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Quaternionic Kähler Structures on the Tangent Bundle of a Complex Space Form

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SUMMARY. - *We construct a class of quaternionic Kähler structures on the tangent bundle of a complex space form of dimension* $2n(n>2)$, *giving a generalization of the result in [13].*

1. Introduction

The purpose of the present paper is to construct a class of quaternionic Kähler structures on the tangent bundle of a complex space form when its real dimension is more than six. If the holomorphic sectional curvature is a positive constant, then this class includes a one-parameter family of hyperkähler structures constructed in [13] and further other quaternionic Kähler structures, which are hyperhermitian but not hyperkählerian.

An interesting feature of the almost hypercomplex or quaternionic Hermitian metrics which was considered on the tangent bundle of a complex space form is that they are of cohomogeneity one with respect to a semi-simple Lie group (equal to $U(n+1)$ or $U(1,n)$ according as the scalar curvature is positive or negative).

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In Section 2, we recall some structures discussed here and review some preliminaries concerning tangent bundles. Then, in Section 3, we introduce almost hyperhermitian structures on the tangent bundle of an almost Hermitian manifold, constructed in [13]. In Section 4, we construct a class of quaternionic Kähler structures on the tangent bundle of a complex space form. The last section is devoted to give some examples.

Our main purpose is to present these examples in a concise but wide way. For that reason we delete most of the tedious but rather straightforward computations.

2. Preliminaries

First, we recall some basic facts concerning the Riemann geometry and some structures. Let (M, g) be an n-dimensional connected Riemannian manifold with Levi Civita connection ∇. The Riemannian curvature tensor R is given by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

for all vector fields $X, Y \in \mathfrak{X}$, the Lie algebra of smooth vector fields on M. We denote by T_pM the tangent space of M at $p \in M$, by TM the tangent bundle of M and by π the natural projection of TM onto M.

An almost Hermitian structure on a manifold M is, by definition, a pair (J, g) formed by a tensor field J of type (1,1) and a Riemannian metric g, satisfying

$$J^2(X) = -X, \quad g(JX, JY) = g(X, Y)$$

for all vector fields $X, Y \in \mathfrak{X}$. J (resp. g) is called an almost complex structure (resp. an almost Hermitian metric). We denote by F the Kähler form, given by $F(X, Y) = g(JX, Y)$ for all vector fields $X, Y \in \mathfrak{X}$. The almost Hermitian structure (J, g) is said to be almost Kählerian if F is closed, that is, dF=0.

Furthermore, if J is an almost complex structure on M and $p \in M$, let $T_p^+(M, J)$ denote the eigenspace of J_p corresponding to the eigenvalue $\sqrt{-1}$ and $\chi^+(M, J)$ the set of complex tangent vector fields of type (1,0) on M. We note that J is a complex structure if and only if $[\chi^+(M,J),\ \chi^+(M,J)] \subset \chi^+(M,J)$ (cf. [7, Chap.IX, Theorems 2.5 and 2.8]). Such an almost complex structure is called integrable. If an almost complex structure is integrable, then the almost Hermitian structure (J,g) is said to be Hermitian. Moreover, if the Kähler form of such a structure is closed, it is called a Kähler structure. It is well known that an almost Hermitian structure (J,g) is a Kähler structure if and only if J is parallel with respect to ∇, that is, $\nabla J = 0$. Furthermore, a connected Kähler manifold (M,J,g) is called a complex space form of constant holomorphic sectional curvature $4c$ if the curvature tensor R satisfies

$$\begin{aligned} R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y \\ -g(Y,JZ)JX + g(X,JZ)JY + 2g(X,JY)JZ\} \end{aligned} \tag{1}$$

for all vector fields $X,Y,Z \in \mathfrak{X}$. It is a manifold which is locally isometric to the complex projective n-space $\mathbb{C}P(n)$ (if $c>0$), the complex hyperbolic n-space $\mathbb{C}H(n)$ (if $c<0$) or the Euclidean n-space $\mathbb{C}^n$ (if $c=0$).

Following Alekseevsky-Marchiafava [1] and Besse [2], we recall the definitions of almost hypercomplex, hypercomplex, almost quaternionic Hermitian and quaternionic Kähler structures. An almost hypercomplex structure on a manifold M of dimension $4n$ is, by definition, a triple $H = (J_\lambda)_{\lambda=1,2,3}$ of almost complex structures, satisfying

$$J_\lambda J_\mu = J_\nu$$

where (λ,μ,ν) is a circular permutation of $(1,2,3)$. It generates a subbundle $Q = \langle H \rangle$ of the bundle $\mathrm{End}(TM)$ of endomorphisms where the fiber $Q_p = \mathbb{R}J_1|_p + \mathbb{R}J_2|_p + \mathbb{R}J_3|_p$ in a point $p \in M$ is isomorphic to the Lie algebra $\mathfrak{sp}_1$. Such a subbundle is called an almost quaternionic structure generated by H. When each J_λ is a complex structure, H is said to be a hypercomplex structure on M. More generally, an almost quaternionic structure on a manifold M is defined as a subbundle $Q \subset \mathrm{End}(TM)$ of the bundle of endomorphisms which is locally generated by an almost hypercomplex structure H. We shall refer to such H as an almost hypercomplex structure compatible with Q. Let $H = (J_\lambda)_{\lambda=1,2,3}$ (resp. Q) be an almost hypercomplex (resp. almost quaternionic) structure on M. Then M can be equipped with a H-Hermitian (resp. Q-Hermitian) metric g, that is the endomorphisms J_λ, $\lambda = 1,2,3$, (resp. all endomorphisms from Q) are skew-symmetric with respect to g. Note that an H-Hermitian metric is a Q-Hermitian one with respect to $Q = \langle H \rangle$.

An almost hyperhermitian (resp. almost quaternionic Hermitian) structure on a manifold M of dimension $4n$ is, by definition, a pair (H, g) (resp. (Q, g)) formed by an almost hypercomplex structure H (resp. almost quaternionic structure Q) and a H-Hermitian (resp. Q-Hermitian) metric g. When each J_λ is a complex structure, an almost hyperhermitian manifold (M, H, g) is said to be a hyperhermitian manifold. A hyperhermitian manifold (M, H, g) is called a hyperkähler manifold if the complex structures J_λ, $\lambda = 1,2,3$, are parallel (*that is* g is a kähler metric for each one).

An almost quaternionic Hermitian manifold (M, Q, g) of dimension $4n$ is called a quaternionic Kähler manifold if the Levi Civita connection ∇ of g preserves Q. In an almost quaternionic Hermitian manifold (M, Q, g), we define the non-degenerate 4-form by

$$\Omega = \sum_\lambda F_\lambda \wedge F_\lambda,$$

where the 2-forms F_λ are defined by

$$F_\lambda(X, Y) = g(J_\lambda X, Y), \quad \lambda = 1,2,3$$

for all vector fields $X, Y \in \mathfrak{X}$. This 4-form Ω is globally defined on M. It is well known that M is quaternionic Kähler if and only if $\nabla\Omega = 0$. Furthermore if M is quaternionic Kähler, then $d\Omega = 0$ (cf. [1]). Conversely, we introduce the following Swann's result (cf. [1], [11]), which plays an essential role in the present paper.

THEOREM 2.1. *Let (M, Q, g) be an almost quaternionic Hermitian manifold of dimension $4n (n > 2)$. If $d\Omega = 0$, then M is a quaternionic Kähler manifold.*

Next we review some preliminaries concerning tangent bundles. Let (M, g) be an n-dimensional connected Riemannian manifold and $K : TTM$
$\longrightarrow TM$ the connection map corresponding to ∇ [6]. For each $u \in T_pM$, $p \in M$, we denote by $T_u^H TM$ (resp. $T_u^V TM$) the kernel of

$K|_{T_uTM}$ (resp. $d\pi|_{T_uTM}$). This is an n-dimensional subspace of T_uTM called the horizontal subspace (resp. the vertical subspace) of T_uTM. We then have a direct sum decomposition

$$T_uTM = T_u^H TM \oplus T_u^V TM.$$

The elements of $T_u^H TM$ (resp. $T_u^V TM$) are said to be horizontal vectors (resp. vertical vectors) at u. For each u, $X \in T_pM$, X_u^H (resp. X_u^V) denotes the horizontal lift (resp. the vertical lift) of X to T_uTM. These lifts are determined by

$$K(X_u^H) = 0,\ d\pi(X_u^H) = X,\ K(X_u^V) = X,\ d\pi(X_u^V) = 0. \qquad (2)$$

The canonical almost complex structure J_0 on the tangent bundle TM of a Riemannian manifold (M, g) is defined by

$$J_0(X_u^H) = X_u^V,\ J_0(X_u^V) = -X_u^H.$$

J_0 is integrable if and only if M is locally flat [6]. Furthermore, Sasaki [10] introduced the well known canonical metric G_0 on the tangent bundle of a Riemannian manifold (M, g), given by

$$G_0(X^H, Y^H) = G_0(X^V, Y^V) = g(X, Y) \circ \pi,\ G_0(X^H, Y^V) = 0$$

for all vector fields $X, Y \in \mathfrak{X}$.

Tachibana and Okumura [12] pointed out that the almost Hermitian structure (J_0, G_0) is an almost Kähler structure which is not Kählerian unless the base manifold M is locally flat. Musso and Tricerri [8] showed that (TM, G_0) has constant scalar curvature if and only if (M, g) is locally Euclidean and also remarked that the metric induced on the fiber is the Euclidean metric. But recently, by a deformation of the canonical structure (J_0, G_0), many new structures are constructed on the tangent bundles of some Riemannian manifolds (*cf.* [14], [9]).

Lastly for the calculation in this paper it will be useful to take account of the following formulas for the brackets of vertical and horizontal lifts (see [6]): For each $u \in T_pM$, $p \in M$ and for all vector fields $X, Y \in \mathfrak{X}$, we have

(i) $[X^H, Y^H] = [X, Y]^H - v\{R(X, Y)u\}$,

(ii) $[X^H, Y^V] = (\nabla_X Y)^V$,

(iii) $[X^V, Y^V] = 0$,

where $v\{R(X,Y)u\}$ is the vertical lift of $R(X,Y)u$.

3. Almost Hyperhermitian structures on TM

Let (M, g) be a connected Riemannian manifold and TM the tangent bundle of M. When F is a function on some interval, we define a function $\tilde{F}$ on TM, given by $\tilde{F} := F(g_x(u,u))$ for each $x \in M, u \in T_xM$. In what follows, we may simply write $F = F(t)$ instead of $\tilde{F}$, where $t = \|u\|^2$.

Let (M, J, g) be a connected almost Hermitian manifold of real dimension $2n$ and TM the tangent bundle of M. Let us consider functions $f = f(t), h = h(t)$ and $k = k(t)$ on $[0, \infty)$, satisfying the conditions:

$$\begin{cases} f, h \text{ and } k \text{ are positive and smooth,} \\ \text{the functions } \dfrac{h-f}{t} \text{ and } \dfrac{k-f}{t} \text{ are differentiable at } t = 0. \end{cases} \tag{3}$$

Then the functions $\frac{h-f}{t}$ and $\frac{k-f}{t}$ are smoothly extendible on the whole of TM.

We now define the three kinds of almost complex structures

$J_\lambda, \lambda = 1,2,3$ on TM as follows (see [13], [14]):

$$\begin{cases} J_1(X_u^H) = fX^V + \dfrac{h-f}{t} g(X,u)u^V + \dfrac{k-f}{t} g(X,Ju)(Ju)^V, \\ J_1(X_u^V) = -\dfrac{1}{f}X^H + \dfrac{h-f}{tfh} g(X,u)u^H + \dfrac{k-f}{tfk} g(X,Ju)(Ju)^H, \end{cases}$$

$$\begin{cases} J_2(X_u^H) = f(JX)^V + \dfrac{k-f}{t} g(X,u)(Ju)^V - \dfrac{h-f}{t} g(X,Ju)u^V, \\ J_2(X_u^V) = \dfrac{1}{f}(JX)^H + \dfrac{f-h}{tfh} g(X,u)(Ju)^H - \dfrac{f-k}{tfk} g(X,Ju)u^H, \end{cases}$$

$$\begin{cases} J_3(X_u^H) = -(JX)^H, \\ J_3(X_u^V) = (JX)^V + \dfrac{k-h}{th} g(X,u)(Ju)^V + \dfrac{k-h}{tk} g(X,Ju)u^V \end{cases} \tag{4}$$

for all vectors $X \in T_{\pi(u)}M$.

Next, for the same functions f, h and k in the above, we define a Riemannian metric G on the tangent bundle TM of an almost Hermitian manifold (M, J, g), compatible with these $J_\lambda, \lambda = 1,2,3$, given by

$$\begin{cases} G(X^H, Y^H) = \alpha g(X,Y) + \beta\{g(X,u)g(Y,u) + \\ \qquad\qquad g(X,Ju)g(Y,Ju)\}, \\ G(X^V, Y^V) = \dfrac{\alpha}{f^2} g(X,Y) + \dfrac{(f^2-h^2)\alpha + tf^2\beta}{tf^2h^2} g(X,u)g(Y,u) \\ \qquad\qquad + \dfrac{(f^2-k^2)\alpha + tf^2\beta}{tf^2k^2} g(X,Ju)g(Y,Ju), \\ G(X^H, Y^V) = 0 \end{cases} \tag{5}$$

for all vectors $X, Y \in T_{\pi(u)}M$, where the smooth functions α and β on $[0,\infty)$ satisfy the conditions of positive definiteness of G

$$\alpha > 0, \quad \alpha + t\beta > 0. \tag{6}$$

Then by a direct computation we can show the following.

PROPOSITION 3.1. *$(J_\lambda, G)_{\lambda=1,2,3}$ mentioned above is an almost hyperhermitian structure on TM.*

In particular, we also have the following (see [13]).

PROPOSITION 3.2. *Suppose that (M, J, g) is a complex space form of holomorphic sectional curvature $4c$. The functions f, h and k satisfy*

$$h = \frac{f^2 - ct}{f - 2tf'}, \quad k = \frac{ct + f^2}{f}$$

if and only if $(J_\lambda, G)_{\lambda=1,2,3}$ is a hyperhermitian structure constructed on the tangent bundle TM.

4. Quaternionic Kähler structures on TM

Now, suppose that (M, J, g) is a complex space form of the holomorphic sectional curvature $4c$ and that dimension of M is $2n (n > 2)$. Then we construct a quaternionic Kähler structure on TM.

THEOREM 4.1. *Let (M^{2n}, J, g), $n > 2$, be a complex space form of the holomorphic sectional curvature $4c$. Then the almost hyperhermitian structure (H, G) on TM in Proposition 3.1, where $H = (J_\lambda)_{\lambda=1,2,3}$ and G are defied by (4), (5) respectively, is a quaternionic Kähler structure if and only if the functions f, h, k, α and β given by the conditions (3), (6) satisfy the following:*

$$\begin{cases} (i) \quad h = \dfrac{f^2 - ct}{f - 2tf'}, & (ii) \quad k = \dfrac{ct + f^2}{f}, \\[2ex] (iii) \quad \alpha' = \dfrac{c}{fh}\alpha, & (iv) \quad \alpha + t\beta = \dfrac{k}{f}\alpha. \end{cases} \tag{7}$$

Proof. We compute the exterior derivative $d\Omega$ of the 4-form $\Omega = \sum F_\lambda \wedge F_\lambda$, given by (H, G) on TM. These tedious computations, which we delete here and which are performed by using Theorem 2.1 and Dombrowsky's formulas for brackets in Section 2, show that the condition $d\Omega = 0$ is equivalent to the following:

$$
\begin{aligned}
&(i) && ct\frac{\alpha}{f^2} - 2ct\frac{A}{fk} - \alpha + A = 0,\\
&(ii) && 2\frac{A}{k} - \frac{\alpha}{f} + ct\frac{\alpha}{f^3} - \frac{A}{f} = 0,\\
&(iii) && \alpha' - c\frac{A}{hk} = 0,\\
&(iv) && 2\left(\frac{\alpha}{f}\right)' t + \frac{\alpha}{f} - \frac{A}{h} = 0,\\
&(v) && \alpha\left\{\frac{A}{hk} - \left(\frac{\alpha}{f^2}\right)' t - \frac{\alpha}{f^2}\right\} - \frac{\alpha}{f^2}\left(\alpha' - c\frac{A}{hk}\right) = 0,\\
&(vi) && \frac{A}{hk} - \left(\frac{\alpha}{f^2}\right)' t - \frac{\alpha}{f^2} = 0,\\
&(vii) && 2ct\frac{A}{h}\left(\frac{\alpha}{f} - \frac{A}{k}\right) + A\left(\alpha' - c\frac{A}{hk}\right)t + \alpha\left\{(At)' - A - 2ct\frac{A}{hk}\right\}\\
& && = 0,\\
&(viii) && 2\frac{A}{h}\left(\frac{A}{k} - \frac{\alpha}{f}\right) + A\left\{\frac{A}{hk} - \left(\frac{\alpha}{f^2}\right)' t - \frac{\alpha}{f^2}\right\} -\\
& && \frac{\alpha}{f^2}\left\{(At)' - A - 2ct\frac{A}{hk}\right\} = 0,\\
&(ix) && \frac{A}{h}\left(\frac{\alpha}{f} - \frac{A}{k}\right) - \frac{A}{k}\left\{-2\left(\frac{\alpha}{f}\right)' t + \frac{A}{h} - \frac{\alpha}{f}\right\}\\
& && -\frac{\alpha}{f}\left\{-2\left(\frac{A}{k}t\right)' + \frac{A}{k} + \frac{A}{h}\right\} + \frac{A}{hk}\left(\alpha + ct\frac{\alpha}{f^2} - A\right) = 0,
\end{aligned}
\tag{8}
$$

where we set $A = \alpha + t\beta$ and the prime ($'$) means the differentiation with respect to t.

Furthermore, by a long computation, we can see that (8) implies the equations (7). Conversely, (7) implies all equations in (8). □

REMARK 4.2. *(a) The conditions (i) and (ii) in (7) show that* $(J_\lambda, G)_{\lambda=1,2,3}$ *is a hyperhermitian structure, as mentioned in Proposition 3.2.*
(b) From the condition (iii) in (7), we see that $\alpha = \mu \exp \int \frac{c}{fh} dt$ *where* μ *is a positive constant.*

Let $f = f(t)$ be a smooth positive function defined on $[0, \infty)$. Denote $H(f) = (J_\lambda(f))_{\lambda=1,2,3}$ and $G(f)$ the almost hypercomplex structure and the Hermitian metric defined on TM by (4) and (5) respectively, and moreover by using the functions $h = h(t)$, $k = k(t)$, $\alpha = \alpha(t)$ and $\beta = \beta(t)$ given in (7). Then by Theorem 2.1 we obtain the following.

THEOREM 4.3. *Let TM be the tangent bundle of a complex space form (M, J, g) of holomorphic sectional curvature $4c$ and dimension $2n(n > 2)$. If f is a positive function on $[0, \infty)$, satisfying the conditions $f^2 > |c|t$ and $f > 2tf'$, then $(H(f), G(f))$ is a quaternionic Kähler structure on TM.*

5. Examples

Let (M, J, g) be a complex space form of holomorphic sectional curvature $4c$ and TM the tangent bundle of M. Suppose that dimension of M is $2n(n > 2)$. It seems to us that Theorem 4.3 gives many examples on TM. In fact, we first review an example, given in [13].

EXAMPLE 5.1. *For positive constants c, a and μ, we set*

$$f = \frac{1}{2}\left(a + \sqrt{4ct + a^2}\right), \quad h = k = \sqrt{4ct + a^2},$$

$$\alpha = \mu\left(a + \sqrt{4ct + a^2}\right), \quad \beta = \frac{\mu}{t}\left(-a + \sqrt{4ct + a^2}\right),$$

where we put $\beta(0) = \frac{2\mu c}{a}$. Then it is easily seen that these functions satisfy all conditions of (7). Furthermore $(H(f), G(f))$ is a one-parameter family of hyperkähler structures on TM, since these functions satisfy the following conditions (see [13, (5.6) and (5.7)]):

$$\begin{cases} \alpha = \eta f \ (\eta\text{=positive constant}), & \dfrac{\alpha}{f} = \dfrac{\alpha + t\beta}{h} = \dfrac{\alpha + t\beta}{k}, \\ \\ (f^2 + ct)f' = cf. & \end{cases} \tag{9}$$

Note that the last equation of (9) is exactly the case where $h = k$ (see (5.8) of [13] for general solution). It is well known that the complex projective space $\mathbb{C}P(n)$ with the canonical Kähler metric is the complex space from of positive holomorphic sectional curvature $4c$. We can see that the case of $a = 1$ and $\mu = \frac{1}{2}$ in the above coincides with

Calabi's example [3, p.299] constructed on the holomorphic cotangent bundle $T^*\mathbb{C}P(n)$, naturally identified with the tangent bundle $T\mathbb{C}P(n)$.

Next, we give two examples, which are quaternionic Kählerian, but not hyperkählerian.

EXAMPLE 5.2. *For positive constants a and μ, we set*

$$f = \sqrt{|c|t + a},\ h = \frac{\{(|c| - c)t + a)\}\sqrt{|c|t + a}}{a},\ k = \frac{(|c| + c)t + a}{\sqrt{|c|t + a}},$$

$$\alpha = \frac{\mu a(|c|t + a)}{(|c| - c)t + a},\ \beta = \frac{\mu ca}{(|c| - c)t + a}.$$

Then f, h and k satisfy the conditions (3), and α, β do the condition (6). Then it is easily seen that these functions satisfy all conditions of (7). Therefore $(H(f), G(f))$ is a quaternionic Kähler structure, but not a hyperkähler structure on TM, since they do not satisfy (9).

EXAMPLE 5.3. *For positive constants a and μ, we set*

$$f = \sqrt{e^{-t} + |c|t}, \quad h = \frac{\{e^{-t} + (|c| - c)t\}\sqrt{e^{-t} + |c|t}}{e^{-t}(1 + t)},$$

$$k = \frac{e^{-t} + (|c| + c)t}{\sqrt{e^{-t} + |c|t}}, \alpha = \frac{\mu(e^{-t} + |c|t)}{e^{-t} + (|c| - c)t}, \quad \beta = \frac{\mu c}{e^{-t} + (|c| - c)t}.$$

Then f, h and k satisfy the conditions (3), and α and β do the conditions (6). Then it is easily seen that these functions satisfy all conditions of (7). Therefore $(H(f), G(f))$ is a quaternionic Kähler strucure, but not a hyperkähler structure on TM, since they do not satisfy (9).

REMARK 5.4. *Let (M, J, g) be an almost Hermitian mainifold and $\mathfrak{G} = \mathrm{Aut}(M)$ the group of automorphisms of M, that is $\varphi^* g = g$ and $\varphi_* J = J\varphi_*$ for each $\varphi \in \mathfrak{G}$. There is a natural left action on TM defined by $\Phi(x, u) = (\varphi(x), \varphi_*(u))$ for $x \in M$, $u \in T_x M$ (cf. Musso and Tricerri [8]). For the structures $J_\lambda\,(\lambda = 1, 2, 3)$ and G constructed in (4) and (5), it is easily seen that $\Phi_* J_\lambda = J_\lambda \Phi_*\,(\lambda = 1, 2, 3)$ and $\Phi^* G = G$.*

Let $\mathbb{C}P(n)$ be the complex projective $n-$space with the canoincal Kähler structure. We also regrad $\mathbb{C}P(n)$ as a homogeneous space $U(n+1)/U(n) \times U(1)(n > 2)$ and $U(n+1)$ acts transitively on the tangent bundle $T\mathbb{C}P(n)$ (it is well known that $\mathbb{C}P(n)$ is a 2-point homogeneous space). Therefore the example stated above are all irreducible quaternionic Kähler manifolds of cohomogeneity one with respect to the compact semi-simple Lie group $U(n+1)$. When $c < 0$, we can discuss similarly. We refer the readers to [4] about a classification of hyperkähler metrics of cohomogeneity one with respect to a compact simple Lie group.

Acnowledgement

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