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Quasi-invariant Measures on a Group of Diffeomorphisms of an Infinite-dimensional Real Manifold and Induced Irreducible Unitary Representations

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Summary. - *Quasi-invariant strongly differentiable measures on a group of diffeomorphisms of an infinite-dimensional real Banach manifold, relative to dense subgroups, are constructed. These measures are used for the investigation of irreducible unitary representations.*

1. Introduction.

For a compact real Riemannian manifold M measures on a group of diffeomorphisms of M were constructed, such that measures were quasi-invariant relative to dense subgroups [38]. Such groups are not locally compact, therefore, they can not possess non-zero measures quasi-invariant relative to the entire groups [42].

On the other hand, the group G of diffeomorphisms appear naturally in mathematical physics and in quantum mechanics [20, 31, 30]. In such theories a description of irreducible unitary representations and quasi-invariant measures on G is necessary. For a finite-dimensional Riemannian manifold there is a natural measure on it

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known as the Riemannian volume element. In particular for the Euclidean space $\mathbf{R^n}$ it is simply the Lebesgue measure. But for an infinte-dimensional real Banach space there is not any non-trivial quasi-invariant measure realtive to shifts of the entire space X [15]. There are only measures quasi-invariant relative to the dense subspace X_0, $X_0 \ne X$, $X_0 \subset X$. These measures have not such unique characteristic as the Riemannian volume element. Moreover, different quasi-invariant measures on the Banach space X or on a manifold M modelled on X may be non-equivalent or even orthogonal ([8] Theorem II.4.1, [3]). This is very important difference with the finite-dimensional case, when all left-quasi-invariant measures on a locally compact group are equivalent to the Haar measure ([4], Proposition VII.1.9.11). This circumstance is the reason for the existence of vast families of inequivalent irreducible unitary representations on a group of diffeomorphisms apart from the case of a locally compact group. In [23] regular irreducible unitary representations of G' for the one-dimensional manifold M associated with the Gaussian quasi-invariant measures on G relative to the left action of the dense subgroup G' were constructed. But his proof was strongly related with one-dimensionality of M.

In the previous paper of the author the group $G := Diff^t_{\beta,\gamma}(M)$ of diffeomorphisms of the infinite-dimensional real Banach $E^{\infty}_{\omega,\delta}$-manifold M was defined and investigated, where $\omega \ge \beta$, $\delta > \gamma + 2$ ([29] §2.5, §2.8). In that article irreducible unitary representations associated with the quasi-invariant measures ν on M relative to the action of G were constructed. This article is devoted to the construction of strongly differentiable Gaussian quasi-invariant measures μ on G relative to the left action of a dense subgroup G'. Then such measures are used for producing of irreducible unitary representations of G'. It also is proved that on G there is not any non-trivial quasi-invariant measure relative to both left L_h and rigt R_h actions of a dense subgroup G', as well as to the action of G' by inner automorphisms α_h, where $L_h(g) := hg$, $R_h(g) := gh$, $\alpha_h(g) := h^{-1}gh$, $h \in G'$, $g \in G$. This is another difference from the theory of quasi-invariant measures on locally compact groups. In particular results of this paper encompass the case of the finite-dimensional Riemannian manifold besides infinite-dimensional M.

§2 contains some necessary specific isomorphisms of Banach spaces assosiated with parameter-elliptic pseudodifferential operators. This section is based on [1, 6, 17, 18, 16, 40, 41], therefore it is given shortly. Quasi-invariant measures on the group G of diffeomorphisms are produced in §3. §4 is devoted to irreducible unitary representations.

2. Some specific isomorphisms of Banach spaces.

NOTE 2.1. Let M be a complete connected Riemannian C^∞-manifold, which is Euclidean at infinity [6] and without a boundary. That is, $M \setminus \bar{M}_R$ is a finite union of disjoint connected components Ω_j diffeomorphic with $\mathbf{R^n} \setminus \bar{B}$ by diffeomorphisms $\psi_j : \Omega_j \to \mathbf{R^n} \setminus \bar{B}$, where $\bar{B}$ is a closed ball in $\mathbf{R^n}$, $\bar{M}_R := \{x \in M : d(x,0) \le R\}$, $0 < R < \infty$, 0 is a fixed point in M, $d(x,y)$ is a Riemannian distance along a geodesic joining x and $y \in M$, and $\partial M = \emptyset$. As usually $d(x,y)$ is induced by a Riemannian metric g on M, where $g : M \to S_2M$ is a section in the bundle $\sigma_2 : S_2M \to M$ with values $g(p) \in Pos\, T_pM$, that is, $g(p)$ is the symmetric positive definite bilinear form on T_pM for each $p \in M$ (see §1.8 in [21]). We also assume that M is with a finite atlas $At(M) = \{(U_i, \phi_i) : i = 1, ..., k\}$ with charts (U_i, ϕ_i), where U_i are diffeomorphic to $\mathbf{R^n}$. Since $\bar{M}_R$ is compact, then $\phi_i \circ \phi_j^{-1} =: \kappa$ and all their derivatives are bounded and $C_1|x-y| \le |\kappa(x) - \kappa(y)| \le C_2|x-y|$ for each $U_j \cap U_i \ne \emptyset$ and each $x, y \in \phi_j(U_j \cap U_i)$, where C_1 and C_2 are positive constants. Hence this M Euclidean at infinity is also admissible by [18].

NOTE 2.2. *For integer $s \ge 0$ and $\beta \in \mathbf{R}$ let $C^s_\beta(TM)$ denotes a completion of a space of sections f of a vector Riemannian tangent bundle TM with $f_i := f \circ \phi_i^{-1} \in \mathsf{L}(\mathbf{R^n})$ for each U_i relative to the following norm*

(i) $\|f\|_{C^s_\beta(TM)} := \sum_{b=0}^s \sup_{x\in M}[\|\sigma(x)^{b+\beta+n\eta(\beta)}\nabla^b f(x)\|]$, *where ∇ is a covariant differential for M, $n = dim_{\mathbf{R}} M$, $\eta(\beta) = 0$ for $\beta \ge 0$ and $\eta(\beta) = 1$ for $\beta < 0$, $\sigma(x)^\beta$ is a weight factor, $\sigma(x) := (1+|x|^2)^{1/2}$, $|x| := d(x,0)$; $\mathsf{L}(M)$ is a Schwarz space of functions on M with values in $\mathbf{R^n}$ (see $\mathsf{L}(M)$ in [35]). For $t = s + q$ with an integer s and $0 < q < 1$ the weighted Hölder space $C^t_\beta(TM)$ is the linear space of sections f of the tangent bundle TM such that*

for each compact canonical closed subset $V \subset M$, $V = cl(Int\ V)$, $f|V \in C^t$ and are satisfied conditions $(1, 2)$*:*

$$(1)\ \|f\|_{C^t_\beta(TM)} := \|f\|_{C^s_\beta(TM)} + \sup\{[\sigma(\tilde{x})]^{t+\beta+n\eta(\beta)}\|\nabla^s f(x)$$

$$-\tau(x,x')\nabla^s f(x')\|/[d(x,x')]^q :\ d(x,x') < \rho(x),\ x \in M\} < \infty,$$

$$(2)\ \lim\{\|f|M^c_R\|_{C^t_\beta(T(M^c_R))} :\ R \to \infty\} = 0,$$

where $M^c_R := \{x : x \in M,\ d(x,0) > R\}$, $\rho(x)$ denotes the injectivity radius of exp_x for the exponential mapping at $x \in M$, $exp : \tilde{T}M \to M$, $\tilde{T}M$ is a neighbourhood of the submanifold M in $TM := \bigcup_{x\in M} T_xM$, $exp_x := exp|_{T_xM}$, $\sigma(\tilde{x}) := \min(\sigma(x), \sigma(x'))$ and $\tau(x,x')$ is the bitensor of parallel transport as in [6, 21].

Let $C^t_\beta(M,N)$ denote a space of C^t_β mappings $f : M \to N$ topogolized with the help of a metric

$$d^t_\beta(f,g) := \sum_{i,j} \|f_{i,j} - g_{i,j}\|_{C^t_\beta(\mathbf{R^n},\mathbf{R^m})},$$

where $f_{i,j} = \tilde{\phi}_i \circ f \circ \phi_j^{-1}$ and it is implied that $f_{i,j}$ is defined on $\phi_j(U_j \cap f^{-1}(\tilde{U}_i))$ if it is nonvoid and $f_{i,j}$ is zero otherwise, $(\tilde{U}_j, \tilde{\phi}_j)$ is a finite atlas of N, $dim_{\mathbf{R}}N = m$, M and N are C^∞-manifolds satisfying conditions of §2.1. Then $C^\infty_\beta(M,N) := \bigcap_{l=1}^\infty C^l_\beta(M,N)$ is supplied with the topology given by a family of metrics $\{d^l_\beta : l \in \mathbf{N}\}$.

In view of the fundamental Theorem of Riemannian geometry on M exists the unique Levi-Civita connection $w = w_g$ for g. We assume that g is of the same class of smoothness as M and while consideration of the Hölder space $C^t_\beta(TM)$ let g be satisfying condition (ii):

(ii) $(g - e) \in C^\infty_\beta(M, S_2M)$, where e corresponds to the standard scalar product in $\mathbf{R^n}$. Let also g be elliptic, that is, there exists $c > 0$ such that

(iii) $ce(x)(\xi,\xi) \le g(x)(\xi,\xi)$ for each $\xi \in T_xM$ and $x \in M$.

NOTE 2.3. Let M be a Riemannian locally compact C^∞-manifold satisfying conditions of §2.1 and §2.2. Suppose P is the pseudodifferential operator corresponding to $(1+\Delta_g)$, where Δ_g is the Beltrami-Laplace operator defined on $C^2_\beta(TM)$ for M with the Riemannian metric g. There are pseudodifferential operators P^ϵ and $P^{1-\epsilon}$ such that $(P^\epsilon P^{1-\epsilon}-P)\in OPS^1_{1,0}$ by Theorems II.3.8 and II.4.4 [41]. Let $C^{-l-\epsilon,\delta}_\omega(TM)$ be a Banach space equal to the completion of a space $C^\infty_\omega(TM)$ with $\omega\in\mathbf{R}$, $\omega=\omega'+l+\epsilon+\delta$, relative to a norm

$$\|f\|_{C^{-l-\epsilon,\delta}_\omega(TM)}=\|g\|_{C^\delta_{\omega'}(TM)}$$

for each f with $f_{i,j}=\tilde P^{(l+\epsilon+\delta)/2}g_{i,j}$ for each i and j, where $g\in C^\infty_{\omega'}(TM)$, $\infty>l\ge 0$, $\tilde P$ corresponds to $(1+\Delta_e)$ for $\mathbf{R^n}$ as P for M.

LEMMA 2.4. *Let $C^{-l-\epsilon_i,\delta_i}_\omega(TM)=:X_{i,\omega}$ be given by Note 2.3 with $\epsilon_1+\delta_1=\epsilon_2+\delta_2$. Then $X_{1,\omega}$ is isomorphic with $X_{2,\omega}$.*

Proof. If $Q\in OPS^0_{1,0}$, then $Q:C^\epsilon(TM)\to C^\epsilon(TM)$ for compact M and $Q:C^\epsilon_\omega(TM)\to C^\epsilon_\omega(TM)$ for noncompact M is a continuous linear operator due to Theorems XI.2.1 and XI.2.2 in [41]. In fact, P^ϵ is given by Theorem XII.1.3 [41] as the function of the self-adjoint elliptic operator. For each pseudodifferential operator $P^{(\epsilon+\delta)/2}$ corresponding to the elliptic operator $(1+\Delta)^{(\epsilon+\delta)/2}$ there exists a decomposition for $\delta_1<\delta_2$:

$$P^{(\epsilon_1+\delta_2)/2}=P^{(\epsilon_1+\delta_1)/2}P^{(\delta_2-\delta_1)/2}\ (mod\ OPS^{\epsilon_1+\delta_2-1})$$

[37]. At the same time

$$\tilde P^\alpha : C^\beta_\gamma(T\mathbf{R^n})\to C^{\beta-\alpha}_{\gamma+\alpha}(T\mathbf{R^n})$$

is the isomorphism for $\gamma\ge 0$, $\mathbf{R}\setminus\mathbf{N}\ni\alpha>0$, $\mathbf{R}\setminus\mathbf{N}\ni\beta>\alpha$ by Theorem XI.2.5 [41]. On the other hand, there exists a linear topological isomorphism $J_\zeta:C^t_\beta(TM)\to C^t_{\beta+\zeta}(TM)$ given by the following formula $J_\zeta f(x):=\sigma(x)^{-\zeta}f(x)$, where $\zeta/2>t\ge 0$. Using charts (U_j,ϕ_j) and Note 2.3 we get the statement of this Lemma. □

NOTATION 2.5. In view of Lemma 2.4 we denote $C^{-l-\epsilon,\delta}_\gamma(TM)$ simply by $C^{-l-\epsilon}_\gamma(TM)$.

THEOREM 2.6. *Let M be a C^∞-manifold fulfilling conditions of §2.1 and §2.2. Suppose $s\in\mathbf{Z}$, $0<q<1$, $t=s+q$, $l>2j>s\ge 0$, $l\in\mathbf{N}$, $j\in\mathbf{N}$, $\beta\ge 0$, $2(\beta+j)>n$, $n=dim_{\mathbf{R}}M$ and*

$$\mathsf{A}_b|C^l_\beta(TM)=(\bar A_{1,b}+A_0),$$

where $0 \le b \in \mathbf{R}$, $\bar{A}_{1,b}$ *and* A_0 *are pseudodifferential operators corresponding to the following restrictions*

$$\bar{A}_{1,b}|C^l(TM)_\beta =< b >^{2j} +\Delta_g^j|C^l(TM)_\beta \text{ and}$$

$$A_0|C^l(TM) = \sum c(x;p(1),...p(n))\nabla_1^{p(1)}...\nabla_n^{p(n)},$$

$0 < p(1) + ... + p(n) = p < 2j$, $c(x;p(1),...,p(n)) \in C^z(T^*M)$ *with an order* $ord(A_0) < 2j$ *and* $z = l - 2j + p$, *where* T^*M *denotes a cotangent bundle for* M, $< b >:= (1 + b^2)^{1/2}$. *Then there are an extension of a linear operator* A_b *and* b_0 *with* $0 \le b_0 \in \mathbf{R}$ *such that*

$$\bar{\mathsf{A}}_b : C^t_\beta(TM) \to C^{t-2j}_{\beta+2j}(TM)$$

is an isomorphism for each $b \ge b_0$.

Proof. Let at first

$$A_{1,b}|C^l(TM)_\beta = (< b >^2 +\Delta_g)^j|C^l(TM)_\beta$$

and $\mathsf{A}_b|C^l_\beta(TM) = (A_{1,b} + A_0)$, where $0 \le b \in \mathbf{R}$. The corresponding pseudodifferential operator A_b is uniformly parameter-elliptic by Definition 1.3 in [17]. For weighted Sobolev and Hölder spaces there are inclusions $H^s_{2,\alpha}(TM) \supset C^{t'}_\beta(TM)$ for $\beta > \alpha + n/2$, $t' \ge s \in \mathbf{N}$, $\beta \in \mathbf{R}$, $t' > 0$; and $H^{[t]+[n/2]+1}_{2,\beta+[n/2]+1}(TM) \subset C^t_\beta(TM)$ for $\beta \in \mathbf{R}$ and $0 < t \in \mathbf{R} \setminus \mathbf{Z}$, where $[t]$ is the integral part of t, $[t] \le t$, $H^s_{2,\alpha}(TM)$ is defined as the completion of $C^s_{\alpha+n/2+1}(TM)$ relative to the norm

$$(\|f\|_{H^s_{2,\alpha}(TM)})^2 := \sum_{\xi=0}^{s}(\|\sigma(x)^{\alpha+\xi}\nabla^\xi f(x)\|_{L^2(\lambda)})^2,$$

where λ is the Riemannian volume element on M [21, 43]. By Theorem 1.7 [17] and [1] there exists $b_1 \ge 0$ such that

$$\mathsf{A}_b : H^{s,b}_{2,\alpha}(TM) \to H^{s-2j,b}_{2,\alpha+2j}(TM)$$

is the isomorphism for each $b \ge b_1$ and any $s \in \mathbf{N}$, where $H^{s,b}_{2,\alpha}(T\mathbf{R^n})$ is the Sobolev weighted space with parameter b, which has the norm

$$\|f\|_{H^{s,b}_{2,\alpha}(T\mathbf{R^n})} :=< b >^{-n/2+s} \|M_bf\|_{H^s_{2,\alpha}(T\mathbf{R^n})} \text{ and}$$

$$(M_b f)(x) := f(< b >^{-1} x).$$

Then there are inclusions $H^{s,b}_{2,\beta-n/2}(TM) \subset C^m_\beta(TM)$ for each $s > m+n/2$ and $\beta \ge 0$ (with the help of Formula (1.12) in [18]). Therefore, the restriction $\mathsf{A}_b|C^{t'}_\beta(TM)$ of A_b is the invertible operator. Hence there exists $b_1 \ge 0$ such that

$$(i)\ \mathsf{A}_b : C^{t'}_\beta(TM) \to C^{t'-2j}_{\beta+2j}(TM)$$

is the isomorphism for each $b \ge b_1$, where $t' = s' + q$, $l > t' > 2j$, $0 < q < 1$, since the proof in [6] may be easily generalized for each $t' > 2j$ using results of [1, 7].

Let $a(*)$ be a symbol of A_b and

$$a - a_1 a_2 \in S^{2j-1,\nu}_{1,0}(T^*M),$$

where

$$a \in S^{2j,2j-1}_{1,0}(T^*M), \quad a_1 \in S^{t",\nu_1}_{1,0}(T^*M),$$

$$a_2 \in S^{2j-t",\nu_2}_{1,0}(T^*M), \quad \max(\nu_1,\nu_2) \le 2j-1,\ 0 < |t - t"| < 1/4$$

and $0 < t" < 2j-1$ (see Definition 2.1 and Theorems 2.7, 5.1 in [18]).

Now we consider charts and the function $\bar{a}(x,\xi,b) =< \xi, b >^{-\alpha}$, $b \in \mathbf{R}$ with $0 < \alpha < n$, where $< \xi, b >:= (e(\xi,\xi)+ < b >^2)^{1/2}$, $e(*,*)$ is the standard scalar product in $\mathbf{R^n}$. Then the Fourier transform F_ξ in $\mathsf{L}^*(T\mathbf{R^n})$ by ξ is defined,

$$(ii)\ \hat{\bar{a}}(x,k,b) = F_\xi(\bar{a}(x,\xi,b))(k) \in L^{n/(n-\alpha)}_w$$

and for each $b \in \mathbf{R} \setminus \{\mathbf{0}\}$ it decreases exponentially by $k \in \mathbf{R^n}$ (see theorem IX.46 and exer. IX.50 in [35]). In view of Theorem XI.2.5 [41]:

$$(iii)\ \|OP(< \xi, b >^q)f\|_{L^2} \le C \times \|f\|_{C^q(TV)}.$$

Therefore, from the Hölder conditions and (iii) using principal symbols of pseudodifferential operators we have:

$$(iv)\ \|OP(a)f\| \le C' \times \|f\|_{C^t(TM)},$$

where $C' > 0$ is a constant (see also §V.4 [40]). Using Theorems about compositions of pseudodiffererntial operators and convolutions of generalised functions and functions we have, that for each $\alpha \in \mathbf{R}$ and $u = f \circ \phi_i^{-1} \in \mathsf{L}(T\mathbf{R^n})$ for each chart (U_i, ϕ_i) functions $OP(<\xi, b>^{-\alpha})u$ are defined, where $b \neq 0$.

At first we can consider a dense subspace $P(TM)$ of $C^t_\beta(TM)$ consisting of f with $f \circ \phi_i^{-1} \in \mathsf{D}(T\mathbf{R^n})$, for example, for functions f with supports $supp(f) \subset U_i$ and then their finite linear combinations, where $supp(f \circ \phi_i^{-1})$ are compact subsets, $supp(f) := cl\{x \in M : f(x) \neq 0\}$, $f : M \to TM$, $f(x) \in T_xM$. It is possible due to Condition (2) of Definition 2.2, since $\beta \geq 0$.

For $p(\xi, b) = \tilde{a}_1(\xi, b)\tilde{a}_2(\xi, b)$ with symbols $\tilde{a}_1$ and $\tilde{a}_2$ independent from $x \in \mathbf{R^n}$, for each $f, g \in \mathsf{D}(T\mathbf{R^n})$ we have

$$\int_{\mathbf{R^n}} [OP(\tilde{a}_1)f](y)[OP(\tilde{a}_2)g](y)dy =$$

$$= \int_{\mathbf{R^n}} [F^{-1}(\tilde{a}_1 F(f))](y)[F^{-1}(\tilde{a}_2(F(g))](y)dy =$$

$$= \int_{\mathbf{R^n}} \tilde{a}_1(\xi, b)\hat{f}(\xi)\tilde{a}_2(\xi, b)\hat{g}(\xi)d\xi = \int_{\mathbf{R^n}} [OP(\tilde{a}_1\tilde{a}_2)f](y)g(y)dy,$$

since the Fourier transforms F and F^{-1} are unitary operators on L^2, where $\hat{g}(\xi) = F(g)(\xi)$, $(g)^v(x) = [F^{-1}(g)](x)$, $x, y, \xi \in \mathbf{R^n}$. Therefore, for each $f, g \in C^\infty_\beta(TM)$:

$$(v) \quad \int_M [OP(\tilde{a}_1\tilde{a}_2)f](y)g(y)\lambda(dy) =$$

$$= \int_M [OP(\tilde{a}_1)f](y)[OP(\tilde{a}_2)g](y)\lambda(dy)$$

is the bilinear functional by f, g having the continuous extension on $C^{t"}_\beta(TM) \otimes C^{2j-t"}_\beta(TM)$ due to Theorem II.4.4 [40], the Lebesgue-Fubini Theorem and Lemma 2.4, since $2(\beta + j) > n$ and

$$\int_M < x >^{-2(j+\beta)} \lambda(dx) < \infty.$$

Then we take into account pseudodifferential operators dependent on x, using approximation of $a(x, \xi, b)$ by linear combinations of $c_j(x)d_j(\xi, b)$ with $c_j(x) \in C^l(M, \mathbf{R})$ and symbols $d_j(\xi, b)$ independent from x. Consequently, $\tilde{\mathsf{A}}_\beta := OP(a_1(x, \xi, b)a_2(x, \xi, b))$ with

$b \geq b_1$ and b_1 defined by formula (i) has the extension onto $C^t_\beta(TM)$ and

$$(vi)\ \tilde{\mathsf{A}}_b : C^t_\beta(TM) \to C^{t-2j}_{\beta+2j}(TM) \text{ is continuous, since}$$

$$|(\mathsf{A}_b f, g)| \leq C \times \|f\|_{C^{t"}_\beta(TM)} \|g\|_{C^{2j-t"}_\beta(TM)},$$

where $(*,*)$ denotes the scalar product on $L^2(M,\lambda)$, $C = const > 0$. The operator $\tilde{\mathsf{A}}_b$ is injective and differs with A_b on compact operator. There exists a continuous operator $\mathsf{B}_b = (\mathsf{A}_b | C^\infty_\beta(TM))^{-1}$ on $H := \mathsf{A}_b(C^\infty_\beta(TM)) \subset C^\infty_{\beta+2j}(TM)$ such that

$$(vii)\ \|f\|_{C^{t'}_\beta(TM)} \leq \|h\|_{C^{t'-2j}_{\beta+2j}(TM)} \text{ for } f = \mathsf{B}_b h,\ h \in H, \text{ as follows}$$

from the Oskolkov-Tarasov Theorem [33]. In view of formulas (v) and (vi) an operator $\mathsf{B}_b\mathsf{A}_b - I =: K_b$ is compact on $C^t_\beta(TM)$ into $C^t_\beta(TM)$. The operator B_b can be written as $\mathsf{B}_b = \mathsf{B}_{1,b} + \mathsf{B}_{0,b}$, where $\mathsf{B}_{1,b} := OP((< b >^2 + |\xi|^2)^{-j})$ with $|\xi|^2 = g_x(\xi,\xi)$ for each $x \in M$ and $\xi \in T_xM$. Evidently, $\mathsf{B}_{1,b}\mathsf{A}_b - I =: K_{2,b}$ is a compact operator and there exists a constant $C_2 > 0$ such that

$$\|K_{2,b}\|_{L(C^t_\beta(TM), C^t_\beta(TM))} < C_2 < b >^{-1}$$

for each $b > b_0$, since $< \xi >^k (< b >^2 + |\xi|^2)^{-j} \leq (< b >^2 + |\xi|^2)^{-1/2}$ for each $k = 0, 1, ..., 2j-1$ and each $x \in M$, $\xi \in T_xM$, where $L(X,Y)$ denotes the standard space of continuous linear operators from one real Banach space X into another real Banach space Y,

$$\|K\|_{L(X,Y)} := \sup_{x \in X, x \neq 0} (\|Kx\|_Y / \|x\|_X)$$

is a norm of an operator $K \in L(X,Y)$. Therefore, there exists a finite-dimensional subspace $\mathbf{R^m}$ in $C^t_\beta(TM)$ and $b_2 \geq 0$ such that

$$\|K_b | (C^t_\beta(TM) \ominus \mathbf{R^m})\|_{L(C^t_\beta(TM), C^t_\beta(TM))} < \epsilon$$

for each $b \geq b_2$, where $0 < \epsilon < 1/4$. For the restriction $K_b|\mathbf{R^m}$ it is lightly to find $b_3 \geq 0$ such that $\|K_b|\mathbf{R^m}\| < \epsilon$ for each $b \geq b_3$, since $K_b(\mathbf{R^m})$ is a finite-dimensional subspace of $C^t_\beta(TM)$. From

Lemma 2.4 it follows that $\tilde{P}^j C^t_\beta(TM) = C^{t-2j}_{\beta+2j}(TM)$, where e_x is a standard scalar product in $\mathbf{R^n}$, $\tilde{P} := OP((< b >^2 + e_x(\xi,\xi)))$. On the other hand, $\tilde{P}^j - \mathsf{A}_b =: K_{3,b}$ is a compact operator from $C^t_\beta(TM)$ into $C^{t-2j}_{\beta+2j}(TM)$ and the symbol of $K_{3,b}$ belongs to the class $S^{2j-1,2j-2}_{1,0}(T^*M)$, since $\Delta_g - \Delta_e$ is the operator of the first order. Analogously to the case of K_b for $K_{3,b}$ there exists $b_4 \ge 0$ such that

$$\|K_{3,b}\|_{L(C^t_\beta(TM),C^{t-2j}_{\beta+2j}(TM))} < \epsilon \|\mathsf{A}_{3,b}\|_{L(C^t_\beta(TM),C^{t-2j}_{\beta+2j}(TM))}$$

for each $b \ge b_4$, where $0 < \epsilon < 1/4$. Hence we can choose $b_0 = \max_{(j=1,\dots,4)}(b_j)$ such that

$$\mathsf{A}_b(C^t_\beta(TM)) = C^{t-2j}_{\beta+2j}(TM)$$

for each $b \ge b_0$.

If to consider $< b >^{2j} + \Delta_g^j$ instead of $(< b >^2 + \Delta_g)^j$ then these operators differ on the compact operator. Therefore, $\mathsf{A}_b - \bar{\mathsf{A}}_b =: \bar{K}_b$ differ on the compact operator and its symbol $k(x,\xi,b)$ belongs to the class $S^{2j-2,2j-2}_{1,0}(TM)$. Hence there exists $b_0 \ge 0$ such that

$$\|\bar{K}_b\|_{L(C^t_\beta(TM),C^{t-2j}_{\beta+2j}(TM))} < (\|\mathsf{A}_b\|_{L(C^t_\beta(TM),C^{t-2j}_{\beta+2j}(TM))})/2$$

and $\bar{\mathsf{A}}_b C^t_\beta(TM) = C^{t-2j}_{\beta+2j}(TM)$ for each $b \ge b_0$. □

3. Quasi-invariant measures on a group of diffeomorphisms.

At first we give few preliminary definitions and results. Then we formulate the main Theorem 3.10.

Definitions and Notes 3.1. Let U and V be open subsets in l_2, suppose $\theta : U \to V$ is a smooth mapping, $\infty > \delta \ge 0$. We define a uniform space $E^{\{r\},\theta}_{\gamma,\delta}(U,V)$ as a completion of a set Q relative to the family of metrics given below $[\chi_{r,\gamma,\delta} : r = r(n), n \in \mathbf{N}]$,

$$Q := [f : \ f \in E^{\infty,\theta}_{\infty,\delta}(U,V), \text{ there exists } n \in \mathbf{N} \text{ such that}$$

$$supp(f) \subset U \cap \mathbf{R^n}, \chi_{r,\gamma,\delta}(f,\theta) < \infty \text{ for each } r], \text{ where}$$

$$(i)\ \chi_{r,\gamma,\delta}(f,g) := \sup_{n\in\mathbf{N}} \sup_{x\in U} \rho^r_{n,\gamma,\delta}(f,g) < \infty \text{ and}$$

$$(ii)\ \lim_{R\to\infty} \chi_{r,\gamma,\delta}(f|_{U^c_R}, g|_{U^c_R}) = 0,$$

for f in $\rho^r_{n,\gamma,\delta}$ restrictions are taken corresponding to $U\cap\mathbf{R^n}$, that is $f|_{U\cap\mathbf{R^n}} : U\cap\mathbf{R^n}\to f(U)\subset V$,

$$\rho^r_{n,\gamma,\delta}(f,g) = \rho^r_{\gamma,\delta}(f|_{(U_j\cap\mathbf{R^n})}, g|_{(U_j\cap\mathbf{R^n})}),$$

$r = r(n) = t + 4m(n)n$, $2m(n) > n + [n/2] + 1$ for each n, $\gamma \ge 0$ (see §2.1 and §2.3 in [29]). Here $\rho^r_{n,\gamma,\delta}(f,g)$ is the metric by arguments $x^1,...,x^n$ for f and g as functions by $(x^1,...,x^n)$ in $E^r_{\gamma,\delta}(U\cap\mathbf{R^n}, V)$, $\mathbf{R^n} = X_n \hookrightarrow l_2$, X_n are subspaces in l_2, $X_n \hookrightarrow X_{n+1}$ for each n such that $\bigcup_{n\in\mathbf{N}} X_n$ is dense in l_2. Evidently, $E^{\{r\},\theta}_{\gamma,\delta'}(U,V) \subset E^{\infty,\theta}_{\gamma,\delta}(U,V)$ for each $\delta' > \delta + 1$, since

$$\sup_n \rho^l_{n,\gamma,\delta}(f,g) \ge \sup_n d^l_{n,\gamma,\delta}(f,g)$$

for each $0 \le l \in \mathbf{R}$ and

$$\sum_{(m_i\in\mathbf{N}, i=1,...,n; n\in\mathbf{N})} (m_1...m_n n^n)^{-1-\epsilon} < \infty$$

for each $0 < \epsilon \in \mathbf{R}$. We omit θ for $\theta = 0$.
Let $E^{\{r\},\theta}_{\infty,\delta}(U,V) := \bigcap_{\gamma\in\mathbf{N}} E^{\{r\},\theta}_{\gamma,\delta}(U,V)$.

Let a Riemann separable manifold M be modelled either on $\mathbf{R^n}$ or l_2 and fulfils conditions of §2.2 and §2.4 [29], $At(M)$ be finite and

$$(\phi_j\circ\phi_i^{-1} - id_{i,j}) \in E^{\{r'\}}_{\infty,\chi}(U_{i,j}, l_2) \text{ for each } U_i\cap U_j \ne \emptyset,$$

a metric g is of class $E^{\{r'\}}_{\infty,\chi}$, where $U_{i,j}$ are open in l_2 domains of $\phi_j\circ\phi_i^{-1}$, $r'(n) \ge r(n) + 2$, for each n, $\infty > \chi > \delta + 1$.

Definitions and Notes 3.2. Let (M,g) fulfil conditions of §2.2 and §2.4 in [29] and §3.1 above, $1 \le q \le t$, $0 \le \gamma \le \beta$. For $f \in G$ we can define

$$D_\xi\zeta(x) = (f(x), \nabla_{f_*\xi}\theta) \in T^r_s(M)|f(x), \text{ where}$$

$\zeta(x) = (f(x), \theta(x))$, $\theta(x) \in \mathbf{T}^r_s(l_2)$, $r, s \in \mathbf{N_o}$, $\mathbf{N_o} := \{0\}\cup\mathbf{N}$,

$\mathbf{N} := \{1,2,3,...\}$, $\xi \in E^t_{\beta,\delta}(TM) := [\xi \in E^t_{\beta,\delta}(M,TM)|\ \pi(\xi(x)) = x]$

is the space of vector fields on M of the class $E^t_{\beta,\delta}$,

$$\zeta \in {}_fE^q_{\gamma,\delta}(M,T^r_s(M)) := [\zeta \in E^q_{\gamma,\delta}(M,T^r_s(M)) : \pi(\zeta(x)) = f(x) \text{ for each } x \in M],$$

∇ is the covariant differentiation of tensor fields over M, $f_*\xi$ denotes the push-forward of ξ, that is, related with a pull-back f^* by $f^* = f_*^{-1}$ (see §3.9(iv) in [14]), $\mathbf{T}^r_s(l_2)$ is the tensor space of type (r,s) over l_2, $T^r_s(M)$ is the tensor bundle of type (r,s) over M, TM corresponds to $(r,s) = (1,0)$ [21, 22], $(df)\xi := f_*\xi =: D_\xi f$, $(D\zeta)(\xi) := D_\xi(\zeta(x))$, $(\nabla_\xi\theta)(X_1,...,X_s) = (\nabla\theta)(X_1,...,X_s;\xi)$, $X_j(x) \in T_xM$, $X_j \in \Xi(M)$, $j = 1,...,s \in \mathbf{N}$, $r \in \mathbf{N}$, $(df)^{-1}\zeta(x) := (df[\pi(\zeta(x))])^{-1}\zeta(x) \in T_{f^{-1}(\pi(\zeta(x)))}M$; df and $\nabla^m df$ are well defined for $f \in G$ analogously to §4 [10]. Indeed, the differential $f_* = df$ is a section of $T^*M \otimes f^*TM$ (with the induced connection in f^*TM).

LEMMA 3.3. *In the notation of §3.2 $D_\xi\zeta(x) \in {}_fE^{q-1}_{\gamma+1,\delta}(M,T^r_s(M))$ and D_ξ, $(df)^{-1}$ are continuous mappings of ${}_fE^q_{\gamma,\delta}(M,T^r_s(M))$ into ${}_fE^{q-1}_{\gamma+1,\delta}(M,T^r_s(M))$.*

Proof. Let $[M_k : k = k(n), n \in \mathbf{N}]$ be a sequence of submanifolds as in §3.2 [29] with atlases $At(M_k) = [(U_{j,k},\phi_j) : j] = At(M) \cap M_k$, that is $U_{j,k} = U_j \cap M_k$ for each j,k. From Definitions in §2, Lemma 3.2, Theorems 3.1 and 3.3 in [29] and §3.2 above it follows that D_ξ and $(df)^{-1}$ are continuous, since $f_*\xi \in E^{t-1}_{\beta+1,\delta}(M,TM)$ and $(df)^{-1}$ corresponds to $f^* = f_*^{-1}$, $t \ge q \ge 1$. □

NOTE 3.4. Let D_ξ be as in §3.2, f and $\phi \in Diff^t_{\beta,\delta}(M)$ with $0 \le t < \infty$ and $\infty > \beta \ge 0$, $\infty > \delta \ge 0$, where M is a Hilbert manifold and g is a Riemannian metric as in §2.2 and §2.4 [29] and §3.1 above. Then for each f and $\phi \in Diff^t_\beta(M)$ and $[t] \ge l$ in view of Theorem 2.5 in [2], §5.1-5.3 in [36] and Lemma 3.3 it follows the equality (3.1):

$$(3.1)\quad \sum_{\sigma\in S_l} D_{\xi_{\sigma(1)}}...D_{\xi_{\sigma(l)}}(\phi\circ f) = \sum_{\omega(l)}[l!/(i_1!...i_m!(l_1!)^{i_1}...(l_m!)^{i_m})]$$

$$\sum_{\sigma\in S_l}\tilde{S}D^{i_1+...+i_m}\phi(D_{\xi_{\sigma(1)}}...D_{\xi_{\sigma(l_1)}}f,...,D_{\xi_{\sigma((i_1-1)l_1+1)}}...D_{\xi_{\sigma(i_1l_1)}}f,$$

$$D_{\xi_{\sigma(i_1l_1+1)}}...D_{\xi_{\sigma(i_1l_1+l_2)}}f,...,D_{\xi_{\sigma(i_1l_1+...+i_{m-1}l_{m-1}+(i_m-1)l_m+1)}}...D_{\xi_{\sigma(l)}}f),$$

where $D_\xi f = f_*\xi$, $D_{\xi_l}...D_{\xi_2}D_{\xi_1}f = \nabla_{\phi_* f_*\xi_l}...\nabla_{\phi_* f_*\xi_2} f_*\xi_1$ for $l > 1$, $D_\xi \phi \circ f = \phi_* f_* \xi$, S_l is the symmetric group of $\{1,...,l\}$ elements of which are considered as bijective mappings $\sigma : \{1,...,l\} \to \{1,...,l\}$, $\tilde{S}$ is the symmertizer of $(D^{i_1+...+i_m}\phi)(a,...,z)$ by all arguments $(a,...,z) = (D_{\xi_{\sigma(1)}}...D_{\xi_{\sigma(l_1)}}f, ..., D_{\xi_{\sigma(k-l_m+1)}}...D_{\xi_{\sigma(l)}}f)$,

$$\tilde{S}g(a_1,...,a_p) := \sum_{\sigma \in S_p} g(a_{\sigma(1)},...,a_{\sigma(p)})$$

for a function g of arguments $a_1,...,a_p$; $\sum_{\omega(l)}$ denotes the summation by all partitions $\omega(l)$ of l, that is by all representations of l as $l_1 i_1 + ... + l_m i_m = l$, $i_j > 0$ for each $j = 1,...,m$, $m \ge 1$, $l_1 \ge l_2 \ge ... \ge l_m > 0$.

Definition 3.5. Let (M,g) and D_ξ be as in §3.2, let us denote with the help of §3.4 the following expression

$$(3.2)\ D_{f,\xi_1,...,\xi_l}\phi(\zeta) := \sum_{\omega(l)} [l!/(i_1!...i_m!(l_1!)^{i_1}...(l_m!)^{i_m})] \times$$

$$\sum_{\sigma \in S_l} \tilde{S} D^{i_1+...+i_m+1}\phi(\zeta, D_{\xi_{\sigma(1)}}...D_{\xi_{\sigma(l_1)}}f, ..., D_{\xi_{\sigma(k-l_m+1)}}...D_{\xi_{\sigma(l)}}f,$$

where $\zeta \in {}_fE^q_\gamma(M, T^r_s(M))$, $[q] \ge l$, $\beta \ge \gamma \ge 0$.

Lemma 3.6. *Let (M,g) and $\{M_k : k = k(n), n \in \mathbf{N}\}$ be as in §2.2 and §2.4 [29] and §3.1 above with each atlas $At(M_k)$ inherited from $At(M)$. Then there exists the locally finite partition of unity $\{\psi_i : i \in J\}$, $J \subset \mathbf{N}$, for M such that*

(i) $V_i \subset supp(\psi_i) \subset U_{p(i)}$, V_i are open, $\phi_{p(i)}(V_i)$ are locally convex, $p = p(i) \in \{i,...,s\}$, $\bigcup_{i \in J} V_i = M$;

(ii) vector fields $[\xi_{l,i} : i \in \mathbf{N}]$ are in $\Xi(M)$ of class $E^{\{r'\}}_{\gamma',\chi}$, $supp(\xi_{l,i}) \subset supp(\psi_l)$, $\xi_{l,i} \in \Xi(M_k)$ for each $i \le k$;

(iii) $[\xi_{l,i}(x) : i \in \mathbf{N}]$ is a linear basis in T_xM for each $x \in V_l$;

(iv) for each $l \in J$ there exists $x \in V_l$ with $\xi_{l,i}(x) = e_i$ for every i, where e_i is the standard basis in l_2;

(v) $$1/2 \le \inf_{l,i} \|\xi_{l,i}\|_{E^{t'}{}_{\beta',\chi}(TV_l)} \le \sup_{l,i} \|\xi_{l,i}\|_{E^{t'}{}_{\beta',\chi}(TV_l)} \le 2 \text{ and}$$

$\sup_{i \ne j,\ l \in J,\ x \in V_l,\ i}$ *and* $_{j \in \mathbf{N}}$ $|(\xi_{l,i}(x), \xi_{l,j}(x))_{l_{2,\chi}}| < 1/2$ *for some $t' \ge t$, $\beta' \ge \gamma' \ge 0$.*

Proof. The manifold (M, g) is Riemannian and modelled on l_2, hence it possesses the partition of unity $\{\xi_l : l \in J\}$, $J \subset \mathbf{N}$, of class $E^{\{r'\}}_{\infty,\chi}$ fulfilling (i) due to §2.3 in [25], that is, $\sum_{l\in J} \psi_l(x) = 1$ for each $x \in M$, $\psi_l(x) \ge 0$ for each l and x, $supp(\psi_l) := cl(x \in M : \psi_l(x) \ne 0) \subset U_{p(l)}$, $cl(B)$ denotes the closure of $B \subset M$. Let $\{M_k : k = k(n), n \in \mathbf{N}\}$ be as in 3.2 then exp for M induces exp for M_k as restrictions on corresponding neighbourhoods of the zero sections in TM_k. Therefore, the Gaussian coordinates in M induce corresponding coordinates in M_k, since each $M_{k(n)}$ has tubular neighbourhoods in $M_{k(n+j)}$ (for $j > 0$) and in M (§4.4-4.6 [25]). Hence for each $\xi \in \Xi(M_k)$ there is the equality $\tilde{\xi}(x) = \sum_{l\in J} \tilde{\xi}_l(x)$, where $\tilde{\xi}_l(x) = \psi_l(x)\xi(x)$. There are embeddings $\Xi(M_{k(n)}) \hookrightarrow \Xi(M_{k(n+1)}) \hookrightarrow ...\Xi(M)$ due to conditions of being Hilbertian at infinity for M and conditions of §2.4 [29] on g.

Then with the help of Gaussian coordinates, the base $\{e_j : j \in \mathbf{N}\}$ in l_2 and parallel translation along geodesics we can choose by induction $[\xi_i : \xi_i(x) \in T_xM_i$ for each $x \in M_i$ and of class $E^{\{r'\}}_{\{\gamma'\},\chi}$ on $M]$ with $\xi_{l,i} = \psi_l\xi_i$ such that to satisfy (ii-v), since M_k and M are geodesically complete, $(\phi_j \circ \phi_i^{-1} - id)$ are in the class $E^{\{r'\}}_{\gamma',\chi}$ for each $U_i \cap U_j \ne \emptyset$, $T_{\phi_j,\phi_i} := D(\phi_i \circ \phi_j^{-1})$, $T_{\phi_j,\phi_i} - I$ are in the class $E^{\{r'(n)-1:n\}}_{\gamma'+1,\chi}$, T_{ϕ_j,ϕ_i} are the unitary operators in l_2 for each z in the domain of $\phi_i \circ \phi_j^{-1}$, fibers in the tangent bundle TM are isomorphic to l_2, fibers in TM_i are isomorhic to $\mathbf{R^i}$, $\mathbf{R^i} \hookrightarrow l_2$ (see Chapter VII in [25] and Chapter I in [14]), where $I : l_2 \to l_2$ is the identity operator. □

Definitions and Notes 3.7. Let $[\xi_{l,i} : l, i]$ be the same as in Lemma 3.6 $f \in Diff^q_{\beta,\delta}(M)$, $q > deg(A_{n;m(n)})$. Let us define operators:

$$(3.3)\ A_{n;m(n)}(f(x)) =$$

$$< b(n) > \tilde{E}^{-1}(f_n) + \sum_{l\in J} \psi_l(x) A_{n;m(n),f}(\xi_{l,1}, ..., \xi_{l,n}),$$

where

$$A_{n;m(n),f}(\xi_{l,1}, ..., \xi_{l,n}) := \sum_{s=1}^{n} F_{s,n,m(n)}(g^{q(1),j(1)}_{x,n}, ..., g^{q(a),j(a)}_{x,n};$$

$$D^{2i(1)}_{\xi_{l,i}},...,D^{2i(s-1)}_{\xi_{l,s-1}},D^{2i(s+1)}_{\xi_{l,s+1}},...,D^{2i(n)}_{\xi_{l,n}})\times$$

$$\sum_{p=0}^{2m(n)-1}\alpha(p,2i(s))D^p_{\xi_{l,s}}[(df_n)^{-1}\circ D^{2i(s)-p}_{\xi_{l,s}}f_n]$$

and $f_n := f|M_n$ are restrictions of f on the submanifolds M_n with $s := \min_{i_j\ge 2m(n)} j$. Also $deg(A_{n;m(n)})$ is a degree of $A_{n;m(n)}$ as the differential operator, n and $m(n)\in \mathbf{N}$, $\alpha(p,j)\in \mathbf{Q}$, $a=i(1)+...+i(n)=2m(n)n$, $g^{i,j}_{x,n}$ are components of $g^{-1}_{x,n}$ on M_n, the Riemannian metric $g_{x,n}$ on M_n is induced by g_x on M for each $x\in M_n$, $(g_{x,n})^{i,j}=g_{x,n}(\partial/\partial x^i,\partial/\partial x^j)$, $F_{s,n,m(n)}$ are operators of polynomial types by $g^{i,j}_{x,n}$ and $D_{\xi_{r,p}}$, $0\le b(n)\in \mathbf{R}$. Let also $\tilde{\Delta}_n$ be the Beltrami-Laplace operator for M_n given with the help of §3.2 [29] (see also [22]) and Ω_n be a linear differential operator by the first argument

$\zeta\in E^{\infty}_{\infty}(M_n|TM) := \{\xi : M_n\to TM|\pi(\xi(x))=x$ for each $x\in M_n$,

$\xi\in E^{\infty}_{\infty}(M_n,TM)\}$ with $deg(\Omega_n)<4m(n)n$, $\phi\in Diff^{q+2m(n)}_{\beta,\delta}(M)$,

$supp(\phi) := cl(x\in M : \phi(x)\ne x)\subset \phi_j^{-1}(\phi_j(U_j)\cap \mathbf{R^n})$ for some $j\in\{1,...,s\}$, W is some open neighbourhood of id in $Diff^q_{\beta,\delta}(M)$, $\phi\in W$, $f\in W$, $\tilde{E}(Y)=W$, Y is an open neighbourhood of 0 in $T_{id}Diff^q_{\beta,\delta}(M)$ (see §3.3 in [29]). When $q=t<deg(A_{n,m(n)})$, then $A_{n,m(n)}$ is considered as the corresponding pseudodifferential operator.

LEMMA 3.8. *Let the operator $A_{n;m(n)}(f)$ and f be the same as in §3.7. Then there exists $F_{s,n,m(n)}$ and $\alpha(p,j)$ such that $A_{n;m(n)}(\tilde{E}(\hat{V}))$ is the continuously Frechét differentiable by $\hat{V}$ mapping from $Y\otimes(E^q_{\beta,\delta}(TM))^{\otimes 4m(n)n}$ into $E^{q-4m(n)n}_{\beta+4m(n)n,\delta}(M,TM)$ with*

$$\nabla_{\hat{V}}A_{n;m(n)}\circ\tilde{E}|_{\hat{V}=0}=$$

$$=\tilde{\Delta}^{2m(n)n}_n+\Omega_n :\ E^q_{\beta,\delta}(TM)\otimes(E^q_{\beta,\delta}(TM_n))^{\otimes 4m(n)n}\to$$

$$E^{q-4m(n)n}_{\beta+4m(n)n,\delta}(TM),$$

$$\nabla_{\hat{V}}A_{n;m(n)}(\tilde{E}(\hat{V}))|_{(f=\tilde{E}(\hat{V}))} :\ {}_fE^q_{\beta,\delta}(TM)\otimes(E^q_{\beta,\delta}(TM_n))^{\otimes 4m(n)n}\to$$

$$ {}_fE^{q-4m(n)n}_{\beta+4m(n)n,\delta}(TM), $$

and

$$ A_{n;m(n)}(\phi\circ f)-A_{n;m(n)}(f)\in E^{q-4m(n)n+2m(n)}_{\beta+4m(n)n-2m(n),\delta}(M,TM). $$

Proof. For the submanifold M_n in M with the covariant differentiation ${}_n\nabla$ and the torsion tensor $T^p_{l,r}=0$ there is the equality ${}_n\nabla_X g^{i,j}_{x,n}={}_n\nabla_X(g_{x,n})_{i,j}=0$ for each $X\in\Xi(M_n)$. In view of Proposition III.7.6 [22] the curvature tensor field for M_n is given by the equation

$$ R^i_{j,k,l}=(\partial\Gamma^i_{l,j}/\partial x^k-\partial\Gamma^i_{k,j}/\partial x^l)+\sum_m(\Gamma^m_{l,j}\Gamma^i_{k,m}-\Gamma^m_{k,j}\Gamma^i_{l,m}) $$

in local coordinates (x^j), where

$$ \sum_l g_{l,k}\Gamma^l_{j,i}=(\partial g_{k,i}/\partial x^j+\partial g_{j,k}/\partial x^i-\partial g_{j,i}/\partial x^k)/2, $$

here $g=g_{x,n}$ (see corollary IV.2.4 [22] and §3.3 above); or

$$ R(X,Y,Z)_{\phi(p)}=D\Gamma_{\phi(p)}X_{\phi(p)}(Y_{\phi(p)},Z_{\phi(p)})-D\Gamma_{\phi(p)}Y_{\phi(p)}(X_{\phi(p)}, $$

$$ Z_{\phi(p)})+\Gamma_{\phi(p)}(X_{\phi(p)},\Gamma_{\phi(p)}(Y_{\phi(p)},Z_{\phi(p)}))- $$

$$ \Gamma_{\phi(p)}(Y_{\phi(p)},\Gamma_{\phi(p)}(X_{\phi(p)},Z_{\phi(p)})) $$

for M in local coordinates in infinite-dimensional case (§8.3 in [14] and Lemma 1.5.3 in [21]). Consequently, the Riemannian connection Γ in M_k and R are in the class $E^\infty_{\infty,\chi}$, $\chi\ge\beta$. For tensor fields $S_{j(1),...,j(b)}$ on M_k in the normal local coordinates we have

$$ [\nabla_r,\nabla_k]S_{j(1),...,j(b)}= $$

$$ -R^p_{j(1),r,k}S_{p,j(2),...,j(b)}-...-R^p_{j(b),r,k}S_{j(1),...,j(b-1),p}, $$

where $[\nabla_r,\nabla_k]=\nabla_r\nabla_k-\nabla_k\nabla_r$. Then

$$ D^j_\xi(\phi\circ f)(x)=(\phi_*)(f(x))(D^j_\xi f(x))+ $$

$$ \sum_{k=1}^{z-1}\binom{j-1}{k}[D^k_\xi(\phi_*)(f(x))]D^{j-k}_\xi f(x)+B_{p,j}(x), $$

where

$$B_{p,j}(x)=\sum_{k=z}^{j}\binom{j-1}{k}(D_\xi^k[(\phi_*)(f(x))])(D_\xi^{j-k}f(x)),$$

$z=2m(n)$, $\xi\in\Xi(M_n)$, $\sum_1^0 :=0$ for $j=1$.

There are constant coefficients $\alpha(j,u)$ fulfilling the following system of linear algebraic equations

$$\sum_{j=d}^{p}\binom{u-j}{w-d}\binom{j}{d}\alpha(j,u)=0$$

with $d=0,...,w-1$, $w=1,..,\min(2m(n)-1,u)=:p$ and

$$\sum_{j=0}^{p}\alpha(j,u)=1,$$

since this system is equivalent to

$$\sum_{j=k}^{p}\binom{u-k}{j-k}\alpha(j,u)=1$$

for $k=0,1,...,p$, where $u\ge p$,

$$det\{\binom{u-k}{j-k}\}_{j,k}\ne 0,$$

$$\binom{a}{d}=0 \text{ for } a<d \text{ or } d<0,\ \binom{a}{d}:=a!/(d!(a-d)!)$$

for $d=0,...,a$ are the binomial coefficients.

Using the following facts (i-vi):

(i) the equality $[\nabla_i^p,\nabla_j]=\sum_{a=0}^{p-1}\nabla_i^a[\nabla_i,\nabla_j]\nabla_i^{p-a-1}$ for $p=2,3,...$, $\nabla^0:=I$ for infinitely differentiable vector fields;

(ii) the corresponding to ∇ pseudodifferential operators with additional terms belonging to $S_{1,0}^{-\infty}$ with the well-known rules for their compositions [18, 16];

(iii) the coefficients $\alpha(j,u)$ as above;

(iv) smoothness of Γ and R;

(v) the expression of the Beltrami-Laplace operator in normal coordinates for the Levi-Civita connection in M_k: $\tilde{\Delta}_k = (g_{x,k})^{i,j}\nabla_i\nabla_j$ (see Note 14 in v.2 [22]);

(vi) Lemma 3.2 [29] and Lemma 3.3 above -

we can find polynomials $F_{s,n,m(n)}$ by $D_{\xi_{l,i}}$ and g with coefficients depending on x as functions in $E^{\infty}_{\infty,\chi}(M_k, \mathbf{R})$ such that to fulfil demands of Lemma 3.8, since $B_{p,i}(x)$ are polynomials of $D^j_\xi f$ with $j = 1, ..., i - 2m(n)$ and $(D^j_\xi \phi)(f(x))$ with $1 \le j \le i$.

The differentiability by $\hat{V}$ follows from the existence of $E^{\{r'(n)-2:n\}}_{\infty,\chi}$-mapping of some neighbourhood Y of 0 in the Banach space $T_{id}Diff^q_{\beta,\delta}(M)$ onto a neighbourhood W of id in $Diff^q_{\beta,\delta}(M)$, where $E^{\{r'(n)-2:n\}}_{\{\infty,\chi} \subset E^{\infty}_{\infty,\chi}$. Indeed, there is a neighbourhood W of $id \in Diff^q_{\beta,\delta}(M)$ such that $W^2 \subset U$ and it is given with the help of the mapping $\tilde{E}$ from §3.3(v) [29] (analogously for the class of the smoothness of M considered here). Hence the differentiability by $f \in W$ reduces to the differentiability of $A_{n;m(n)} \circ \tilde{E}$ by $\hat{X} \in Y$ (see also Chapter 1 in [21]). □

Definitions and Notes 3.9. Let $G := Diff^t_{\beta,\gamma}(M)$ be a group of diffeomorphisms. It has also a structure of a real smooth Banach manifold. Let TG be its tangent space, $T^k f := (f, Df, ..., D^k f)$; $f \in G$, where $T^k f := T(T^{k-1} f)$ for each $k \in \mathbf{N}$, $T^0 f = f$, $Df := f_*$ is the differential of f (see [21, 25] and §3.3). For a σ-additive measure $\mu : Af(G, \mu) \to [0, \infty) \subset \mathbf{R}$ a σ-additive measure, its left shifts $\mu_\phi(E) := \mu(\phi^{-1} \circ E)$ are considered for each $E \in Af(G, \mu)$ and $\phi \in G$, where $Af(G, \mu)$ is the completion of the Borel σ-field $Bf(G)$ on G by μ-null sets, $\phi \circ E := \{(\phi \circ h) : h \in E\}$. Then μ is called quasi-invariant if there exists a dense subgroup G' such that μ_ϕ is equivalent to μ for each $\phi \in G'$. Henceforth, we assume that a quasi-invariance factor $\rho_\mu(\phi, g) = \mu_\phi(dg)/\mu(dg)$ is continuous by $(\phi, g) \in G' \times G$, $\mu(V) > 0$ for some (open) neighbourhood $V \subset G$ of id, where $id = e$ is the unit element in G and $\mu(G) < \infty$.

Let (M, F) be a space M of measures on $(G, Bf(G))$ with values in $\mathbf{R}$ and G" be a dense subgroup in G such that a topology T on M is compatible with G", that is, $\mu \to \mu_h$ is the homeomorphism of (M, F) into itself for each $h \in G$". Let T be the topology of convergence for each $E \in Bf(G)$. Let $\Xi(G")$ denotes the set of all differentiable vector fields X on $G"$, that is, X are sections of the tangent bundle $TG"$. We say that the measure μ is continuously differentiable if there exists its tangent mapping $T_\phi \mu_\phi(E)(X_\phi)$ corresponding to the strong differentiability relative to the Banach structures of the manifolds $G"$ and $TG"$. Its differential we denote $D_\phi \mu_\phi(E)$, so $D_\phi \mu_\phi(E)(X_\phi)$ is the σ-additive real measure by $E \in Af(G,\mu)$ for each $\phi \in G"$ and $X \in \Xi(G")$ such that $D\mu(E) : TG" \to \mathbf{R}$ is continuous for each $E \in Af(G,\mu)$, where $D_\phi \mu_\phi(E) = pr_2 \circ (T\mu)_\phi(E)$, $pr_2 : p \times \mathbf{F} \to \mathbf{F}$ is the projection in TN, $p \in N$, $T_pN = \mathbf{F}$, N is another real Banach differentiable manifold modelled on a Banach space $\mathbf{F}$. For a differentiable mapping $F : G" \to N$ by $TF : TG" \to TN$ is denoted the corresponding tangent mapping, $(T\mu)_\phi(E) := T_\phi \mu_\phi(E)$. Then by induction μ is called n times continuously differentiable if $T^{n-1}\mu$ is continuously differentiable such that $T^n\mu := T(T^{n-1}\mu)$, $(D^n\mu)_\phi(E)(X_{1,\phi},...,X_{n,\phi})$ are the σ-additive real measures by $E \in Af(G,\mu)$ for each X_1,...,$X_n \in \Xi(G")$, where $(X_j)_\phi =: X_{j,\phi}$ for each $j = 1,...,n$ and $\phi \in G"$, $D^n\mu : Af(G,\mu) \otimes \Xi(G")^n \to \mathbf{R}$.

THEOREM 3.10. *Let M be a $E^{\{r'\}}_{\infty,\chi}$-manifold as in §3.1 and $G := Diff^t_{\beta,\gamma}(M)$. Then G has quasi-invariant infinitely differentiable probability measures μ relative to dense subgroups G', where $\chi > \gamma + 2$, $\gamma \ge 0$.*

Proof. Let at first $1 < t \in \mathbf{R} \setminus \mathbf{Z}$. In view of Theorem 2.6 and Lemma 3.8 each operator $A_{n;m(n)} \circ \tilde{E}$ from $Y \otimes (E^t_{\beta,\gamma}(TM))^{\otimes 4m(n)n}$ into $E^{t(n)}_{\beta(n),\gamma}(M,TM)$ is continuously differentiable by $\hat{V} \in Y$, where $t(n) = t - s(n)$, $\beta(n) = \beta + s(n)$, $s(n) = 4m(n)n$, $n = n(p)$, $p \in \mathbf{N}$, Y is an open neighbourhood of id in $T_{id}Diff^t_{\beta,\gamma}(M)$. Suppose that $b(n) \ge 0$ are chosen in accordance with Theorem 2.6 and Lemma 3.8 such that

$$\tilde{\Delta}^{2m(n)n}_n + \Omega_n : C^t_\beta(T\dot{M}_n) \to C^{t-4m(n)n}_{\beta+4m(n)n}(T\dot{M}_n)$$

are the linear topological isomorphisms for $\dot{M}_n := M_n \setminus \partial M_n$ Euclidean at infinity due to §§2.1, 2.2 and 3.1 for each $n \in \mathbf{N}$.

There exists a subgroup G_0 in G such that G_0 consists of finite compositions of elements $g \in G$ with supports $supp(g) \subset U_{j,n}$, where

$U_{j,n} = U_j \cap M_n$, $j = 1,..,k$ and $n \in \mathbf{N}$ are dependent from g, $supp(g) := cl\{x \in M : g(x) \neq x\}$, $cl(v)$ denotes the closure of a subset v in M. Then G_0 is dense in G. An operator A defined below is written at first for a neighbourhood $Y \cap T_{id}G_0$ of 0 in $T_{id}G_0$, then it is extended onto a neighbourhood Y of 0 in $T_{id}G$. Choose a sequence $\{B_p : B_p > 0, p \in \mathbf{N}\}$ such that the following operator $A(\psi)$ is well defined on Y,

$$A(\psi) := \sum_{p=1}^{\infty} B_p A_{n(p);m(n(p))}(\psi) {e'}_p \in H,$$

where $\{{e'}_p : p \in \mathbf{N}\}$ is the standard orthonormal base in l_2,

$$H := \{\xi = ({e'}_p B_p A'_{n(p);m(n(p))}\zeta : \ p \in \mathbf{N}) | \zeta \in E^t_{\beta,\gamma}(TM)\}$$

is a Banach space with the following norm

$$\|\xi\|_H := \|\zeta\|_{E^t_{\beta,\gamma}(TM)}$$

and B_n are chosen such that $A : Y \to H$ is the local uniform isomorphism, $At(M)$ is finite, $\chi > \gamma" > \gamma' + 2$. In H is dense a direct sum of spaces

$$\bigoplus_{p=1}^{\infty} (E^{t(n(p))}_{\beta(n(p)),\gamma}(TM) \otimes {e'}_p) =: Z$$

[32]. From Lemma 3.2 and Theorem 3.3 [29] it follows that there are U and W such that $G' := Diff^{\infty}_{\infty,\gamma"}(M)$ acts uniformly continuous from the left on $W \subset U$, where U and W are neighbourhoods of id in $G := Diff^t_{\beta,\gamma}(M)$. There is a neighbourhood P of id in G' such that $PW \subset U$. In view of Lemma 3.8 for each $\phi \in P$ the operator $S_\phi(h) := A[\phi(A^{-1}(h))] - h$ is nuclear on $V_H := A(W)$ with values in H, where $h \in V_H$. There are $B_p > 0$ such that $A : Y \to H$ is the local uniform diffeomorphism, since $\chi > \gamma" > \gamma' + 2$.

Let $\{H_n : n = n(p), p \in \mathbf{N}\}$ be a sequence of Hilbert spaces over $\mathbf{R}$, then $l_{2,\gamma}\{H_n : n = n(p), p \in \mathbf{N}\}$ denotes a Hilbert space with elements $x = (x_n : x_n \in H_n$ for each $n)$ having a finite norm

$$\|x\| := (\sum_{p \in \mathbf{N}, n=n(p)} \|n^\gamma x_n\|^2_{H_n})^{1/2} < \infty,$$

where $\gamma \ge 0$. Therefore, H contains a dense Hilbert subspace

$$X = l_{2,\gamma+1+\epsilon}\{H_n : n = n(p), p \in \mathbf{N}\}$$

for each $0 < \epsilon \in \mathbf{R}$, where H_n are isomorphic with the corresponding weighted Sobolev spaces $H^{[t+1]+[n/2]+1-4m(n)n,b(n)}_{2,\beta+4m(n)n-n/2}(TM_n)$ as in §2.6. Let

$$\|f\|'_{H^{s,b}_{2,\beta}(\mathbf{R^n},\mathbf{R^n})} :=$$

$$(\sum_{|\alpha|\le s} < b(n) >^{s-n/2} \|\bar{n}^{\alpha\gamma} < x >^{\beta+|\alpha|} D^\alpha_x(\mathsf{M}_b f)(x)\|^2_{L^2(\mathbf{R^n},\lambda,\mathbf{R^n})})^{1/2},$$

where $0 \le s \in \mathbf{Z}$, $0 \le \gamma \in \mathbf{R}$, $\beta \in \mathbf{R}$, $\alpha = (\alpha^1, ..., \alpha^n)$, $0 \le \alpha^j \in \mathbf{Z}$, $|\alpha| = \alpha^1 + ... + \alpha^n$, $\bar{n}^{\alpha\gamma} = 1^{\alpha^1\gamma}2^{\alpha^2\gamma}...n^{\alpha^n\gamma}$, λ is the Lebesgue measure on $\mathbf{R^n}$, $f : \mathbf{R^n} \to \mathbf{R^n}$, $(\mathsf{M}_b f)(x) := f(< b >^{-1} x)$. Using charts $(U_{j,n}, \phi_j)$ of the atlas $At(M_n) = \{(U_{j,n}, \phi_j) : j = 1, ..., k\}$ we get

$$\|f\|'_{H^{s,b}_{2,\beta}(TM_n)} := (\sum_{i,j=1}^{k} (\|\tilde{f}_{i,j}\|'_{H^{s,b}_{2,\beta}(\mathbf{R^n},\mathbf{R^n})})^2)^{1/2},$$

where $f_{i,j} = \phi_i \circ f \circ \phi_j^{-1}$, $\tilde{f}_{i,j}$ are extensions of $f_{i,j}$ from the open domain $V_{i,j} := \phi_j(U_{j,n} \cap f^{-1}(U_{i,n}))$ of $f_{i,j}$ on $\mathbf{R^n}$, $\tilde{f}_{i,j}|_{(\mathbf{R^n}\setminus V_{i,j})} = 0$. Then

$$\|g\|_{H_n} := \|B_p A'_{n;m(n)} f\|'_{H^{[t+1]+[n/2]+1-4m(n)n,b(n)}_{2,\beta+4m(n)n-n/2}(TM_n)}$$

for each $g = B_p A'_{n;m(n)} f$ and $f \in H^{[t+1]+[n/2]+1,b(n)}_{2,\beta-n/2}(TM_n)$, $n = n(p) \in \mathbf{N}$, $p \in \mathbf{N}$.

The Hilbert spaces $H^{[t+1]+[n/2]+1-4m(n)n,b(n)}_{2,\beta+4m(n)n-n/2}(M_n, \mathbf{R})$ and $H^{[t+1]+[n/2]+1-4m(n)n,b(n)}_{2,\beta+4m(n)n-n/2}(M_n, l_{2,\gamma+1+\epsilon})$ have the natural embeddings $\theta_{1,p,i}$ and $\theta_{2,p}$ respectively into X, where $n = n(p)$. There are also embeddings

$$(\bigotimes_{l=1}^{n} H^{\eta,b}_\zeta(\mathbf{R}e_l, \mathbf{R})) \hookrightarrow H^{\eta,b}_\zeta(\mathbf{R^n}, \mathbf{R}),$$

where $\{e_1, .., e_n\}$ is the standard orthonormal basis in $\mathbf{R^n}$. The space X has an orthonormal base which we denote by $\{\bar{e}_{(p,i;l_1,...,l_n)} : p, i, l_1, ..., l_n \in \mathbf{N}; n = n(p) \in \mathbf{N}\}$, where

$$\{\bar{e}_{(p,i;l_1,...,l_n)} : l_1, ..., l_n \in \mathbf{N}\} \subset$$

$$\subset \theta_{1,p,i}(H^{[t+1]+[n/2]+1-4m(n)n,b(n)}_{2,\beta+4m(n)n-n/2}(M_n, \mathbf{R}));$$

$$\{\bar{e}_{(p,i;l_1,...,l_n)} : l_1, ..., l_n \in \mathbf{N}; i \in \mathbf{N}\} \subset$$

$$\subset \theta_{2,p}(H^{[t+1]+[n/2]+1-4m(n)n,b(n)}_{2,\beta+4m(n)n-n/2}(M_n, l_{2,\gamma+1+\epsilon})).$$

Let $J : X \to X$ be a non-degenerate symmetric positive definite nuclear correlation operator, for example,

$$J\bar{e}_{(p,i;l_1,...,l_n)} = (l_1...l_n n^n i)^{-(1+\epsilon)}\bar{e}_{(p,i;l_1,...,l_n)},$$

$$\text{since} \sum_{(i,n,l_1,...,l_n \in \mathbf{N})} (l_1...l_n n^n i)^{-(1+\epsilon)} < \infty$$

for each $0 < \epsilon \in \mathbf{R}$ (see [34]), where $n = n(p) \in \mathbf{N}$, $p \in \mathbf{N}$. Therefore, J indices a Gaussian measure ν on H from $J(X)$ such that ν is quasi-invariant relative to shifts from a linear subspace X_0, $X_0 \subset J(X)$ (see Chapter III in [8] and §I.4 in [24]). There exist $0 < \zeta \in \mathbf{R}$ and a sufficiently small ϵ, $0 < \epsilon \in \mathbf{R}$, such that $S_\phi(V_H) \subset J^{1+\zeta}(X) \subset X_0$ for each $\phi \in P$. In view of Theorems 26.1 and 26.2 [39] and Chapter IV [8] the measure ν on V_H is quasi-invariant and infinitely differentiable relative to the operators $(I + S_\phi)$ for each $\phi \in P$, since the operators L_ϕ and $(I + S_\phi)$ are infinitely differentiable by $\phi \in G'$ [9, 21, 29]. The measure ν on H with the correlation operator J and a zero mean value induces a measure $\tilde{\mu}$ on $Bf(W)$, $\tilde{\mu}(Q) := \nu(A(Q))$ for each $Q \in Bf(W)$. The spaces G and G' in their own uniformities are separable, Lindelöf and paracompact, consequently, there exist locally finite open coverings $\{g_iW(i) : i \in \mathbf{N}\}$ of G and $\{g_iW'(i) : i \in \mathbf{N}\}$ of G' with $g_i \in G'$ and $W(i) \subset W$, $W(0) := W$, $W'(i) \subset W(i) \cap G'$. Hence

$$\bar{\mu}(Q) := \sum_{j=0}^{\infty} 2^{-j}\tilde{\mu}((L_{g_i^{-1}}Q) \cap W(i))$$

is countably additive and quasi-invariant relative to G' on $Bf(G)$. Consequently, $\mu(Q) := \bar{\mu}(Q)/\bar{\mu}(G)$ is the probability quasi-invariant infinitely differentiable measure.

Now let either $0 \le t(1) < 1 \le t$, or $t \ge t(1) \in \mathbf{Z}$ and $\{z(i) : i = 1, 2, ...\}$ be dense in M. This is possible, since M is separable. We may define the following subsets of W and $W(1) \subset Diff^{t(1)}_{\beta,\delta}(M) =:$

$G(1)$, $W(1)\cap G =: W$, $W(k,t(1),c;f) := [g\in W(1) : \rho(k;g,f)\le c]$, $W(k,t,c;f) := [g\in W : \rho(k;g,f)\le c]$, where $\infty > c > 0$, $k\in \mathbf{N}$, $f\in W$, the mappings

$$\rho(k,k';g,f) := \sum_{a,b} \sup[[\tilde\sigma(x)]^{j+\beta}|\nabla^j(f^{-1}g-id)_{a,b}(x)|_{l_{2,\delta}} : j =$$

$$0,1,...,s(1), x\in F(k,k')]+\sup[[\tilde\sigma(x)]^{t(1)+\beta}[\nabla^{s(1)}(f^{-1}\circ g-id)_{a,b}(x)-$$

$$\tau(x,y)\nabla^{s(1)}(f^{-1}\circ g-id)_{a,b}(y)|_{l_{2,\delta}}]/[d(x,y)]^{q(1)} : d(x,y)<\rho(x);$$

$$(x,y)\in F^2(k,k') \text{ and for } (x,y) \text{ a chart exists } U_i\ni x, U_i\ni y, x\ne y]$$

are continuous on $G(1)$ relative to the metric $\rho_{G(1)}(g,f)$ in $G(1)$, $F(k,k') := [z(k),...,z(k')]$ for each $k'>k$; $\rho(k;g,f) := \rho(1,k;f,g)$, $t(1) = s(1)+q(1)$, $0\le s(1)\in\mathbf{Z}$, $0\le q(1)<1$, $\tilde\sigma(x) := \min[\sigma(x),\sigma(y)]$ for a pair (x,y), $h_{a,b} = \phi_a\circ h\circ\phi_b^{-1}$, so $W(k+1,t(1),c;f)\subset W(k,t(1),c;f)$ for each $k\in\mathbf{N}$. Therefore, $\cap\{W(k,t(1),c;f) : k\in \mathbf{N}\} = B_\rho(G(1),f,c)\cap W(1)$, where

$$B_\rho(G(1),f,c) := \{g\in G(1) : \rho_{G(1)}(g,f)\le c\},$$

whence the least σ-field A generated by the family $\mathsf{V}(1) := \{W(k,t(1),c;f) : c>0, k\in\mathbf{N}, f\in W\}$ is such that $\mathsf{A} \supset Bf(W(1))$. Moreover,

$$\bigcap_{k=1}^{\infty}(\bigcap_{m=1}^{\infty}(\bigcup_{n>m} W(k,t(1),1/k;f_n))) = \{f\}$$

for each $f\in W(1)$ and each sequence $\{f_n\}\subset W$ converging to $f\in G(1)$.

Then we put $\mu_1(W(k,t(1),c;f)) := \mu(W(k,t,c;f))$ for each $c>0$, $f\in W$, $k\in N$, whence μ_1 is finitely additive, since from $E(1)\cap L(1)=\emptyset$ in $G(1)$ it follows $E\cap L=\emptyset$ in G, where $E(1)$ and $L(1)$ are in $\mathsf{V}(1)$, E and L are corresponding sets in $\mathsf{V} := \{W(k,t,c;f) : c> 0, k\in\mathbf{N}, f\in W\}$. From the definition of μ it follows that

$$\mu(\bigcap_{k=1}^{\infty}[\bigcap_{m=1}^{\infty}[\bigcup_{n>m} W(k,t,1/k;f_n)]]) = 0,$$

consequently, μ_1 is countably additive on $Bf(W(1))$. Each $\rho(k;g,f)$ is left-invariant: $\rho(k;hg,hf) = \rho(k;g,f)$, hence $hW(k,t,c;f) =$

$W(k,t,c;hf)$ and $hB'(f,c) = B'(hf,c)$ for each $h, f \in G$, where $B'(f,c) := \bigcap_{k=1}^{\infty} W(k,t,c;f)$, $c > 0$. Therefore, μ_1 is extendable to the quasi-invariant infinitely differentiable measure on $G(1)$ relative to G' (see above the analogous case of μ on G and also [28]). □

THEOREM 3.11. *Let (1) G be a group of diffeomorphisms as in Theorem 3.10 or in [26], or (2) G be an infinite-dimensional over the corresponding field (**R** or the local field **K**) Banach-Lie group such that for its Banach-Lie algebra g there is not any dense subalgebra g' such that $ad(h)$ are nuclear in the case of **R** or compact in the case of **K** operators for each $h \in \mathsf{g}'$. Assume that G' is a dense subgroup of G. Then it does not have non-trivial quasi-invariant measure μ with values in **R** or **F** correspondingly which is quasi-invariant relative to (a) the left L_ψ and right R_ψ shifts simultaneously or (b) relative to inner automorphisms $\alpha_h(f) := h^{-1}fh$ for each $h \in G'$, where **F** is a local field.*

Proof. For a Banach-Lie group there exists the exponential mapping $\tilde{exp} : V \to W$ which is the local isomorphism of open V and W, where $0 \in V \subset \mathsf{g}$ and $e \in W \subset G$. Therefore, $\tilde{ln}(W \cap G') =: V'$ is dense in V (see the Hausdorff series, §§II.6-8, ch. II, §VII.3 in [5]).

For a group of diffeomorphisms there exists a refinement $At"(M)$ of $At(M)$ such that $At"(M)$ provides a locally finite covering of M by charts $U"_j$. Therefore, $U := U_1 \setminus (\bigcup_{i \ne 1} \bar{U}"_i)$ is open M [11]. Let G_U denote a subgroup of G consisting of $f \in G$ with $supp(f) \subset U$. The set $M \setminus U$ is closed in M, hence G_U is closed in G. From the definition of topology in G it follows that $G' \cap G_U =: G'_U$ is dense in G_U. In view of [11, 19] a retraction r exists of G onto G_U. Hence each quasi-invariant measure μ on G relative to G' induces a quasi-invaraint measure $\nu(S) = \mu(r^{-1}S)$ for each $S \in Bf(G_U)$ relative to G'_U. But U is the flat manifold and exp_x is trivial for each $x \in U$, consequently, $\tilde{E}$ is trivial on a sufficiently small neighbourhood W of $id \in G_U$. We denote ν and G_U again by μ and G.

To each local one-parameter subgroup of G a vector field on U corresponds for the group of diffeomorphisms or there corresponds an element of g for the Banach-Lie group. If G is the non-Archimedean group of diffeomorphisms then $ad(h)$ is not a compact operator for each smooth of class C^∞ element h in $\tilde{E}^{-1}(W')$, where $W' = W \cap G'$.

For the real group of diffeomorphisms it is not a nuclear operator for each such h. This follows from the consideration of the algebra of smooth vector fields on U and the fact that the group of diffeomorphisms is simple and perfect.

If μ is a quasi-invariant measure on G and fulfils either condition (a) or (b) then $\tilde{l}n$ or $\tilde{E}^{-1}$ induce a measure λ on V that is quasi-invariant relative to $ad(h)$ for each $h \in V'$, $V' = \tilde{l}n(W')$ or $V' = \tilde{E}^{-1}(W')$ respectively. But this contradicts Theorems 2.31, 3.12, Lemma 3.26 [27] in the non-Archimedean cases, the Minlos-Sazonov Theorem and Theorem 19.1 [39] in the cases of the Banach-Lie group or M over $\mathbf{R}$ for the group of diffeomorphisms. □

NOTE 3.12. Let N and M be two manifolds such that N is a Hilbert manifold and M is a Banach manifold as in Definition 2.8 [29]. In view of corollary 2.9 [29] there exists a dense subgroup $Diff^{t'}_{\beta,\gamma'}(N)$ in $Diff^t_{\beta,\gamma}(M)$ with topologies τ' and τ respectively such that $\tau|Diff^{t'}_{\beta,\gamma'}(N) \subset \tau'$, since $\delta > \gamma + 2$. Therefore, quasi-invariant and infinitely differentiable measure λ on $Diff^t_{\beta,\gamma'}(N)$ relative to G' (see Theorem 3.10) induces quasi-invariant and infinitely differentiable measure μ on $Diff^t_{\beta,\gamma}(M)$. This justifies the consideration of Hilbert manifolds only in Theorem 3.10.

4. Irreducible unitary representations.

THEOREM 4.1. *Let μ be a quasi-invariant relative to G' measure on $Bf(G)$ with $G := Diff^t_{\beta,\delta}(M)$ as in Theorem 3.10. Assume also that $\bar{H} := L^2(G, \mu, \mathbf{C})$ is the standard Hilbert space of equivalence classes of square-integrable (by μ) functions $f : G \to \mathbf{C}$. Then there exists a strongly continuous injective homomorphism $T : G' \to U(\bar{H})$, where $U(\bar{H})$ is the unitary group on $\bar{H}$ in a topology induced from a Banach space $L(\bar{H}, \bar{H})$ of continuous linear operators supplied with the operator norm.*

Proof. Let f and h be in $\bar{H}$, their scalar product is given by

$$(f, h) := \int_G \bar{h}(g) f(g) \mu(dg),$$

where f and $h : G \to \mathbf{C}$, $\bar{h}$ denotes complex conjugated h. There exists the regular representation $T : G' \to U(\bar{H})$ defined by the following formula:

$$T(z)f(g) := (\rho(z,g))^{1/2} f(z^{-1}g),$$

where $\rho(z,g) = \mu_z(dg)/\mu(dg)$, $\mu_z(S) = \mu(z^{-1}S)$ for each $S \in Bf(G)$, $z \in G'$. For each fixed z the quasi-invariance factor $\rho(z,g)$ is continuous by g, hence $T(z)f(g)$ is measurable, if $f(g)$ is measurable relative to $Af(G,\mu)$ and $Bf(\mathbf{C})$. Therefore,

$$(T(z)f(g), T(z)h(g)) = \int_G \bar{h}(z^{-1}g) f(z^{-1}g) \rho(z,g) \mu(dg) = (f,h),$$

consequently, T is unitary. From

$$\mu_{z'z}(dg)/\mu(dg) = \rho(z'z,g) = \rho(z,(z')^{-1}g)\rho(z',g)$$

$$= [\mu_{z'z}(dg)/\mu_{z'}(dg)][\mu_{z'}(dg)/\mu(dg)]$$

it follows that $T(z')T(z) = T(z'z)$ and $T(id) = I$, $T(z^{-1}) = T^{-1}(z)$.

For each $v > 0$ and a finite family of continuous functions $f_j : G \to \mathbf{C}$ with $\|f_j\|_{\bar{H}} = 1$, $j = 1, ..., m$, there is an open neighbourhood V of id in G' in the topology of G', such that $|\rho(z,g) - 1| < v$ for each $z \in V$ and each $g \in F$ for some open F in G, $id \in F$ with $\mu_z^f(G \setminus F) < v$ for each $z \in V$ and $f \in \{f_1, ..., f_m\}$, where $\mu^f(dg) := |f(g)|\mu(dg)$, $\mu_z^f(S) := \mu^f(z^{-1}S)$ for each $S \in Bf(G)$ (see Theorems 26.1, 26.2 in [39] and the proof of Theorem 3.10).

In $\bar{H}$ continuous functions $f(g)$ are dense, hence

$$\int_G |f(g) - f(z^{-1}g)(\rho(z,g))^{1/2}|^2 \mu(dg) < 4v$$

for each $f \in \{f_1, ..., f_m\}$ and $z \in V' = V \cap V"$, where $V"$ is an open neighbourhood of id in G' such that $\|f(g) - f(z^{-1}g)\|_{\bar{H}} < v$ for each $z \in V"$, $0 < v < 1$. Consequently T is strongly continuous, that is, T is continuous relative to the strong topology on $U(\bar{H})$ induced from the strong topology on $L(\bar{H}, \bar{H})$, (see its definition in [13]). Moreover, T is injective, since for each $g \neq id$ there is $f \in C^0(G, \mathbf{C}) \cap \bar{H}$, such that $f(id) = 0$, $f(g) = 1$, and $\|f\|_{\bar{H}} > 0$, so $T(f) \neq I$. □

NOTE 4.2. In general T is not continuous relative to the norm topology on $U(\bar{H})$, since for each $z \ne id \in G'$ and each $1/2 > v > 0$ there is $f \in \bar{H}$ with $\|f\|_{\bar{H}} = 1$, such that $\|f - T(z)f\|_{\bar{H}} > v$, when $supp(f) =: J_f$ is sufficiently small with $zJ_f \cap J_f = \emptyset$.

THEOREM 4.3. *Let G be a group of diffeomorphisms with a real probability quasi-invariant measure μ relative to a dense subgroup G' as in Theorem 3.10. Then μ may be chosen such that the associated regular unitary representation (see §4.1) of G' is irreducible.*

Proof. Let a measure ν on a Banach space H be of the same type as in the proof of Theorem 3.10. Let a ν-measurable function $f : H \to \mathbf{C}$ be such that $\nu(\{x \in H : f(x+y) \ne f(x)\} = 0$ for each $y \in X_0$ with $f \in L^1(H, \nu, \mathbf{C})$. Let also $P_k : l_2 \to L(k)$ be projectors such that $P_k(x) = x_k$ for each $x = (\sum_{j \in \mathbf{N}} x^j e_j)$, where $x_k := \sum_{j=1}^k x^j e_j$, $x_k \in L(k)$, $L(k) := sp_{\mathbf{R}}(e_1, ..., e_k)$, $sp_{\mathbf{R}}(e_j : j \in \mathbf{N}) := \{y : y \in l_2; y = \sum_{j=1}^n x^j e_j; x^j \in \mathbf{R}; n \in \mathbf{N}\}$. Since the dense subspace X in H is isomorphic with l_2, then each finite-dimensional subspace $L(k)$ is complemented in H [32]. From the proof of Proposition II.3.1 [8] in view of the Fubini Theorem there exists a sequence of cylindrical functions

$$f_k(x) = f_k(x^k) = \int_{H \ominus L(k)} f(P_k x + (I - P_k)y) \nu_{I-P_k}(dy)$$

which converges to f in $L^1(H, \nu, \mathbf{C})$, where $\nu = \nu_{L(k)} \otimes \nu_{I-P_k}$, ν_{I-P_k} is the measure on $H \ominus L(k)$. Each cylindrical function f_k is ν-almost everywhere constant on H, since $L(k) \subset X_o$ for each $k \in \mathbf{N}$, consequently, f is ν-almost everywhere constant on H. Let $A : W \to V_H$ be the same as in §3.10. From the construction of G' and μ with the help of the local diffeomorphism A and ν it follows that, if a function $f \in L^1(G, \mu, \mathbf{C})$ satisfies the following condition $f^h(g) = f(g)$ $(mod\ \mu)$ by $g \in G$ for each $h \in G'$, then $f(x) = const$ $(mod\ \mu)$, where $f^h(g) := f(hg)$, $g \in G$.

Let $f(g) = ch_U(g)$ be the characteristic function of a subset U, $U \subset G$, $U \in Af(G, \mu)$, then $f(hg) = 1 \Leftrightarrow g \in h^{-1}U$. If $f^h(g) = f(g)$ is true by $g \in G$ μ-almost everywhere, then $\mu(\{g \in G : f^h(g) \ne f(g)\}) = 0$, that is $\mu((h^{-1}U) \triangle U) = 0$, consequently, the measure μ satisfies the condition (P) from §VIII.19.5 [13], where $A \triangle B :=$

$(A \setminus B) \cup (B \setminus A)$ for each $A, B \subset G$. For each subset $E \subset G$ the outer measure is bounded, $\mu^*(E) \le 1$, since $\mu(G) = 1$ and μ is non-negative [4], consequently, there exists $F \in Bf(G)$ such that $F \supset E$ and $\mu(F) = \mu^*(E)$. This F may be interpreted as the least upper bound in $Bf(G)$ relative to the latter equality. In view of the Proposition VIII.19.5 [13] the measure μ is ergodic, that is for each $U \in Af(G, \mu)$ and $F \in Af(G, \mu)$ with $\mu(U) \times \mu(F) \neq 0$ there exists $h \in G'$ such that $\mu((h \circ E) \cap F) \neq 0$.

From Theorem I.1.2 [8] it follows that $(G, Bf(G))$ is a Radon space, since G is separable and complete. Therefore, a class of compact subsets approximates from below each measure μ^f, $\mu^f(dg) := |f(g)|\mu(dg)$, where $f \in L^2(G, \mu, \mathbf{C}) =: \bar{H}$. Due to the Egorov Theorem 2.3.7 [12] for each $\epsilon > 0$ and for each sequence $f_n(g)$ converging to $f(g)$ for μ-almost every $g \in G$, when $n \to \infty$, there exists a compact subset K in G such that $\mu(G \setminus \mathsf{K}) < \epsilon$ and $f_n(g)$ converges on K uniformly by $g \in \mathsf{K}$, when $n \to \infty$. In each Hilbert space $L^2(\mathbf{R^n}, \lambda, \mathbf{R})$ the linear span of functions $f(x) = exp[(b, x) - (ax, x)]$ is dense, where b and $x \in \mathbf{R^n}$, a is a real symmetric positive definite $n \times n$ matrix, $(*, *)$ is the standard scalar product in $\mathbf{R^n}$ and λ is the Lebesgue measure on $\mathbf{R^n}$. If a non-linear operator U on X satisfies conditions of Theorem 26.1 [39], then

$$\nu^U(dx)/\nu(dx) = |detU'(U^{-1}(x))|\rho_\nu(x - U^{-1}(x), x),$$

where $\nu^U(B) := \nu(U^{-1}B)$ for each $B \in Bf(X)$,

$$\rho_\nu(z, x) = exp\{\sum_{l=1}^{\infty}[2(z, e_l)(x, e_l) - (z, e_l)^2]/\lambda_l\}$$

by Theorem 26.2 [39], where λ_l and e_l are eigenvalues and eigenfunctions of the correlation operator J on X enumerated by $l \in \mathbf{N}$, $z \in X_0$, $\rho_\nu(z, x) := \nu_z(dx)/\nu(dx)$, $\nu_z(B) := \nu(B - z)$ for each $B \in Bf(X)$. Since the Gaussian measure ν induces with the help of subalgebras of cylinder subsets in $Bf(H)$ and $Bf(X)$ the corresponding Gaussian measure on H, which is also denoted by ν, then analogous formulas of quasi-invariance factor are true for ν on H [8]. Hence in view of the Stone-Weierstrass Theorem A.8 [13] an algebra $\mathsf{V}(Q)$ of finite pointwise products of functions from the following space $sp_{\mathbf{C}}\{\psi(g) := (\rho(h, g))^{1/2} : h \in G'\} =: Q$ is dense in

$L^2(G, \mu, \mathbf{C})$, since $\rho(e, g) = 1$ for each $g \in G$ and $L_h : G \to G$ are diffeomorphisms of the manifold G, $L_h(g) = hg$.

For each $m \in \mathbf{N}$ there are C^∞-curves $\phi_j^b \in G' \cap W$, where $j = 1, ..., m$ and $b \in (-2, 2) := \{a : -2 < a < 2; a \in \mathbf{R}\}$ is a parameter, such that $\phi_j^b|_{b=0} = e$ and $\phi_j := \phi_j^1$ and vectors $(\partial\phi_j^b/\partial b)|_{b=0}$ for $j = 1, ..., m$ are linearly independent in T_eG'. Then the following condition $det(\Psi(g)) = 0$ defines a submanifold G_Ψ in G of codimension over $\mathbf{R}$,

(i) $codim_{\mathbf{R}} G_\Psi \geq 1$, where $\Psi(g)$ is a matrix dependent from $g \in G$ with matrix elements

$$\Psi_{l,j}(g) := D^{2l}_{\phi_j}(\rho(\phi_j, g))^{1/2}.$$

If $f \in \bar{H}$ is such that $(f(g), (\rho(\phi, g))^{1/2})_{\bar{H}} = 0$ for each $\phi \in G' \cap W$, then differentials of these scalars products by ϕ are zero. But $\mathsf{V}(Q)$ is dense in $\bar{H}$ and in view of condition (i) this means that $f = 0$, since for each m there are $\phi_j \in G' \cap W$ such that $det\Psi(g) \neq 0$ μ-almost everywhere on G, $g \in G$. If $\|f\|_{\bar{H}} > 0$, then $\mu(supp(f)) > 0$, consequently, $\mu(G'supp(f)) = 1$, since $G'U = G$ for each open U in G and for each $\epsilon > 0$ there exists an open U, $U \supset supp(f)$, such that $\mu(U \setminus supp(f)) < \epsilon$.

This means that the vector f_0 is cyclic, where $f_0 \in \bar{H}$ and $f_0(g) = 1$ for each $g \in G$. From the construction of μ it follows that for each $f_{1,j}$ and $f_{2,j} \in \bar{H}$, $j = 1, ..., n$, $n \in \mathbf{N}$ and each $\epsilon > 0$ there exists $h \in G'$ such that $|(T_h f_{1,j}, f_{2,j})_{\bar{H}}| \leq \epsilon |(f_{1,j}, f_{2,j})_{\bar{H}}|$, when $|(f_{1,j}, f_{2,j})_{\bar{H}}| > 0$, since G is the Radon space by Theorem I.1.2 [8] and G is not locally compact. This means that there is not any finite-dimensional G'-invariant subspace H' in $\bar{H}$ such that $T_h H' \subset H'$ for each $h \in G'$ and $H' \neq \{0\}$. Hence if there is a G'-invariant closed subspace H' in $\bar{H}$ it is isomorphic with the subspace $L^2(V, \mu, \mathbf{C})$, where $V \in Bf(G)$.

Let A_G denotes a $*$-subalgebra of $L(\bar{H}, \bar{H})$ generated by the family of unitary operators $\{T_h : h \in G'\}$. In view of the von Neumann double commuter Theorem (see §VI.24.2 [13]) A_G" coincides with the weak and strong operator closures of A_G in $L(\bar{H}, \bar{H})$, where A_G' denotes the commuting algebra of A_G and $\mathsf{A}_G" = (\mathsf{A}_G')'$. Suppose that λ is a probability Radon measure on G' such that λ has not any atoms and $supp(\lambda) = G'$. Let $a(x) \in L^\infty(G, \mu, \mathbf{C})$, f and $g \in \bar{H}$, $\beta(h) \in L^2(G', \lambda, \mathbf{C})$. Since $L^2(G', \lambda, \mathbf{C})$ is infinite-dimensional, then for each finite family of $a \in \{a_1, ..., a_m\} \subset L^\infty(G, \mu, \mathbf{C})$, $f \in \{f_1, ..., f_m\} \subset \bar{H}$ there exists $\beta(h) \in L^2(G', \lambda, \mathbf{C})$, $h \in G'$, such that

$$\beta \text{ is orthogonal to } \int_G \bar{f}_s(g)[f_j(h^{-1}g)(\rho(h,g))^{1/2} - f_j(g)]\mu(dg)$$

for each $s, j = 1, ..., m$. Hence each operator of multiplication on $a_j(g)$ belongs to A_G", since due to cyclicity of f_0 there exists $\beta(h)$ such that

$$(f_s, a_j f_l) = \int_G \int_{G'} \bar{f}_s(g)\beta(h)(\rho(h,g))^{1/2} f_l(h^{-1}g)\lambda(dh)\mu(dg)$$

$$= \int_G \int_{G'} \bar{f}_s(g)\beta(h)(T_h f_l(g))\lambda(dh)\mu(dg),$$

$$\int_G \bar{f}_s(g) a_j(g) f_l(g)\mu(dg) =$$

$$\int_G \int_{G'} \bar{f}_s(g)\beta(h) f_l(g)\lambda(dh)\mu(dg) = (f_s, a_j f_l).$$

Hence A_G" contains subalgebra of all operators of multiplication on functions from $L^\infty(G, \mu, \mathbf{C})$.

Let us remind the following. A Banach bundle B over a Hausdorff space G' is a bundle $< B, \pi >$ over G', together with operations and norms making each fiber B_h $(h \in G')$ into a Banach space such that conditions $BB(i - iv)$ are satisfied:

$BB(i)$ $x \to \|x\|$ is continuous on B to $\mathbf{R}$;

$BB(ii)$ the operation $+$ is continuous as a function on $\{(x,y) \in B \times B : \pi(x) = \pi(y)\}$ to B;

$BB(iii)$ for each $\lambda \in \mathbf{C}$, the map $x \to \lambda x$ is continuous on B to B;

$BB(iv)$ if $h \in G'$ and $\{x_i\}$ is any net of elements of B such that $\|x_i\| \to 0$ and $\pi(x_i) \to h$ in G', then $x_i \to 0_h$ in B, where $\pi : B \to G'$ is a bundle projection, $B_h := \pi^{-1}(h)$ is the fiber over h (see §II.13.4 [13]). If G' is a Hausdorff topological group, then a Banach algebraic bundle over G' is a Banach bundle $\mathsf{B} =< B, \pi >$ over G' together with a binary operation $\bullet$ on B satisfying conditions $AB(i - v)$:

$AB(i)$ $\pi(b \bullet c) = \pi(b)\pi(c)$ for b and $c \in B$;

$AB(ii)$ for each x and $y \in G'$ the product $\bullet$ is bilinear on $B_x \times B_y$ to B_{xy};

$AB(iii)$ the product $\bullet$ on B is associative;

$AB(iv)$ $\|b \bullet c\| \le \|b\| \times \|c\|$ $(b, c \in B)$;

$AB(v)$ the map $\bullet$ is continuous on $B \times B$ to B (see §VIII.2.2 [13]). With G' and a Banach algebra A the trivial Banach bundle $\mathsf{B} = \mathsf{A} \times G'$ is associative, in particular let $\mathsf{A} = \mathbf{C}$ (see §VIII.2.7 [13]).

The regular representation T of G' gives rise to a canonical regular $\bar{H}$-projection-valued measure $\bar{P}$: $\bar{P}(W)f = Ch_W f$, where $f \in \bar{H}$, $W \in Bf(G)$, Ch_W is the characteristic function of W. Therefore,

$$T_h \bar{P}(W) = \bar{P}(h \circ W) T_h$$

for each $h \in G'$ and $W \in Bf(G)$, since $\rho(h, h^{-1} \circ g)\rho(h, g) = 1 = \rho(e, g)$ for each $(h, g) \in G' \times G$, $Ch_W(h^{-1} \circ g) = Ch_{h \circ W}(g)$ and $T_h(\bar{P}(W)f(g)) = \rho(h, g)^{1/2} \bar{P}(h \circ W) f(h^{-1} \circ g)$. Thus $< T, \bar{P} >$ is a system of imprimitivity for G' over G, which is denoted T^μ. This means that conditions $SI(i - iii)$ are satisfied:

$SI(i)$ T is a unitary representation of G';

$SI(ii)$ $\bar{P}$ is a regular $\bar{H}$-projection-valued Borel measure on G and

$SI(iii)$ $T_h \bar{P}(W) = \bar{P}(h \circ W) T_h$ for all $h \in G'$ and $W \in Bf(G)$.

For each $F \in L^\infty(G, \mu, \mathbf{C})$ let $\bar{\alpha}_F$ be the operator in $L(\bar{H}, \bar{H})$ consisting of multiplication by F: $\bar{\alpha}_F(f) = Ff$, $f \in \bar{H}$. The map $F \to \bar{\alpha}_F$ is an isometric $*$-isomorphism of $L^\infty(G, \mu, \mathbf{C})$ into $L(\bar{H}, \bar{H})$ (see §VIII.19.2[13]). Therefore, Propositions VIII.19.2,5[13] (using the approach of this particular case given above) are applicable in our situation.

If $\bar{p}$ is a projection onto a closed T^μ-stable subspace of $\bar{H}$, then $\bar{p}$ commutes with all $\bar{P}(W)$. Hence $\bar{p}$ commutes with multiplication by all $F \in L^\infty(G, \mu, \mathbf{C})$, so by VIII.19.2 [13] $\bar{p} = \bar{P}(V)$, where $V \in Bf(G)$. Also $\bar{p}$ commutes with all T_h, $h \in G'$, consequently, $(h \circ V) \setminus V$ and $(h^{-1} \circ V) \setminus V$ are μ-null for each $h \in G'$, hence $\mu((h \circ V) \triangle V) = 0$ for all $h \in G'$. In view of ergodicity of μ and proposition VIII.19.5 [13] either $\mu(V) = 0$ or $\mu(G \setminus V) = 0$, hence either $\bar{p} = 0$ or $\bar{p} = I$, where I is the unit operator. Hence T is the irreducible unitary representation. □

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