its solutions are of the form $(B, q_1, p_1) : S \to M$, B normal framing of S.

Observe that

$$\frac{1}{g(v)^2} + \frac{k}{6} \geq e_1 \tag{42}$$

and that equality occurs if and only if $\dot{g}(y) = 0$. Moreover, $\frac{1}{g(v)^2} + \frac{k}{6}$ increases (decreases) monotonically when $\dot{g}(v) < 0$ ($\dot{g}(v) > 0$). Let ω_1 denote the real half period of $\wp(z)$. As $t \in \mathbb{R}$ increases from 0 to ω_1, $\wp(t)$ decreases monotonically from $+\infty$ to e_1, and as t increases from ω_1 to $2\omega_1$, $\wp(t)$ increases monotonically from e_1 to $+\infty$.

Accordingly, we define $s(v) : \mathbb{R} \to (0, 2\omega_1)$ by

$$s(v) = \begin{cases} \left(\wp_{|(0,\omega_1)}\right)^{-1}\left(\frac{1}{g(v)^2} + \frac{k}{6}\right) & \dot{g}(v) > 0 \\ \omega_1 & \dot{g}(v) = 0 \\ \left(\wp_{|(\omega_1,2\omega_1)}\right)^{-1}\left(\frac{1}{g(v)^2} + \frac{k}{6}\right) & \dot{g}(v) < 0. \end{cases} \tag{43}$$

s is a continuous function which is smooth on $\{v \in \mathbb{R} : \dot{g}(v) \neq 0\}$. Moreover, a direct calculation shows that

$$\frac{ds}{dv} = \frac{\sqrt{2}}{g}.$$

Thus, s is a smooth function. Let now $\sigma(z)$ and $\zeta(z)$ be the sigma and zeta functions associated to the elliptic function $\wp(z)$. Fix z_0 such that $\wp(z_0) = -\frac{k}{3}$ (note that we may choose $z_0 = t_0\omega_1 + \omega_3$, where $t_0 \in (0, 1]$ and ω_3 is the imaginary half period) and let $L : \mathbb{R} \to \mathbb{R}$ be given by

$$\begin{cases} L(t) = -\dfrac{\sqrt{|k|}}{2\dot{\wp}(z_0)}\left(\log\dfrac{\sigma(t - z_0)}{\sigma(t + z_0)} + 2t\zeta(z_0)\right) & k \neq 0 \\ L(t) = -\dfrac{1}{2\dot{\wp}(z_0)}\left(\log\dfrac{\sigma(t - z_0)}{\sigma(t + z_0)} + 2t\zeta(z_0)\right) & k = 0. \end{cases} \tag{44}$$

The function L arises from a Weierstrass's integral of the third kind and satisfies

$$\begin{cases} \dfrac{dL(t)}{dt} = -\dfrac{3\sqrt{|k|}}{2}\dfrac{1}{3\wp(t) + k} & k \neq 0 \\ \dfrac{dL(t)}{dt} = -\dfrac{1}{2\wp(t)} & k = 0. \end{cases}$$

We then define $\lambda(v) : \mathbb{R} \to \mathbb{R}$ by

$$\lambda(v) = L(s(v)), \tag{45}$$

and compute

$$\begin{cases} \dfrac{d\lambda}{dv} = -\dfrac{\sqrt{2|k|}g}{kg^2+2} & k \neq 0 \\ \dfrac{d\lambda}{dv} = -\dfrac{g}{\sqrt{2}} & k = 0. \end{cases} \tag{46}$$

Three cases may occur: $k > 0$, $k = 0$, $k < 0$.

THEOREM 5.1. *Let $S = f(\mathbb{R}^2) \subset \mathbb{E}^3$ be a Willmore canal surface of positive type and reduced modulus k, and let s and L be the functions defined by (43) and (44) respectively.*
Case $k > 0$. *S is equivalent to the surface of revolution $f(u,v) = (x^1, x^2, x^3) \in \mathbb{E}^3$ given by*

$$\frac{e^{L[s(v)]}}{\sqrt{6\wp[s(v)]+2k}} \left(\sqrt{6k}, \sqrt{2(6\wp[s(v)]-k)}\sin\sqrt{k}u, \right.$$
$$\left. -\sqrt{2(6\wp[s(v)]-k)}\cos\sqrt{k}u\right).$$

Case $k = 0$. *S is equivalent to the cylinder $f(u,v) = ue_1 + \beta[s(v)]$ with directrix curve $\beta = (0, x^2, x^3) : (0, 2\omega_1) \to \mathbb{E}^3$ given by*

$$\beta(t) = \left(0, \frac{1}{\sqrt{2\wp(t)}}, -L(t)\right),$$

and generating lines orthogonal to the plane $x^1 = 0$. Case $k < 0$. *S is equivalent to the cone $f(u,v) = u\gamma[s(v)]$ with vertex in the origin and directrix curve $\gamma = (x^1, x^2, x^3) : (0, 2\omega_1) \to S^2(\sqrt{2}) \subset \mathbb{E}^3$ defined by*

$$\gamma(t) = \frac{1}{\sqrt{6\wp(t)-k}} \left(\sqrt{-6k}, -\sqrt{(12\wp(t)+4k)}\cos L(t), \right.$$
$$\left. -\sqrt{(12\wp(t)+4k)}\sin L(t)\right).$$

Proof of Theorem 5.1. **Case** $k > 0$. Let

$$\eta_1 = \frac{1}{\sqrt{2}}\left(\frac{B_0}{g^2} - g^2 B_4\right), \qquad \eta_2 = B_1, \\ \eta_3 = \frac{1}{\sqrt{k}}\left(\frac{B_0}{g^3} + \frac{dg}{dv}B_2 + gB_3 + gB_4\right). \tag{47}$$

From (34, 35), (36) it follows that $\langle \eta_a, \eta_b \rangle = \delta_{ab}$, $a, b = 1, 2, 3$, and by differentiating (47) and using (38), we find

$$d\eta_1 = 0, \quad d\eta_2 = \sqrt{k}\eta_3 du, \quad d\eta_3 = -\sqrt{k}\eta_2 du. \tag{48}$$

This shows that there exist constant vectors $E_a \in \mathbb{L}^5$, $a = 1, 2, 3$, such that

$$\langle E_a, E_b \rangle = \delta_{ab}, \eta_1 = E_1 \\ eta_2 = \cos\sqrt{k}uE_2 + \sin\sqrt{k}uE_3, \eta_3 = -\sin\sqrt{k}uE_2 + \cos\sqrt{k}uE_3. \tag{49}$$

We now define

$$\zeta_1 = \frac{1}{\sqrt{k(2g^{-2}+k)}}\left\{\frac{1}{\sqrt{2}g^2}\left(\frac{2}{g^2}+k\right)B_0 + \sqrt{2}\frac{1}{g}\frac{dg}{dv}B_2 + \right. \\ \left. +\sqrt{2}B_3 + \frac{g^2}{\sqrt{2}}\left(\frac{2}{g^2}+k\right)B_4\right\}, \tag{50}$$

$$\zeta_2 = \frac{g^2}{\sqrt{2+kg^2}}\left\{B_2 - \frac{1}{g}\frac{dg}{dv}B_3\right\}. \tag{51}$$

Differentiating (50) and (51), we get, using (34, 35), (36) and (38),

$$d\zeta_1 = \frac{\sqrt{2k}g}{2+kg^2}\zeta_2 dv, \quad d\zeta_2 = \frac{\sqrt{2k}g}{2+kg^2}\zeta_1 dv.$$

From (46) it now follows that

$$\zeta_1 = \frac{1}{\sqrt{2}}\left(e^{-\lambda}E_0 + e^{\lambda}E_4\right), \quad \zeta_2 = \frac{1}{\sqrt{2}}\left(e^{-\lambda}E_0 - e^{\lambda}E_4\right), \tag{52}$$

where E_0, E_4 are constant positive null vector in $\mathbb{L}^5$ such that $E = (E_0, \ldots, E_4) \in G$. Replacing, if necessary, f by $E^{-1}f$, we may assume that E is the standard basis. By solving (47), (49), (50), (51) and (52) for B_0, we then calculate

$$f(u,v) = \frac{e^{\lambda(v)}}{\sqrt{2g(v)^{-2}+k}} \left(\sqrt{2k}, \frac{2\sin\sqrt{k}u}{g(v)}, -\frac{2\cos\sqrt{k}u}{g(v)} \right), \tag{53}$$

which furnishes the required expression for f according to (43) and (45).

Case $k = 0$. We set

$$\begin{aligned} \eta_1 &= \frac{1}{\sqrt{2}} \left(\frac{B_0}{g^2} - g^2 B_4 \right), \qquad \eta_2 = B_1, \\ \eta_3 &= \frac{B_0}{g^3} + \frac{dg}{dv} B_2 + g B_3 + g B_4. \end{aligned} \tag{54}$$

By (34, 35), (36) and (38), $d\eta_1 = d\eta_3 = 0$, $d\eta_2 = \eta_3 du$, from which we get

$$\eta_1 = E_2, \quad \eta_3 = E_4, \quad \eta_2 = E_1 + uE_4, \tag{55}$$

where $E_a \in \mathbb{L}^5$ and $\|E_1\|^2 = \|E_2\|^2 = 1$, $\|E_4\|^2 = \langle E_a, E_b \rangle = 0$, $a \neq b$. We now set

$$\zeta_2 = \frac{g^2}{\sqrt{2}} \left\{ B_2 - \frac{1}{g}\frac{dg}{dv} B_3 \right\}. \tag{56}$$

Differentiating (56) and using (34, 35), (36) and (38), we have $d\zeta_2 = \frac{g}{\sqrt{2}} E_4 dv$ and then by (46),

$$\zeta_2 = -\lambda E_4 + E_3, \tag{57}$$

where $E_3 \in \mathbb{L}^5$, $\|E_3\|^2 = 1$ and $\langle E_3, E_a \rangle = 0$, $a \neq 3$. We next put

$$\begin{aligned} \zeta_1 = \frac{1}{2} \Bigg\{ & \frac{1}{2} \left(\frac{1}{g} + \frac{\sqrt{2}}{g^3}\frac{dg}{dv}\lambda \right) B_0 + \frac{\sqrt{2}}{g^2}\lambda B_2 + \\ & + \frac{g^4}{2} \left(\frac{1}{g} + \frac{\sqrt{2}}{g^3}\frac{dg}{dv}\lambda \right) B_4 + \left(\frac{\lambda^2}{g^4} - \frac{g^4}{4} \left(\frac{1}{g} \frac{\sqrt{2}\lambda}{g^3}\frac{dg}{dv} \right)^2 \right) \Bigg\}, \end{aligned} \tag{58}$$

Differentiation of (58) yields $d\zeta_1 = (E_1 + uE_4)du$ and then

$$\zeta_1 = E_0 + uE_1 + \frac{1}{2}u^2 E_4, \tag{59}$$

where $E_0 \in \mathbb{L}^5$, $\|E_0\|^2 = \langle E_0, E_a\rangle = 0$ and $\langle E_0, E_a\rangle = -1$. Thus, $E = (E_0, \dots, E_4) \in G$ and, possibly replacing f by $E^{-1}f$, we may assume that E is the coordinate basis. By (54)–(59), we find

$$f(u,v) = \left(u, \frac{g(v)}{\sqrt{2}}, -\lambda(v)\right) = \left(u, \frac{1}{\sqrt{2\wp[s(v)]}}, -L[s(v)]\right).$$

Case $k < 0$. We put

$$\begin{aligned} \eta_1 &= \frac{1}{\sqrt{2}}\left(\frac{B_0}{g^2} - g^2 B_4\right), \qquad \eta_2 = B_1, \\ \eta_3 &= \frac{1}{\sqrt{-k}}\left(\frac{B_0}{g^3} + \frac{dg}{dv}B_2 + gB_3 + gB_4\right). \end{aligned} \tag{60}$$

From (34, 35), (36) we compute $\langle \eta_a, \eta_b\rangle = \delta_{ab},\ a,b = 1,2,3$, and by differentiating (60), $d\eta_1 = 0$, $d\eta_2 = \sqrt{-k}\eta_3 du$, $d\eta_3 = \sqrt{-k}\eta_2 du$. It follows that there exist constant vectors E_0, E_1 and $E_4 \in \mathbb{L}^5$ such that

$$\begin{aligned} &\|E_1\|^2 = 1 = -\langle E_0, E_4\rangle, \qquad \|E_0\|^2 = \|E_4\|^2 = 0, \\ &\eta_1 = E_1, \qquad \eta_2 = -\frac{1}{\sqrt{2}}\left(e^{-\sqrt{-k}u}E_0 - e^{\sqrt{-k}u}E_4\right), \\ &\eta_3 = \frac{1}{\sqrt{2}}\left(e^{-\sqrt{-k}u}E_0 + e^{\sqrt{-k}u}E_4\right). \end{aligned} \tag{61}$$

We now define

$$\begin{aligned} \zeta_1 = \frac{1}{\sqrt{-k(2g^{-2}+k)}}\Big\{ &\frac{1}{\sqrt{2}g^2}\left(\frac{2}{g^2} + k\right)B_0 + \sqrt{2}\frac{1}{g}\frac{dg}{dv}B_2 + \\ &+ \sqrt{2}B_3 + \frac{g^2}{\sqrt{2}}\left(\frac{2}{g^2} + k\right)B_4\Big\}, \end{aligned} \tag{62}$$

$$\zeta_2 = \frac{g^2}{\sqrt{2 + kg^2}}\left\{B_2 - \frac{1}{g}\frac{dg}{dv}B_3\right\}. \tag{63}$$

Using (34, 35), (36) and (38), we get $\langle \eta_a, \zeta_i \rangle = 0$, $\langle \zeta_i, \zeta_j \rangle = \delta_{ij}$, and

$$d\zeta_1 = -\frac{\sqrt{-2kg}}{2+kg^2}\zeta_2 dv, \quad d\zeta_2 = \frac{\sqrt{-2kg}}{2+kg^2}\zeta_1 dv. \tag{64}$$

Thus, from (46) we deduce

$$\zeta_1 = \cos\lambda E_2 + \sin\lambda E_3, \quad \zeta_2 = -\sin\lambda E_2 + \cos\lambda E_3, \tag{65}$$

where $E = (E_0, \ldots, E_4) \in G$. Replacing, if necessary, f by $E^{-1}f$, we may assume that E is the standard basis. By (60)–(64) and (65) we find that

$$f(u,v) = e^{\sqrt{-k}u}\left(\sqrt{-k}g, -g\sqrt{2g^{-2}+k}\cos\lambda(v),\right.$$
$$\left. -g\sqrt{2g^{-2}+k}\sin\lambda(v)\right).$$

Introducing new adapted coordinates $(e^{\sqrt{-k}u}, v)$ and using (43) and (45), the proof is completed. □

6. Willmore Canal Surfaces of Negative Type

Preparatory Material. Let (u,v) be an adapted coordinate system such that $h = -1$. In this case, the reduced modulus can be any real number $k > \sqrt{8}$. Let

$$e_1 = \frac{k}{6} > e_2 = -\frac{1}{2}\left(\frac{k}{6} - \frac{1}{2}\sqrt{k^2-8}\right) > e_3 = -\frac{1}{2}\left(\frac{k}{6} + \frac{1}{2}\sqrt{k^2-8}\right)$$

and let $\wp(z)$ denote the Weierstrass's elliptic function with roots $e_\alpha = \wp(\omega_\alpha)$, $\alpha = 1,2,3$. Note that $e_1 - e_2 < \frac{1}{g(v)^2} < e_1 - e_3$ and that the function

$$T(t) = (e_1 - e_2)\frac{\wp(t) - e_3}{\wp(t) - e_2} \tag{66}$$

increases from $e_1 - e_2$ to $e_1 - e_3$ as t increases from 0 to ω_1, while decreases from $e_1 - e_3$ to $e_1 - e_2$ as t increases from ω_1 to $2\omega_1$. We now define $s(v) : \mathbb{R} \to (0, 2\omega_1)$ to be the function

$$s(v) = \begin{cases} \left(T_{|(0,\omega_1)}\right)^{-1}\left(\dfrac{1}{g(v)^2}\right) & 2v \in [0,\pi] \\ \left(T_{|(\omega_1,2\omega_1)}\right)^{-1}\left(\dfrac{1}{g(v)^2}\right) & 2v \in [\pi, 2\pi]. \end{cases} \tag{67}$$

We have that

$$\frac{ds}{dv} = \frac{\sqrt{2}}{g}.$$

Let consider the function

$$L(t) = -\frac{\sqrt{k}(\beta e_2 + \gamma)}{2\beta^2 \dot{\wp}(z_0)} \left\{ \log \frac{\sigma(t - z_0)}{\sigma(t + z_0)} + t\left(2\zeta(z_0) - \frac{\beta\dot{\wp}(z_0)}{\beta e_2 + \gamma}\right)\right\}, \tag{68}$$

where $\beta = -2e_1 + e_2$, $\gamma = 3e_1e_2 - e_1e_3 - e_2e_3$, $\wp(z_0) = -\frac{\gamma}{\beta}$. $L(t)$ is the elliptic integral of third kind $\int \frac{\wp(t) - e_2}{\beta\wp(t) - \gamma} dt$. We then set $\lambda(v) = L(s(v))$ and compute

$$\frac{d\lambda}{dv} = -\frac{\sqrt{2k}g}{kg^2 - 2}. \tag{69}$$

We are now in a position to prove

THEOREM 6.1. *A Willmore canal surface $S \subset \mathbb{E}^3$ of negative type is equivalent to the surface of revolution obtained by rotating the curve $\alpha : (0, 2\omega_1) \to \mathbb{E}^3$ given by*

$$\alpha[s(v)] = \frac{1}{\sqrt{k - 2T[s(v)]}\cos L[s(v)] + \sqrt{k}} \cdot \\ \cdot \left(\sqrt{2k - 4T[s(v)]}\sin L[s(v)], 2\sqrt{T[s(v)]}, 0\right)$$

around the x^1-axis. The functions T, s and L are defined by (66), (67) *and* (68) *respectively.*

Proof. Let

$$\eta_1 = \frac{1}{\sqrt{2}}\left(\frac{B_0}{g^2} + g^2 B_4\right), \qquad \eta_2 = B_1, \\ \eta_3 = \frac{1}{\sqrt{k}}\left(-\frac{B_0}{g^3} + \frac{dg}{dv}B_2 + gB_3 + gB_4\right). \tag{70}$$

(34, 35) and (36) yield $\langle \eta_a, \eta_b \rangle = -\delta_{ab},\ a,b = 1,2,3$. Differentiating (70) and using (38),

$$d\eta_1 = 0, \quad d\eta_2 = \sqrt{k}\eta_3 du, \quad d\eta_3 = -\sqrt{k}\eta_2 du.$$

This shows that there exist constant vectors $A, E_2, E_3 \in \mathbb{L}^5$, such that

$$\begin{aligned} \eta_1 &= A, \\ \eta_2 &= \cos\sqrt{k}uE_2 + \sin\sqrt{k}uE_3, \\ \eta_3 &= -\sin\sqrt{k}uE_2 + \cos\sqrt{k}uE_3, \end{aligned} \tag{71}$$

where $\langle A, E_a \rangle = 0$, $\langle E_a, E_b \rangle = \delta_{ab}$, $a, b = 2, 3$, and $\|A\|^2 = -1$. We then set

$$\begin{aligned} \zeta_1 = \frac{1}{\sqrt{k(-2g^{-2}+k)}} &\left\{ \frac{1}{\sqrt{2}g^2}\left(\frac{2}{g^2}+k\right)B_0 + \sqrt{2}\frac{1}{g}\frac{dg}{dv}B_2 + \right. \\ &\left. +\sqrt{2}B_3 - \frac{g^2}{\sqrt{2}}\left(\frac{2}{g^2}+k\right)B_4 \right\}, \end{aligned} \tag{72}$$

$$\zeta_2 = \frac{g^2}{\sqrt{-2+kg^2}}\left\{ B_2 - \frac{1}{g}\frac{dg}{dv}B_3 \right\}. \tag{73}$$

Using (34, 35), (36) and (38), we get $\langle \eta_a, \zeta_b \rangle = 0$, $\langle \zeta_a, \zeta_b \rangle = \delta_{ab}$, and

$$d\zeta_1 = \frac{\sqrt{2k}g}{kg^2-2}\zeta_2 dv, \quad d\zeta_2 = -\frac{\sqrt{2k}g}{kg^2-2}\zeta_1 dv. \tag{74}$$

Thus, from (69) it follows that

$$\zeta_1 = \cos\lambda B + \sin\lambda C, \quad \zeta_2 = \sin\lambda B - \cos\lambda C, \tag{75}$$

where $B, C \in \mathbb{L}^5$, $\|B\|^2 = \|C\|^2 = 1$ and $\langle B, C \rangle = 0$. Setting $E_1 = C$, $E_0 = \frac{1}{\sqrt{2}}(A+B)$, $E_4 = \frac{1}{\sqrt{2}}(A-B)$, we have $E = (E_0, \ldots, E_4) \in G$. We then assume that E is the standard basis of $\mathbb{R}^5$. By (70)–(73) and (75) we obtain

$$\begin{aligned} f(u,v) = &\frac{1}{\sqrt{k-2g^{-2}}\cos\lambda + \sqrt{k}} \cdot \\ &\cdot \left(\sqrt{2k-4g^{-2}}\sin\lambda, \frac{2\sin\sqrt{k}u}{g}, -\frac{2\cos\sqrt{k}u}{g} \right), \end{aligned}$$

which implies the required result by (67). □

7. Willmore Canal Surfaces of Null and Special Type

THEOREM 7.1. *A Willmore canal surface of null type is equivalent to the catenoid.*

Proof. Let (u, v) be adapted coordinates suth that $h = 0$ and $k = 1$. Setting $q_1 = \sinh\tau$, we have $\cosh\tau = g^{-1}$. We put

$$\begin{aligned}&\eta_0 = (1+{q_1}^2)B_0, \quad \eta_1 = B_0,\\&\eta_2 = \frac{dg}{dv}B_2 + gB_3 + gB_4, \quad \eta_3 = -gB_2 + \frac{dg}{dv}B_3, \quad \eta_4 = \frac{1}{1+{q_1}^2}B_4.\end{aligned} \tag{76}$$

By differentiating (76) and using (34, 35), (36) and (38), we obtain

$$\begin{aligned}&d\eta_0 = -dq_1B_2 + \sqrt{1+{q_1}^2}\eta_1 du, \quad d\eta_1 = du\eta_2, \quad d\eta_2 = -du\eta_1,\\&d\eta_3 = \frac{dq_1}{\sqrt{1+{q_1}^2}}\eta_4, \quad \eta_4 = E_4,\end{aligned} \tag{77}$$

where E_4 is a null vector in $\mathbb{L}^5$. This shows that there exist constant vectors E_1, E_2, $E_3 \in \mathbb{L}^5$, such that

$$\begin{aligned}&\eta_1 = \cos uE_1 + \sin uE_2, \quad \eta_2 = -\sin uE_1 + \cos uE_2,\\&\eta_3 = \sinh^{-1}(q_1)E_4 + E_3,\end{aligned} \tag{78}$$

where $\langle E_a, E_b\rangle = \delta_{ab}$, $a, b = 1, 2, 3$. The third equation of (78) implies

$$\eta_3 = \tau E_4 + E_3. \tag{79}$$

(76), (78) and (79) yield

$$B_2 = \frac{\sinh\tau}{\cosh\tau}\eta_2 - \frac{1}{\cosh\tau}\eta_3 - \sinh\tau E_4. \tag{80}$$

By (77)–(80), we deduce that

$$d\eta_0 = d(\sin(u)\cosh(\tau))E_1 - d(\cos(u)\cosh(\tau))E_2 + d\tau E_3 \mod E_4, \tag{81}$$

and hence

$$\eta_0 = E_0 + \sin(u)\cosh(\tau)E_1 - \cos(u)\cosh(\tau)E_2 + \tau E_3 \quad \mathrm{mod}\ E_4, \tag{82}$$

where $E = (E_0, \ldots, E_4) \in G$. Then, we assume that E is the standard basis of $\mathbb{L}^5$ and obtain

$$f(u, v) = (\sin(u)\cosh(\tau), -\cos(u)\cosh(\tau), \tau).$$

□

Eventually we state

THEOREM 7.2. *A Willmore canal surface of special type is equivalent to the stereographic projection of the minimal Clifford torus in S^3, i.e.,*

$$f(u, v) = \sqrt{2}\left(\frac{\cos\sqrt{2}v}{\sin\sqrt{2}v + \sqrt{2}}, \frac{\cos\sqrt{2}u}{\sin\sqrt{2}v + \sqrt{2}}, \frac{\sin\sqrt{2}u}{\sin\sqrt{2}v + \sqrt{2}}\right).$$

Proof. Let (u, v) be an adapted coordinate system such that $h = -\frac{1}{2}$ and $k = 2$. We then have $q_1 = 0$, $p_1 = -\frac{1}{2}$ and $g = 1$. By (38),

$$d(B_0 + 2B_4) = 0, \quad d(A_0 + B_3) = 2B_2 dv, \quad d(B_0 - B_3) = 2B_1 du,$$
$$\frac{d^2}{dv^2}(B_0 + B_3) + 2(B_0 + B_3) = B_0 + 2B_4,$$
$$\frac{d^2}{du^2}(B_0 - B_3) + 2(B_0 - B_3) = B_0 + 2B_4.$$

It follows that

$$B_0 + 2B_4 = A, \quad B_0 + B_3 = \cos\sqrt{2}vB_1 + \sin\sqrt{2}vB_2 + \frac{A}{2},$$
$$B_0 - B_3 = \cos\sqrt{2}uC_1 + \sin\sqrt{2}uC_2 + \frac{A}{2},$$

where A, B_i, $C_i \in \mathbb{L}^5$. Define $E = (E_0, \ldots, E_4) \in G$ by

$$A = \frac{1}{\sqrt{2}}(E_0 + E_4), \quad B_1 = \frac{1}{\sqrt{2}}E_1, \quad B_2 = E_0 - E_4,$$
$$C_1 = \frac{1}{\sqrt{2}}E_2, \quad C_2 = \frac{1}{\sqrt{2}}E_3.$$

Replacing, if necessary, f by $E^{-1}f$, we may assume that E is the standard basis of $\mathbb{L}^5$. We then compute B_0 in terms of E and obtain the required expression for f. □

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