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An Example of Two Compact Spaces with Different Topological Dimensions

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Summary. - *In this paper we give two compact spaces* X, Y *with* $dim(X) = dim(Y) = 1$, $ind(X) = ind(Y) = Ind(X) = Ind(Y) = 3$, *where* dim *is the covering dimension,* ind *and* Ind *are the small and large inductive dimensions respectively.*

1. Introduction

Compact spaces with different topological dimensions have been studied. Filippov found a compact space T_{mn} with $dim(T_{mn}) = 1$, $ind(T_{mn}) = m$, $Ind(T_{mn}) = n$ for each m, n such that $m \leq n \leq 2m - 1$ (see [5]). In this paper we prove the existence of two non homeomorphic compact spaces X, Y with $dim(X) = dim(Y) = 1$, $ind(X) = ind(Y) = 3$, $Ind(X) = Ind(Y) = 3$. In order to get these spaces, we take into account the space T_{23}. In order to evaluate the covering dimension of X, Y, we use the local dimension *locdim* define in [1], and we give a slight modification of the Ind–dimension, the $Indc$–dimension, to state the equalities $Ind(X) = Ind(Y) = 3$.

We ensure that the spaces X, Y are not homeomorphic by defining a topological dimension, named K, such that $K(X) \neq K(Y)$. Since X and Y are topologically distinct, at least one of them is not homeomorphic to the space T_{33}, another compact space 1–dimensional

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for the *dim*–dimension and *three*–dimensional for the dimensions *ind*, *Ind*.

2. Basic concepts and notations

In this paper we consider only Hausdorff spaces.
We begin by defining an inductive dimension by means of separating points:

DEFINITION 2.1. *Let X be a topological space, $n = 0, 1, 2, \ldots$. The following conditions define inductively the dimension K:*

1. $K(X) = -1 \Leftrightarrow X = \varnothing$.

2. *If* $|X| = 1$, *then* $K(X) = 0$.

3. *On the assumption that* $|X| > 1$, $K(X) \leq n$ *if for every pair* x, y *of distinct points of* X *there exist two open sets* $U, V \subset X$ *such that* $x \in U$, $y \in V$, $U \cap V = \varnothing$, $X = U \cup V \cup L$ *with* $K(L) < n$ *where* L *is* $X - (U \cap V)$ *(we said that* L *is a separation of* X *between* x, y*).*

4. $K(X) = n$ *if* $K(X) \leq n$ *and the inequality* $K(X) < n$ *does not hold.*

5. $K(X) = \infty$ *if* $K(X) > n$ *for every* n *(this dimension is similar to the one defined in [6]).*

As an immediate consequence of the definition $K(X) = K(Y)$ for homeomorphic spaces X, Y:

THEOREM 2.2. *If* Y *is a subspace of a space* X *then* $K(Y) \leq K(X)$.

THEOREM 2.3. *For every regular space* X, $K(X) \leq ind(X)$.

We also define the *Indc*–dimension:

DEFINITION 2.4. *Let X be a topological space $n = 0, 1, 2, \ldots$ the following conditions define inductively the dimension Indc:*

1. $Indc(X) = -1 \Leftrightarrow X = \varnothing$

2. *$Indc(X) \leq n$ for every compact space C included in X, for every open subset of X, V, there exists an open subset of X, U, with $C \subset U \subset V$, $Indc(Bd(U)) < n$. ($Bd(U)$ is the boundary of U).*

3. *$Indc(X) = n$ if $Indc(X) \leq n$ and the inequality $Indc(X) < n$ does not hold.*

4. *$Indc(X) = \infty$ if $Indc(X) > n$ for every n.*

We have the following result:

PROPOSITION 2.5. *Let V' be an open subset of X, then $Indc(V') \leq Indc(X)$.*

Proof. We apply induction with respect to $Indc(X)$, assuming that $Indc(X) < \infty$. If $Indc(X) = -1$, then $X = V' = \varnothing$, so $Indc(V') = -1$.

Assume that the result is true for every space X with $Indc(X) < n$, $n \geq 0$.

Let X be a space with $Indc(X) = n$, C a compact included in V', V an open subset of V' such that $C \subset V$. As V is an open subset of the normal space X, there exists an open subset of X, U with $C \subset U \subset cl(U) \subset V$, $Indc(Bd(U)) < n$, where $cl(U)$ is the closure of U in X.

Since $cl(U)$ is included in V, $Bd_{V'}(U) = Bd(U)$, where $Bd_{V'}(U)$ is the boundary of U in V', and the $Indc(Bd_{V'}(U)) < n$. This implies that $Indc(V') \leq n = Indc(X)$. □

Since the unitary sets are compact spaces, we have the next proposition:

PROPOSITION 2.6. *For every regular space X, $ind(X) \leq Indc(X)$.*

3. The spaces X, Y

In order to establish the spaces X, Y with the required properties, we need some preliminary lemmas:

LEMMA 3.1. *For a compact space X with $K(X) \leq 1$ we have that $ind(X) = K(X)$.*

Proof. For $K(X) = 0$, let x be a point of X and F a closed subset of X such that $x \notin F$. We can find, for each point y in F, an open subset and closed subset $U(y)$ such that $x \notin U(y)$. Consequently, as F is a compact space, there exists an open and closed subset of X containing F, say U, such that $x \notin U$. This implies that $ind(X) = 0$.

For $K(X) = 1$, let F be a closed subset of X, x a point of X such that $x \notin F$. Since F is compact and $K(X) = 1$, we can find an open subset of X, U, with $F \subset U$, $x \notin U$, $Bd(U) \subset \{Bd(U_i)/i = 1, \dots, n\}$, $K(Bd(U_i)) < 1$ for $i = 1, \dots, n$.

Since $Bd(U_i)$ are compact spaces,we get $ind(Bd(U_i)) = K(Bd(U_i)) < 1$ for $i = 1, \dots, n$. Now, the subspace theorem and the sum theorem for compact spaces with ind–dimension 0 yield $ind(Bd(U)) < 1$ (see [3], theorem 2.2.7). This implies that $ind(X) \leq 1$. □

LEMMA 3.2. *Let X be a locally compact, noncompact space, then $K(w(X)) \leq K(X) + 1$, $ind(w(X)) \leq ind(X) + 1$.*

Proof. We have that $w(X) = X \cup \{p\}$, with $p \notin X$. Let x, y be two distinct point of $w(X)$, and, say $x \in X$.

As X is an open subset of the regular space $w(X)$, we can find and open subset of $w(X)$, say U, such that $x \in U \subset cl(U) \subset U$, $y \notin cl(U)$, where $cl(U)$ means the closure of U in $w(X)$.

Then $w(X) - Bd(U) = U \cup (X - cl(U))$, where U, $X - cl(U)$ are disjoint open subsets of $w(X)$ such that $x \in U$, $y \in X - cl(U)$. Moreover, applying the Theorem 2.2 we have that $K(Bd(U)) \leq K(cl(U)) \leq K(X) < K(X) + 1$.

This implies that $K(w(X)) \leq K(X) + 1$.

The proof for the ind–dimension is analogous (we just have to substitute the point y for a closed subset of X, say F). □

LEMMA 3.3. *Let X be a locally compact, noncompact space with $K(X) = ind(X)$. Then $K(w(X)) = K(X)$.*

Proof. We are going to see this lemma for every space X under the hypothesis of the lemma with $K(X) < \infty$.

Let x, y be two distinct point of $w(X)$. If, say, $x \in X$, we can find an open subset of $w(X)$, V, such that $x \in V$, $y \in H = w(X) - cl(V)$, $cl(V) \subset X$. Then V is an open subset of X.

Furthermore, as $ind(X) = K(X)$ and V is an open subset of X, there exists an open subset of X, U, such that $x \in U \subset cl_X(U) \subset V$, $ind(Bd_X(U)) < K(X)$, and then $K(Bd_X(U)) \leq ind(Bd_X(U)) < K(X)$.

On the other hand, as $cl(U) \subset cl(V) \subset X$, we have that $cl_X(U) = cl(U)$, $Bd_X(U) = Bd(U)$.

Then $w(X) = U \cup H' \cup Bd(U)$, where U, $H' = w(X) - cl(U)$ are open subsets of $w(X)$ such that $x \in U$, $y \in H'$ (H is included in H').

So we have that $Bd(U)$ is a separation of $w(X)$ between x, y with $K(Bd(U)) = K(Bd_X(U)) < K(X)$, and then $K(w(X)) \leq K(X)$. □

Lemma 3.4. *Let X be a compact space with $ind(X) = 3$. If the equality $K(w(U)) = K(U)$ holds for every open and non–closed subset U of X such that $K(U) = 2$, then $K(X) = 3$.*

Proof. As $ind(X) = 3$, we can find a point of X, x, and an open neighbourhood of x, $V(x)$ such that for every open neighbourhood of x included in $V(x)$, $U(x)$, we have that $ind(Bd(U))) \geq 2$.

If $K(Bd(U(x))) < 2$, since $Bd(U(x))$ is a compact space, by Lemma 3.1 we get $ind(Bd(U(x))) = K(Bd(U(x))) < 2$, a contradiction, so $K(Bd(U(x))) \geq 2$ for every open subset $U(x)$ of X included in $V(x)$.

If we now consider $V(x)$, $V(x)$ is an open subset of X such that it's not closed in X (it has boundary), so it's locally compact, non-compact space.

For every set $U(x)$ such that $U(x)$ is open in $V(x)$ and $cl_{V(x)}(U(x))$ is a compact space, $U(x)$ is an open subset of X included in $V(x)$, and therefore $K(Bd(U(x))) \geq 2$.

On the other hand, as $cl_{V(x)}U(x)$ is a closed subset of X and then $cl_{V(x)}U(x) = cl(U(x))$, $Bd_{V(x)}U(x) = Bd(U(x))$.

We have proved the following assertion: "If $U(x)$ is an open subset of $V(x)$ such that $cl_{V(x)}U(x)$ is a compact space, then

$$K(Bd_{V(x)}(U(x))) \geq 2''$$

Now, let L be a separation of $w(X(x))$ between x, p, we have that $w(V(x)) = U'(x) \cup V'(p) \cup L$, where $U'(x)$, $V'(p)$ are disjoint open subsets of $w(V(x))$.

Since $p \notin U'(x)$, the set $U'(x)$ is an open subset of $V(x)$ with $cl_{V(x)}U'(x) \subset w(V(x)) - V'(p)$, so we have that $cl_{V(x)}U'(x)$ is a compact space and then $K(Bd_{V(x)}(U'(x))) \geq 2$. If we apply the subspace theorem, we get: $K(L) \geq K(Bd_{V(x)}(U'(x))) \geq 2$, and then $K(w(V(x))) > 2$.

On the other hand, if we assume that $K(X) \leq 2$, we have that $K(V(x)) \leq K(X) \leq 2$, $K(w(V(x))) > 2$. Since $K(w(V(x))) \leq K(V(x))+1$ (Lemma 3.2), we have that $K(V(x)) = 2$, $K(w(V(x))) > 2$, a contradiction with the hypothesis.

Consequently, $K(X) = 3$. □

LEMMA 3.5. *Let Y be a compact space with $Ind(Y) = 3$. If the equality $ind(w(U)) = ind(U)$ holds for every open, non–closed subset U of Y with $ind(U) = 2$, then $ind(Y) = 3$.*

Proof. The inequality $ind(Y) \leq Ind(Y)$ for every normal space Y is known (see [3], Theorem 1.6.3). Consequently, we only need to prove the inequality $ind(Y) > 2$.

As $Ind(Y) > 2$, we can find a closed subset of Y, say F, and an open subset of Y including F, say V, such that $Ind(Bd(U)) \geq 2$ for every set U open in Y with $F \subset Y \subset V$. Since $Bd(U)$ is a compact space, the inequality $ind(Bd(U)) < 2$ would imply that $Ind(Bd(U)) = ind(Bd(U)) < 2$ (see [3], Theorems 2.4.2, 2.4.3), so we have that $ind(Bd(U)) \geq 2$.

If we take now the open and non–closed subset of X, V following the argument of the proof of the Lemma 3.4 we can see that every partition L of $w(V)$ between F, p has $ind(L) \geq 2$. As F is a closed subset of $w(V)$ (it's a compact space included in $w(V)$), this implies that $ind(w(V)) > 2$.

If we now assume that $ind(Y) \leq 2$, we get that $ind(V) \leq 2$, $ind(w(V)) > 2$. Since $ind(w(V)) \leq ind(V) + 1$ (Lemma 3.2) we that $ind(V) = 2$, $ind(w(V)) > 2$, a contradiction with the hypothesis. Then we have that $ind(Y) = 3$. □

As we have the equivalence, for a compact space X, between closed subset of X and compact space included in X, we have the next lemma:

LEMMA 3.6. *If X is a compact space, then $Indc(X) = Ind(X)$.*

LEMMA 3.7. *Let X be a compact space, if there exists a partition L of X between C, F with $Indc(L) < n$ for all disjoint compact spaces C, F included in X, then $Indc(X) \leq n$.*

Proof. Let C be a compact in X and let V be an open subset of X such that $C \subset V$. Since C, $X - V$, are disjoint compact spaces, we can find a partition L of X between C, $X - V$ with $Indc(L) < n$. So we have that there exists an open subset of X, say U, such that $C \subset U$, $Bd(U) \subset L$, $U \cap (X - V) = \varnothing$. Applying the Lemma 3.6 and the closed subspace theorem for the Ind–dimension, we have that $Indc(Bd(U)) \leq Indc(L) < n$, and the $Indc(X) \leq n$. □

LEMMA 3.8. *For every locally compact, noncompact space X,*

$$Indc(w(X)) = Indc(X)$$

Proof. The relation $Indc(X) \leq Indc(w(X))$ is a consequence of the Proposition 2.5, so we only need to prove that $Indc(w(X)) \leq Indc(X)$.

Let C, F be disjoint compact spaces in $w(X)$. At least one of them, say C, is included in X. We can find an open subset of X, V, with $C \subset V \subset X$, $cl_X(V) \cap F = \varnothing$, $cl_X(V)$ being a compact space (see [2], Theorem 3.3.2). Since $cl_X(V)$ is a compact space, we have that $cl_X(V) = cl(V)$.

On the other hand, provided that $Indc(X)$ is a finite number, there exists an open subset of X, say U, such that $C \subset U \subset V$, $Indc(Bd_X(U)) < Indc(X)$. Since $Bd(U) = Bd_X(U)$, we have that $L = Bd(U)$ is a partition between C, F with $Indc(L) < Indc(X)$. Because of Lemma 3.7, this implies that $Indc(w(X)) \leq Indc(X)$. □

LEMMA 3.9. *If X is a locally compact, noncompact space with*

$$dim(w(X)) < \infty$$

then $dim(w(X)) \leq locdim(X)$

Proof. Assume that $locdim(X) = n$, then we can find, for each $x \in X$,an open neighbourhood of x, $U(x)$ with $dim(cl_X(U(x))) \leq n$.

As $cl_X(U(x)) \subset cl(U(X)) \subset cl_X(U(x)) \cup \{p\}$ we have that

$$dim(cl(U(x))) \leq dim(cl_x(U(x))) \leq n \qquad \text{(see [1], [2.2])}$$

Consequently, if we assume that $dim(w(X)) > n$, $C = \{x \in w(X)/ w(X)$ is $dim(w(X))$–dimensional at $x\} \subset \{p\}$ and then $dim(C) \leq 0$.

On the other hand, we know that C is a $dim(w(X))$–dimensional space (see [1], [3.7]), so we have a contradiction and $dim(w(X)) \leq n$. □

Example 3.10. *The compact space* T_{23} *has* $dim(T_{23}) = 1$, $ind(T_{23}) = 2$, $Ind(T_{23}) = 3$. *So there exists an open subset of* T_{23}, *say* V, *with* $ind(V) = 2 < ind(w(V)) = 3$ *(Lemma 3.5). Now, Lemma 3.1 implies that* $K(w(V)) \geq 2$. *If* $K(w(V)) = 3$, *then* $K(V) = 2$ *(Lemma 3.2 and Theorem 2.3), and the Lemma 3.3 yields* $K(w(V)) = K(V) = 2$, *a contradiction.*

So we have that $K(w(V)) = 2 < ind(w(V)) = 3$. *In order to prove that* $Ind(w(V)) = 3$, *we are going to see that* $Ind(w(V)) \leq 3$:

Since $Indc(T_{23}) = Ind(T_{23}) = 3$ *(Lemma 3.6), we have the inequality* $Indc(V) \leq Indc(T_{23}) = 3$ *(Proposition 2.5), so we get* $Ind(w(V)) = Indc(w(V)) \leq Indc(V) \leq 3$ *(Lemma 3.8).*

Now, we are going to see that $dim(w(V)) = 1$:
As $locdim(T_{23}) \leq dim(T_{23}) = 1$ *([1], [1.7]), we have that* $locdim(V) \leq locdim(T_{23}) \leq 1$ *([1], [4.1]), and then* $dim(w(V)) \leq locdim(V) \leq 1$ *(Lemma 3.9).*

Since $w(V)$ *is a compact space, the equality* $dim(w(V)) = 0$ *is not possible (it would imply that* $ind(w(V)) = 0$: *see [3], Theorem 3.1.30).*

Now, as $Y = w(V)$ *is a compact space with* $K(w(V)) = 2 < ind(w(V)) = 3$, *we can apply the Lemma 3.4 to find an open, non–closed subset* U *included in* Y *with* $K(U) = 2 < K(w(U)) = 3$. *The same reasoning for* $X = w(U)$ *as the one we have made for* Y *implies that* $dim(X) = 1$, $ind(X) = 3$, $Ind(X) = 3$, *but* X, Y *are non-homeomorphic spaces because we have that* $K(X) = 3$, $K(Y) = 2$.

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